Tutorial 14

1. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$; that is, the density is therefore:

$$p(x; \theta) = \frac{1}{\theta} \mathbf{1}_{[0,\theta]}(x)$$

Let $l(\theta; x) = -\log p(x; \theta)$.

   (a) Show that $\frac{d}{d\theta} l(\theta, x) = \frac{1}{\theta}$ for $\theta > x$ and is undefined for $\theta \leq x$. If $X \sim U(0, \theta)$, conclude that $\frac{d}{d\theta} l(\theta, X)$ is defined with $\mathbb{P}_\theta$ probability 1, but that

$$\mathbb{E}_\theta \left[\frac{d}{d\theta} l(\theta; X) \right] = \frac{1}{\theta} \neq 0.$$

   (b) Recall that $\widehat{\theta}_{ML} = \max\{X_1, \ldots, X_n\}$. Show that: $n(\theta - \widehat{\theta}) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}_\theta} \text{Exp}(1/\theta)$ ($\mathcal{L}_\theta$ denotes the law when the parameter value is θ).

2. Suppose $\lambda : \mathbb{R} \to \mathbb{R}$ satisfies $\lambda(0) = 0$, is bounded and has bounded second derivative λ''. Show that if $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{E}[X_1] = \mu$ and $\mathbf{V}(X_1) = \sigma^2 < +\infty$, then

$$\left| \sqrt{n} \mathbb{E} \left[\lambda(|\overline{X} - \mu|) \right] - \lambda'(0) \sigma \sqrt{\frac{2}{\pi}} \right| \overset{n\to+\infty}{\longrightarrow} 0$$

3. Let $V_n \sim \chi^2_n$. Show that $(\sqrt{V_n} - \sqrt{n}) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0, \frac{1}{2})$ ($\mathcal{L}$ denotes law).

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. variables each with probability function

$$p_X(0) = \theta^2 \qquad p_X(1) = 2\theta(1 - \theta) \qquad p_X(2) = (1 - \theta)^2$$

   (a) Find a and b (in terms of n and θ) such that $Z_n = \frac{\overline{X} - a}{b} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0, 1)$.

   (b) Find c and d (in terms of n and θ) such that $Y_n = \frac{\sqrt{\overline{X}} - c}{d} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0, 1)$.

5. Let $X_1, \ldots, X_n$ be a sample from a population with mean μ and variance $\sigma^2 < +\infty$. Let h be a function and let $h^{(j)}$ denote its jth derivative. Suppose that h has a second derivative continuous at μ and that $h^{(1)}(\mu) = 0$.

   (a) Show that $\sqrt{n}(h(\overline{X}) - h(\mu)) \overset{n\to+\infty}{\longrightarrow} 0$, while $n \left(h(\overline{X}) - h(\mu) \right) \overset{n\to+\infty}{\longrightarrow} \frac{1}{2} h^{(2)}(\mu) \sigma^2 V$ where $V \sim \chi^2_1$.

   (b) Use part (a) to show that when $\mu = \frac{1}{2}$, then

$$n \left(\overline{X}(1 - \overline{X}) - \mu(1 - \mu) \right) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} -\sigma^2 V \qquad V \sim \chi^2_1$$

6. Show that if $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ and $S^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j - \overline{X})^2$, then

$$\sqrt{n} \begin{pmatrix} \overline{X} - \mu \\ S^2 - \sigma^2 \end{pmatrix} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} \right)$$

7. Let $X_{ij} : i = 1, \ldots, p, j = 1, \ldots, k$ be independent with $X_{ij} \sim N(\mu_i, \sigma^2)$.

(a) Show that the MLEs of μ_i and σ^2 are:

$$\overline{\mu}_i = \frac{1}{k}\sum_{j=1}^{k} X_{ij} \qquad \widehat{\sigma^2} = \frac{1}{kp}\sum_{i=1}^{p}\sum_{j=1}^{k}(X_{ij} - \widehat{\mu}_i)^2$$

(b) Show that if k is fixed and $p \to +\infty$, then

$$\widehat{\sigma}^2 \overset{p\to+\infty}{\longrightarrow_{\mathcal{L}}} \left(1 - \frac{1}{k}\right)\sigma^2.$$

That is, the MLE $\widehat{\sigma}^2$ is not consistent.

Answers

1. (a) Let $l(\theta)$ denote the log likelihood. Then:

$$l(\theta) = \begin{cases} -\log\theta & 0 \le x \le \theta \\ -\infty & \text{other} \end{cases}$$

so

$$\frac{d}{d\theta} l(\theta) = \begin{cases} -\frac{1}{\theta} & 0 \le x \le \theta \\ \text{undefined} & \text{otherwise} \end{cases}$$

so it is defined with probability 1 and

$$\mathbb{E}_\theta\left[\frac{d}{d\theta} l(\theta; X)\right] = -\frac{1}{\theta} \neq 0.$$

(b) Let $Y = \max\{X_1, \ldots, X_n\}$, then

$$\mathbb{P}(Y \le y) = \mathbb{P}(X_1 \le y)^n = \begin{cases} 0 & y \le 0 \\ \left(\frac{y}{\theta}\right)^n & 0 < y \le \theta \\ 1 & y > \theta \end{cases}$$

Let $W = n(\theta - Y)$, then

$$\mathbb{P}(W \le t) = \mathbb{P}(n(\theta - Y) \le t) = \mathbb{P}(Y \ge \theta - \frac{t}{n}) = 1 - \left(1 - \frac{t}{n\theta}\right)^n \stackrel{n \to +\infty}{\longrightarrow} 1 - e^{-t/\theta}$$

so that $n(\theta - \widehat{\theta}_{ML}) \stackrel{n\to+\infty}{\longrightarrow}_{\mathcal{L}} \text{Exp}(1/\theta)$.

2. Taylor's expansion theorem gives:

$$\sqrt{n}\lambda(|\overline{X} - \mu|) = \sqrt{n}\lambda'(0)|\overline{X} - \mu| + \sqrt{n}\,\frac{\lambda''(Y)}{2}|\overline{X} - \mu|^2$$

where $0 \le Y \le |\overline{X} - \mu|$. Using $\sup_x |\lambda''(x)| \le K$, it follows that:

$$|\mathbb{E}[\frac{\sqrt{n}\lambda''(Y)}{2}|\overline{X} - \mu|^2]| \le \frac{\sigma^2}{2\sqrt{n}}K$$

since

$$\mathbb{E}\left[\left|\overline{X} - \mu\right|^2\right] = \mathbf{V}(\overline{X}) = \frac{\sigma^2}{n}.$$

By the central limit theorem,

$$\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \stackrel{n\to+\infty}{\longrightarrow} N(0,1).$$

Let $Z_n = \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma}$. Then from the definition of 'convergence in law', for any *bounded* function f,

$$\mathbb{E}[f(Z_n)] \to \mathbb{E}[f(Z)] \qquad Z \sim N(0,1)$$

For any non negative $N < +\infty$,

$$\mathbb{E}[|Z_n|\mathbf{1}_{\{|Z_n|\geq N\}}] \leq \mathbb{E}[|Z_n|^2]^{1/2}\mathbb{P}(|Z_n| \geq N) = \mathbb{P}(|Z_n| \geq N) \to \mathbb{P}(|Z| \geq N),$$

so that

$$|\mathbb{E}[|Z_n|] - \mathbb{E}[|Z|]| \leq |\mathbb{E}[|Z_n| \wedge N] - \mathbb{E}[|Z| \wedge N]| + (\mathbb{P}(|Z_n| \geq N) + \mathbb{P}(|Z| \geq N)).$$

while $\mathbb{P}(|Z_n| \geq N) \stackrel{n\to+\infty}{\longrightarrow} \mathbb{P}(|Z| \geq N)$. It follows that (taking limit in n first and then limit in N),

$$\lim_{n\to+\infty} |\mathbb{E}[|Z|] - \mathbb{E}[|Z_n|]| \leq 2\mathbb{P}(|Z| \geq N) \stackrel{N\to+\infty}{\longrightarrow} 0.$$

Therefore

$$\sqrt{n}\mathbb{E}\left[\frac{|\overline{X} - \mu|}{\sigma}\right] \stackrel{n\to+\infty}{\longrightarrow} \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-x^2/2}dx = \sqrt{\frac{2}{\pi}},$$

from which the result follows.

3. This is a straightforward application of the Delta method. $V_n = Z_1^2 + \ldots + Z_n^2$ where $Z_1, \ldots, Z_n$ are i.i.d. $N(0, 1)$ variables. Let $h(x) = x^{1/2}$, then $h'(x) = \frac{1}{2}x^{-1/2}$, so $(h'(1))^2 = \frac{1}{4}$.

   Let $Y_j = Z_j^2$ and $\overline{Y} = \frac{1}{n}\sum_{j=1}^n Z_j^2$. Since $\mathbb{E}[Z_1^2] = 1$ and $\mathbf{V}(Z_1^2) = 2$, it follows from the Delta method that

$$\sqrt{V_n} - \sqrt{n} = \sqrt{n}\left(\overline{Y} - 1\right) \stackrel{n\to+\infty}{\longrightarrow} N(0, 2 \times \frac{1}{4}) = N(0, \frac{1}{2}).$$

4.

$$\mu = \mathbb{E}[X_1] = 2\theta(1-\theta) + 2(1-\theta)^2 = 2\theta - 2\theta^2 + 2 - 4\theta + 2\theta^2 = 2(1-\theta)$$
$$\mathbb{E}[X_1^2] = 2\theta(1-\theta) + 4(1-\theta)^2 = 2(1-\theta)(\theta + 2 - 2\theta) = 2(1-\theta)(2-\theta)$$

so that

$$\sigma^2 = \mathbf{V}(X_1) = 2(1-\theta)(2-\theta) - 4(1-\theta)^2 = 2(1-\theta)(2-\theta-2+2\theta) = 2\theta(1-\theta).$$

(a) This is just the central limit theorem:

$$\frac{\overline{X} - 2(1-\theta)}{\sqrt{2\theta(1-\theta)/n}} \stackrel{n\to+\infty}{\longrightarrow} N(0,1)$$

$$a = 2(1-\theta) \qquad b = \sqrt{\frac{2\theta(1-\theta)}{n}}.$$

(b) Delta method: $h(x) = x^{1/2}$, $h'(x) = \frac{1}{2x^{1/2}}$, so that $h'(2(1-\theta)) = \frac{1}{2^{3/2}(1-\theta)^{1/2}}$.

$$\sqrt{n}(\sqrt{\overline{X}} - \sqrt{2(1-\theta)}) \stackrel{n\to+\infty}{\longrightarrow} N(0, \frac{2\theta(1-\theta)}{2^3(1-\theta)})$$

so

$$\frac{\sqrt{\overline{X}} - \sqrt{2(1-\theta)}}{\sqrt{\theta/4n}} \stackrel{n\to+\infty}{\longrightarrow} N(0,1).$$

$$c = \sqrt{2(1-\theta)} \qquad d = \frac{1}{2}\sqrt{\frac{\theta}{n}}.$$

5. (a) By Taylor's expansion,

$$h(\mu + (\overline{X} - \mu)) = h(\mu) + (\overline{X} - \mu)h^{(1)}(\mu) + \frac{(\overline{X} - \mu)^2}{2} h^{(2)}(\mu + z) \qquad |z| \le |\overline{X} - \mu|$$

so

$$n(h(\overline{X}) - h(\mu)) = \sigma^2 \left(\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \right)^2 h^{(2)}(\mu + z) \qquad |z| \le |\overline{X} - \mu|.$$

giving

$$n(h(\overline{X}) - h(\mu)) \stackrel{n\to+\infty}{\longrightarrow_{\mathcal{L}}} \sigma^2 h^{(2)}(\mu) V \qquad V \sim \chi_1^2$$

It follows directly that $\sqrt{n}(h(\overline{X}) - h(\mu)) \stackrel{n\to+\infty}{\longrightarrow_{\mathcal{L}}} 0$.

(b) $h(x) = x(1-x)$, $h^{(1)} = 1 - 2x = 0$ if $x = \frac{1}{2}$.

$$h^{(2)}(x) = -2 \Rightarrow h^{(2)}(\frac{1}{2}) = -2$$

$$n(\overline{X}(1-\overline{X}) - \frac{1}{4}) \stackrel{n\to+\infty}{\longrightarrow} -\sigma^2 V \qquad V \sim \chi_1^2$$

6. Firstly, asymptotic normality. This may be seen by expressing

$$S^2 = \frac{1}{n-1}\sum_{j=1}(X_j - \mu)^2 - \frac{n}{n-1}\left(\overline{X} - \mu\right)^2$$

$$\mathbf{V}((\overline{X} - \mu)^2) = \frac{\sigma^2}{n^2}\mathbf{V}\left(\frac{n(\overline{X} - \mu)^2}{\sigma^2}\right) = \frac{2\sigma^2}{n^2}$$

since $\frac{n(\overline{X}-\mu)^2}{\sigma^2} \sim \chi_1^2$ which has variance 2. Let $Y_i = (X_i - \mu)^2 - \sigma^2$. Then

$$\sqrt{n}(S^2 - \sigma^2) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} Y_j + \epsilon_n$$

where $\epsilon_n \to_{\mathbb{P}} 0$. It follows directly from the central limit theorem (no delta method required here) that the random vector $\sqrt{n}\begin{pmatrix} \overline{X} \\ S^2 - \sigma^2 \end{pmatrix}$ is asymptotically normal. It only remains to compute the covariance matrix. Firstly, $\mathbf{V}\left(\sqrt{n}(\overline{X} - \mu)\right) = \sigma^2$, as required. Secondly, $\overline{X}$ and S^2 are independent, giving the 0 covariance terms. This may be seen as follows: consider the random vector $(\overline{X}, X_1 - \overline{X}, \ldots, X_n - \overline{X})$. This is a normal random vector. Then

$$\mathbf{C}(\overline{X}, X_j - \overline{X}) = \mathbf{C}(X_j, \overline{X}) - \mathbf{V}(\overline{X}) = \frac{1}{n}\mathbf{V}(X_j) - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0.$$

It follows that

$$\overline{X} \perp \{(X_1 - \overline{X}), \ldots, (X_n - \overline{X})\}$$

and hence that

$$\overline{X} \perp \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2.$$

Thirdly,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

hence

$$\mathbf{V}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

so that asymptotically, $\frac{\frac{(n-1)S^2}{\sigma^2}-(n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}(S^2-\sigma^2)}{\sqrt{2}\sigma^2} \overset{n\to+\infty}{\longrightarrow_{\mathcal{L}}} N(0,1)$ giving

$$\sqrt{n}(S^2 - \sigma^2) \overset{n\to+\infty}{\longrightarrow_{\mathcal{L}}} N(0, 2\sigma^4)$$

and the result follows.

7. (a)

$$L(\mu_1, \ldots, \mu_p, \sigma; (x_{ij})) = \frac{1}{(2\pi)^{kp/2}\sigma^{kp}} \exp\left\{-\frac{1}{\sigma^2}\sum_{ij}(x_{ij} - \mu_i)^2\right\}$$

$\widehat{\mu_i}$ from minimising $\sum_{i,j}(x_{ij} - \mu_i)^2$ which gives

$$\widehat{\mu_i} = \frac{1}{k}\sum_{j=1}^{k} X_{ij}$$

$\widehat{\sigma^2}$ from maximising

$$-\frac{kp}{2}\log(\sigma^2) - \frac{1}{(\sigma^2)}\sum_{i=1}^{p}\left(\sum_{j=1}^{k}(x_{ij} - \mu_i)^2\right)$$

giving

$$\widehat{\sigma^2} = \frac{1}{kp}\sum_{i=1}^{p}\sum_{j=1}^{k}(X_{ij} - \widehat{\mu_i})^2$$

(b) For each i,

$$\frac{\sum_{j=1}^{k}(X_{ij} - \widehat{\mu_i})^2}{\sigma^2} \sim \chi^2_{k-1}$$

and these are independent, so

$$\frac{\sum_{i=1}^{p}\sum_{j=1}^{k}(X_{ij} - \widehat{\mu_i})^2}{\sigma^2} = \frac{kp\widehat{\sigma^2}}{\sigma^2} \sim \chi^2_{p(k-1)}$$

$$\mathbf{V}\left(\frac{\widehat{\sigma^2}}{\sigma^2}\right) = \frac{2p(k-1)}{k^2p^2} = \frac{2}{p}\left(\frac{1}{k} - \frac{1}{k^2}\right) \overset{p\to+\infty}{\longrightarrow_{\mathcal{L}}} 0$$

Furthermore,

$$\mathbb{E}[\widehat{\sigma^2}] = \frac{p(k-1)}{kp}\sigma^2 = \left(1 - \frac{1}{k}\right)\sigma^2$$

so, by Chebyshev,

$$\widehat{\sigma^2} \overset{p\to+\infty}{\longrightarrow}_{\mathcal{L}} \left(1 - \frac{1}{k}\right)\sigma^2.$$