Tutorial 9

1. Consider a situation where the parameter space has two elements, $\Theta = \{\theta_0, \theta_1\}$ Suppose we want to test $H_0 : \theta = \theta_0$ versus the alternative, $H_1 : \theta = \theta_1$. One way of doing this is to consider the test statistic

$$\nu(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)},$$

the ratio of the likelihood functions. This is a different formulation, but gives the same test as the Likelihood Ratio statistic. We reject $H_0 : \theta = \theta_0$ in favour of $H_1 : \theta = \theta_1$ if $\nu(x)$ is large.

We have a single observation on a random variable X with distribution F, where F is either $U(0, 1)$ or Exp(1). Construct the test described above, with significance level $\alpha = 0.05$ to test $H_0 : X \sim U(0, 1)$ versus the alternative $H_1 : X \sim \text{Exp}(1)$. Compute the rejection region for the test and compute its power when H_1 is true.

2. We have a single observation on the random variable X with density function

$$p(x, \theta) = \begin{cases} \theta e^{-x} + 2(1 - \theta)e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\theta \in [0, 1]$ is an unknown parameter.

   (a) Construct a test between the null hypothesis $H_0 : \theta = 0$ versus the alternative $H_1 : \theta > 0$ with significance level $\alpha = 0.05$. (Use LRT method).

   (b) Compute the power function of this test.

3. Let $(U_j)_{j\geq 1}$ be a sequence of i.i.d. $U(0, 1)$ random variables. Let X be a random variable. It is required to test

$$H_0 : X = \min\{U_1, \ldots, U_k\} \qquad \text{versus} \qquad H_1 : X = \min\{U_1, \ldots, U_l\} \qquad l < k.$$

   (a) Construct a test with significance level α based on the statistic $\nu(x) := \frac{L(H_1;x)}{L(H_0;x)}$ where $L(H_1; x)$ and $L(H_0; x)$ denote the likelihoods based on H_1 and H_0 respectively (each hypothesis corresponds to a single parameter value).

   (b) What is the largest value of the ratio $\frac{l}{k}$ so that a test with significance $\alpha = 0.05$ has power at least 0.95?

4. Consider a population with three types of individual, labelled 1, 2 and 3, which occur in the Hardy - Weinberg proportions

$$p_\theta(1) = \theta^2 \qquad p_\theta(2) = 2\theta(1 - \theta) \qquad p_\theta(3) = (1 - \theta)^2.$$

For a sample $X_1, \ldots, X_n$ from this population, let $N_1 = \sum_{j=1}^n \mathbf{1}_1(X_j)$, $N_2 = \sum_{j=1}^n \mathbf{1}_2(X_j)$, $N_3 = \sum_{j=1}^n \mathbf{1}_3(X_j)$ denote the number of appearances of 1, 2, 3 respectively in the sample. Let $0 < \theta_0 < \theta_1 < 1$.

(a) Show that $\nu(\underline{x}; \theta_0, \theta_1) = \frac{L(\theta_1; \underline{x})}{L(\theta_0; \underline{x})}$ is an increasing function of $2N_1 + N_2$. (n is fixed).

(b) Show that if $c > 0$ and $\alpha \in (0, 1)$ satisfy

$$\mathbb{P}_{\theta_0}(2N_1 + N_2 > c) = \alpha$$

then a test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with a given significance level α that rejects H_0 if and only if $2N_1 + N_2 > c$ corresponds to the test where $H_0 : \theta = \theta_0$ is rejected for large values of $\nu(\underline{x}; \theta_0, \theta_1)$, defined in the previous part.

5. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$ variables and let $M_n = \max\{X_1, \ldots, X_n\}$. Consider a test of $H_0 : \theta \leq \theta_0$ versus the alternative $H_1 : \theta > \theta_0$ where H_0 is rejected if and only if $M_n > c$ for some value $c > 0$.

   (a) Compute the power function of this test and show that it is monotone increasing in θ.

   (b) For $\theta_0 = \frac{1}{2}$, compute the value of c which would give the test a size exactly 0.05.

   (c) Compute the value of n so that the test of size 0.05 for $\theta_0 = \frac{1}{2}$ has power 0.98 for $\theta = \frac{3}{4}$.

6. Consider a simple hypothesis test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Suppose that the test statistic T has a continuous distribution and the null hypothesis is rejected for $t \geq c$ where t is the observed value of T for some c and that, as a function of c, the size of the test is:

$$\alpha(c) = \mathbb{P}_{\theta_0}(T \geq c).$$

   Prove that, for $\theta = \theta_0$, $\alpha(T) \sim U(0, 1)$.

7. Let $T_1, \ldots, T_r$ be independent test statistics for the same simple $H_0 : \theta = \theta_0$ and that for each j, T_j has a continuous distribution. Let $\alpha_j(c) = \mathbb{P}_{\theta_0}(T_j \geq c)$. Show that, under H_0, $\tilde{T} = -2\sum_{j=1}^r \log \alpha_j(T_j) \sim \chi^2_{2r}$.

8. Let $F_0(y) = \mathbb{P}(Y < y)$ where Y is a non negative random variable representing a survival time. Assume that F_0 has a density f_0. Let $X_1, \ldots, X_n$ be i.i.d. each with an alternative distribution, representing survival time under an alternative treatment. The new distribution is considered to take the form

$$G(y, \Delta) = 1 - (1 - F_0(y))^{\Delta} \qquad y > 0 \quad \Delta > 0.$$

   To test whether the new treatment is beneficial, test $H_0 : \Delta \leq 1$ versus $H_1 : \Delta > 1$. Compute the Likelihood Ratio Test and compute the critical region for a test with significance level α in terms of n and an appropriate χ^2 distribution. (This is known as the *Lehmann alternative*).

Answers

1. $f_0(x) = 1$ for $0 \leq x \leq 1$. $f_1(x) = \exp\{-x\}$ for $x \geq 0$. For the Neyman Pearson test, the ratio is:

$$\nu(x) = \frac{f_1(x)}{f_0(x)} = \begin{cases} e^{-x} & 0 \leq x \leq 1 \\ +\infty & x > 1 \\ \text{undefined} & x < 0 \end{cases}$$

By the Neyman Pearson lemma, a test is a UMP test if and only if there is a k such that

$$x \in \mathcal{R} \quad \text{if} \quad \nu(x) > k \quad \text{and} \quad x \in \mathcal{R}^c \quad \text{if} \quad \nu(x) < k$$

For a 5% significance level,

$$\begin{aligned} 0.05 &= \mathbb{P}(\nu(X) > k | X \sim U(0,1)) \\ &= \mathbb{P}(\{X < -\log k\} \cup \{X > 1\} | X \sim U(0,1)) = -\log k \Rightarrow k = e^{-0.05} \end{aligned}$$

Rejection region $\mathcal{R} = [0, 0.05] \cup [1, +\infty]$. The power of the test when $X \sim Exp(1)$ is

$$\mathbb{P}(\{X < 0.05\} \cup \{X > 1\} | X \sim Exp(1)) = (1 - e^{-0.05}) + e^{-1}.$$

2. (a) LRT First find $\widehat{\theta}_{ML}$;

$$L(\theta; x) = \theta(e^{-x} - 2e^{-2x}) + 2e^{-2x}$$

$$\widehat{\theta}_{ML} = \begin{cases} 0 & x < \log 2 \\ 1 & x > \log 2 \\ \in [0,1] & x = \log 2 \end{cases}$$

$$\lambda(x) = \frac{L(0, x)}{L(\widehat{\theta}_{ML}, x)} = \begin{cases} 1 & 0 \leq x \leq \log 2 \\ \frac{2e^{-2x}}{e^{-x}} = 2e^{-x} & x > \log 2. \end{cases}$$

Reject H_0 if

$$\lambda(x) < c \Rightarrow 2e^{-x} < c \Rightarrow x > -\log \frac{c}{2} = k.$$

k determined by:

$$0.05 = \mathbb{P}_{\theta=0}(X > k) = \int_k^{\infty} 2e^{-2x} dx = e^{-2k} \Rightarrow k = \frac{1}{2} \log 20.$$

$$\mathcal{R} = (\frac{1}{2} \log 20, +\infty).$$

(b)

$$\beta(\theta) = \mathbb{P}_\theta(X > \frac{1}{2} \log 20) = \frac{\theta}{\sqrt{20}} + (1 - \theta)\frac{1}{20}.$$

3. (a) Under H_0, X has density $f_0(x) = k(1-x)^{k-1} \quad 0 \le x \le 1$ and under H_1, X has density $f_1(x) = l(1-x)^{l-1} \quad 0 \le x \le 1$.

The N-P ratio is:

$$\nu(x) = \frac{f_1(x)}{f_0(x)} = \frac{l}{k}(1-x)^{l-k}.$$

For N-P test, reject H_0 for large values of $\lambda(x)$;

$$x \in \mathcal{R} \quad \text{if} \quad \frac{l}{k}(1-x)^{l-k} > c \qquad x \in \mathcal{R}^c \quad \text{if} \quad \frac{l}{k}(1-x)^{l-k} < c.$$

Simplifying gives:

$$\mathcal{R} = \{x|\frac{l}{k}(1-x)^{l-k} > c\} \Rightarrow \mathcal{R} = \{x|x > 1 - \left(\frac{ck}{l}\right)^{1/(l-k)} = K\}$$

where we set $K := 1 - \left(\frac{ck}{l}\right)^{1/(l-k)}$. Then, for a size α test, K satisfies:

$$\alpha = \mathbb{P}_0(X > K) = \int_K^1 k(1-x)^{k-1}dx = (1-K)^k \Rightarrow K = 1 - \alpha^{1/k}.$$

so that H_0 is rejected for $X \in \mathcal{R}$ where

$$\mathcal{R} = [1 - \alpha^{1/k}, 1].$$

(b) We require a power of 0.95 when $\alpha = 0.05$ and H_1 is true. Then:

$$0.95 = \int_{1-0.05^{1/k}}^1 l(1-x)^{l-1}dx = 0.05^{l/k} \Rightarrow \frac{l}{k} = \frac{\log 0.95}{\log 0.05} = \frac{\log(20/19)}{\log 20}.$$

4. (a)

$$\begin{aligned} \nu(\underline{x};\theta_0,\theta_1) &= \left(\frac{\theta_1}{\theta_0}\right)^{2N_1} \left(\frac{\theta_1(1-\theta_1)}{\theta_0(1-\theta_0)}\right)^{N_2} \left(\frac{(1-\theta_1)}{(1-\theta_0)}\right)^{2n-2(N_1+N_2)} \\ &= \left(\frac{\theta_1}{\theta_0}\right)^{2N_1+N_2} \left(\frac{1-\theta_1}{1-\theta_0}\right)^{2n-(2N_1+N_2)} \end{aligned}$$

and since $\frac{\theta_1}{\theta_0} > 1$ and $\frac{1-\theta_1}{1-\theta_0} < 1$, this is increasing in $2N_1 + N_2$ for fixed n.

(b) Let $\mathcal{R}$ denote rejection region from Neyman Pearson lemma, a test is UMP if and only if it satisfies

$$\underline{x} \in \mathcal{R} \quad \text{if} \quad \nu(\underline{x};\theta_0,\theta_1) > k \qquad \underline{x} \in \mathcal{R}^c \quad \text{if} \quad \nu(\underline{x};\theta_0,\theta_1) < k$$

for some k.

$$\nu(\underline{x};\theta_0,\theta_1) > k \quad \Rightarrow \quad (2N_1+N_2)\left(\log\frac{\theta_1}{\theta_0} - \log\frac{1-\theta_1}{1-\theta_0}\right) + 2n\log\frac{1-\theta_1}{1-\theta_0} > \log k$$
$$\Rightarrow \quad 2N_1+N_2 > \frac{\log k - 2n\log\frac{1-\theta_1}{1-\theta_0}}{\log\frac{\theta_1}{\theta_0} - \log\frac{1-\theta_1}{1-\theta_0}} = K.$$

To get the UMP test of significance level α, K has to be chosen such that

$$\mathbb{P}(2N_1+N_2 > K) = \alpha$$

so that $K = c$.

5. (a) $P(\theta) = \mathbb{P}_\theta(M_n > c) = 1 - \left(\frac{c}{\theta}\right)^n \mathbf{1}_{\{c<\theta\}}$ (monotone non-decreasing)
   (b) $0.05 = P(\frac{1}{2}) = 1 - (2c)^n \mathbf{1}_{\{c<\theta\}} \Rightarrow c = \frac{1}{2}(0.95)^{1/n}$.
   (c) $0.98 = P(\frac{3}{4}) = 1 - 0.95\left(\frac{2}{3}\right)^n$ so that $n = \frac{\log(95/2)}{\log(3/2)}$. First integer greater than or equal to this gives $n = 10$.

6. Let $\gamma(c) = 1 - \alpha(c)$ then, since T is continuous, γ is the c.d.f. of T and hence $\gamma(T) \sim U(0,1)$. It follows that $\alpha(T) = 1 - \gamma(T) \sim U(0,1)$.

7. This follows directly from the previous exercise: for each j $\alpha_j(T_j) \sim U(0,1)$ from which it follow directly that $-\log\alpha_j(T_j) \sim \text{Exp}(1)$ and hence $-2\sum_{j=1}^r \log\alpha_j(T_j) \sim \Gamma(r,\frac{1}{2}) = \chi^2_{2r}$.

8. $g(y,\Delta) = \Delta(1-F_0(y))^{\Delta-1} f_0(y)$. It follows that

$$\lambda(\underline{x}) = \frac{\sup_{0\le\Delta\le 1}\Delta^n\left(\prod_{j=1}^n(1-F_0(x_j))\right)^{\Delta-1}}{\sup_{0\le\Delta<+\infty}\Delta^n\left(\prod_{j=1}^n(1-F_0(x_j))\right)^{\Delta-1}} = \begin{cases} 1 & \Delta^* \le 1 \\ \frac{1}{\Delta^{*n}\left(e^{-n+(n/\Delta^*)}\right)} & \Delta^* > 1 \end{cases}$$

where

$$\Delta^* = -\frac{1}{\frac{1}{n}\sum_{j=1}^n \log(1-F_0(x_j))}.$$

This comes from solving

$$\frac{d}{d\Delta} n\log\Delta + (\Delta-1)\log\prod_{j=1}(1-F_0(x_j))\Bigg|_{\Delta=\Delta^*} = 0$$

giving

$$\frac{n}{\Delta^*} + \log\prod_{j=1}(1-F_0(x_j)) = 0 \Rightarrow \Delta^* = -\frac{1}{\frac{1}{n}\sum_{j=1}^n\log(1-F_0(x_j))}$$

The LRT rejects H_0 if the ratio is small; for some k to be determined

$$\mathcal{R} = \left\{ \underline{x} | \lambda(\underline{x}) < \frac{1}{k^n} \right\} = \left\{ \underline{x} | \Delta^* e^{(1/\Delta^*)-1} > k \right\}.$$

For $x > 1$, $xe^{(1/x)-1}$ is *increasing* (take derivative of log) hence critical region is

$$\mathcal{R} = \left\{ \underline{x} | -\sum_{j=1}^{n} \log(1 - F_0(x_j)) < c \right\}$$

for some constant c.

Recall that if X has c.d.f. F for F continuous, then

$$\mathbb{P}(F(X) \leq x) = \mathbb{P}(X \leq F^{-1}(x)) = F(F^{-1}(x)) = x$$

so $F(X) \sim U(0,1)$ and hence $1 - F(X) \sim U(0,1)$.

For a prescribed significance level α, under the assumption of H_0, X_j has distribution F_0 and hence

$$\alpha \quad = \quad \mathbb{P}(\underline{X} \in \mathcal{R}) = \mathbb{P}\left(-\sum_{j=1}^{n} \log U_j < c \right)$$

where $U_1, \ldots, U_n$ are i.i.d. $U(0,1)$ If $U_j \sim U(0,1)$, then $-\log U_j \sim Exp(1)$ and hence

$$-\sum_{j=1}^{n} \log U_j \sim \text{gamma}(n,1)$$

so that $W := -2\sum_{j=1}^{n} \log U_j \sim \chi^2_{2n}$, so

$$\alpha = \mathbb{P}(W < 2c) \Rightarrow c = \frac{1}{2} k_{2n,1-\alpha}$$

where $k_{2n,\gamma}$ is the value such that $\mathbb{P}(W > k_{2n,\gamma}) = \gamma$.