Tutorial 10

1. We have a single observation on a random variable X from a distribution with density

$$p(x;\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

where $\theta > 0$ is unknown. We test $H_0 : \theta = 0$ against the alternative $H_1 : \theta > 0$ and we reject the null hypothesis if the observed value $x \in [c, +\infty) = \mathcal{R}_{\text{crit}}$ for an appropriate $c > 0$.

   (a) Compute c if the test has significance level $\alpha = 0.05$.

   (b) Determine whether or not this test is uniformly most powerful.

2. Let $X_1, \ldots, X_n$ be i.i.d. with distribution $F(x)$ where

$$F(x) = \begin{cases} 1 - e^{-x^\theta} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \theta > 0.$$

Find the most powerful test for $H_0 : \theta = 1$ versus $H_1 : \theta = \theta_1$ for a particular $\theta_1 > 1$. For $\alpha = 0.05$, show that this does not give a UMP test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

3. Let

$$X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right) \qquad i = 1, \ldots, n.$$

and suppose that $X_1, \ldots, X_n$ are independent. Consider the hypothesis test: $H_0 : \mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ versus the alternative $H_1 : \mu_1 \neq \mu_2$ or $\sigma_1 \neq \sigma_2$. Compute the likelihood ratio test statistic.

4. The $F_{n,m}$ distribution is defined as follows: if $V \sim \chi^2_m$, $W \sim \chi^2_n$ and $V \perp W$, then $F := \frac{W/n}{V/m}$ has $F_{n,m}$ distribution. Let $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ be independent exponential $\text{Exp}(\theta)$ and $\text{Exp}(\lambda)$ samples respectively and let $\Delta = \frac{\theta}{\lambda}$.

   (a) Let $f(\alpha)$ denote the value such that $\mathbb{P}(F > f(\alpha)) = \alpha$ where $F \sim F_{2n_1,2n_2}$. Show that $\left[\frac{\overline{Y}}{\overline{X}} f\left(1 - \frac{\alpha}{2}\right), \frac{\overline{Y}}{\overline{X}} f\left(\frac{\alpha}{2}\right)\right]$ is a confidence interval for Δ with confidence coefficient $1 - \alpha$.

   (b) Show that the test with acceptance region (the region where H_0 is not rejected) given by $[f(1 - \alpha/2), f(\alpha/2)]$ for the test $H_0 : \Delta = 1$ versus $H_1 : \Delta \neq 1$ using test statistic $\widehat{\Delta} = \frac{\overline{X}}{\overline{Y}}$ has size α.

5. Let $X_1, \ldots, X_n$ denote the times (in days) to failure of n similar pieces of equipment which is considered to be an $\text{Exp}(\lambda)$ random sample. Consider the hypothesis $H_0 : \frac{1}{\lambda} = \mu \leq \mu_0$ (the average lifetime is no greater than μ_0).

   (a) Show that the test with critical region $\overline{X} \in \left[\mu_0 \frac{k_{2n,\alpha}}{2n}, +\infty\right)$ where $k_{m,\alpha}$ is the value such that $\mathbb{P}(W > k_{m,\alpha}) = \alpha$ for $W \sim \chi^2_m$, is a size α test.

(b) Give an expression for the power function in terms of the χ^2_{2n} distribution.

(c) Use the central limit theorem to show that $\Phi\left(-\frac{\mu_0 z_\alpha}{\mu} + \frac{\sqrt{n}(\mu-\mu_0)}{\mu}\right)$ is an approximation to the power function of the test in part (a). Here z_α is the value such that $\mathbb{P}(Z > z_\alpha) = \alpha$ for $Z \sim N(0,1)$ and $\Phi(z) = \mathbb{P}(Z \leq x)$.

6. Let $X_1, \ldots, X_n$ be a random sample from Poiss(θ), where θ is unknown.

(a) Construct a UMP level α test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

(b) Show that the power function of the test is increasing in θ.

(c) What distribution tables would you need to calculate the power function of the UMP test?

(d) Give an approximate expression, derived using the central limit theorem, for the critical value (above which you reject H_0) if n is large and θ not too close to 0 or $+\infty$.

7. (a) Given a random sample $X_1, \ldots, X_n$ from a distribution with c.d.f. F, let

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{(-\infty,x]}(X_j)$$

denote the empirical distribution. Consider the test of $H_0 : F = F_0$ versus the alternative $H_1 : F \neq F_0$ for a given F_0. Let $D_n = \sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F_0(x)|$ and consider the test: reject H_0 if and only if $D_n \geq k_\alpha$ for k_α such that $\mathbb{P}_{F_0}(D_n \geq k_\alpha) = \alpha$ under H_0. (Recall that asymptotically D_n has the Kolmogorov distribution).

Show that the *power* of this test (for a true distribution F), $\beta(F)$, is bounded below by

$$\beta(F) \geq \sup_x \mathbb{P}_F\left(|\widehat{F}_n(x) - F_0(x)| \geq k_\alpha\right).$$

(b) For $n = 80$, obtain an approximation $\widetilde{k_{0.1}}$ for $k_{0.1}$ and see how well it approximates the true value:

$$k_{0.1} \simeq \frac{1.2}{\sqrt{n}} \simeq 0.134$$

(c) Again, take $\alpha = 0.10$ and $n = 80$. Let F_0 be the $N(0,1)$ c.d.f. and

$$F(x) = \frac{1}{1+\exp\left\{-\frac{x}{\tau}\right\}} \qquad -\infty < x < +\infty \qquad \tau = \frac{\sqrt{3}}{\pi}.$$

With this choice of τ, this is the *logistic* distribution with mean zero and variance 1. Evaluate the lower bound $\mathbb{P}_F(|\widehat{F}_n(x) - F_0(x)| \geq k_\alpha)$ for $\alpha = 0.10$, $n = 80$ and $x = 1.5$ using the normal approximation to the binomial distribution of $n\widehat{F}(x)$ and the approximate critical value of the previous part. (the value of 1.2 may be obtained from tables of the Kolmogorov Smirnov distribution). If F_0 is the c.d.f. for $N(0,1)$ then $F_0(1.5) \simeq 0.93$.

(d) Show that if $F \neq F_0$ and F and F_0 are continuous, then the power of this test tends to 1 as $n \to +\infty$. You may use the fact that $\sqrt{n}D_n$ under the null hypothesis converges to the Kolmogorov Smirnov distribution. In particular, $\sqrt{n}k_{0.1} \overset{n\to+\infty}{\longrightarrow} c_{0.1} = 1.2$.

8. Again, let $X_1, \ldots, X_n$ be a random sample from a distribution with continuous c.d.f. F and let $\widehat{F}_n$ denote the empirical distribution. Let $\psi : (0,1) \to (0, +\infty)$ and $\alpha > 0$. Define the statistics:

$$S_{\psi,\alpha} = \sup_x \psi(F_0(x))|\widehat{F}(x) - F_0(x)|^\alpha$$

$$T_{\psi,\alpha} = \sup_x \psi(\widehat{F}_n(x))|\widehat{F}(x) - F_0(x)|^\alpha$$

$$U_{\psi,\alpha} = \int \psi(F_0(x))|\widehat{F}_n(x) - F_0(x)|^\alpha F_0(dx)$$

$$V_{\psi,\alpha} = \int \psi(\widehat{F}_n(x))|\widehat{F}_n(x) - F_0(x)|^\alpha \widehat{F}_n(dx)$$

For each of these statistics show that the distribution under $H_0 : F = F_0$, does not depend on F_0 (continuous).

Short Answers

1. (a)

$$0.05 = \mathbb{P}_0(X \geq c) = e^{-c} \Rightarrow c = \log 20$$

   (b) This follows from the *Karlin-Rubin* theorem (Theorem 9.2). Clearly X is sufficient for θ and the likelihood ratio satisfies for $\theta_1 < \theta_2$:

$$\lambda(\theta_1, \theta_2; x) := \frac{e^{-(x-\theta_2)}}{e^{-(x-\theta_1)}} \frac{\mathbf{1}_{[\theta_2,+\infty)}(x)}{\mathbf{1}_{[\theta_1,+\infty)}(x)} = \begin{cases} \text{undefined} & x, \theta_1 \\ 0 & \theta_1 \leq x < \theta_2 \\ e^{\theta_1 - \theta_2} & x \geq \theta_2. \end{cases}$$

   This is monotone in x for x in the support of at least one of $p(. : \theta_1)$ or $p(.; \theta_2)$.Hence, by Karlin-Rubin theorem, $\mathcal{R}_{\text{crit}} = \{x : x > c\}$ is UMP.

2. Here the density is $p(x; \theta) = \theta x^{\theta-1} e^{-x^\theta} \mathbf{1}_{[0,+\infty)}(x)$. Therefore, for an NP test of $H_0 : \theta = 1$ against an alternative, we need the ratio:

$$\lambda(\theta; \underline{x}) := \theta^n \left(\prod_{j=1}^n x_j \right)^{\theta-1} \exp\left\{ -\sum_{j=1}^n (x_j^\theta - x_j) \right\}.$$

   where $\underline{x} = (x_1, \ldots, x_n)$ denotes the vector of observations.

   The NP rejection region for H_0: $\theta = 1$ against $H_1 : \theta = \theta_1$ of size α, where $\theta_1 > 1$ is:

$$\mathcal{R}_{\text{crit};\theta_1,\alpha} = \{\underline{x} : \lambda(\theta_1, \underline{x}) > k_{\theta_1,\alpha}\}$$

   where $k_{\theta_1,\alpha}$ satisfies:

$$\mathbb{P}(\lambda(\theta_1, \underline{X}) > k_{\theta_1,\alpha}) = \alpha, \qquad \underline{X} \quad \text{i.i.d.} \quad \text{Exp}(1).$$

   This is the most powerful level α test.

   A test (reject if $\underline{x} \in \mathcal{R}$ for some critical region $\mathcal{R}$) is UMP for all $\theta > 1$ if and only if, for the given level α it satisfies $\mathbb{P}(\underline{X} \in \mathcal{R}) \leq \alpha$ and

$$\beta(\theta) := \mathbb{P}_\theta(\underline{X} \in \mathcal{R}) \geq \mathbb{P}_\theta(\underline{X} \in \widetilde{\mathcal{R}})$$

   for any other $\widetilde{\mathcal{R}}$ such that

$$\mathbb{P}_1(\underline{X} \in \widetilde{\mathcal{R}}) \leq \alpha.$$

   The NP lemma gives an 'if and only if' condition. That is, a test is UMP if the NP tests give the same critical region for all $\theta > 1$. The critical region may be expressed as:

$$\mathcal{R}_\theta = \left\{ \underline{x} : (\theta - 1) \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} (x_j^\theta - x_j) > c_\theta \right\}$$

where c_θ is chosen such that for $X_1, \ldots, X_n$ i.i.d. Exp(1) variables,

$$\mathbb{P}(\underline{X} \in \mathcal{R}_\theta) = 0.05.$$

For the test to be UMP, we need: for all $\underline{x} \in \mathbb{R}_+^n$,

$$(\theta_1 - 1) \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} x_j^{\theta_1} + \sum_{j=1}^{n} x_j \geq c_{\theta_1} \Leftrightarrow (\theta_2 - 1) \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} x_j^{\theta_2} + \sum_{j=1}^{n} x_j \geq c_{\theta_2}.$$

For $n \geq 2$, the shapes of the regions depend on the parameter θ. Indeed, we can see that $\lim_{\theta \to +\infty} F(x; \theta) = \mathbf{1}_{[1,+\infty)}(x)$; for any $\epsilon > 0$,

$$\lim_{\theta \to +\infty} \mathbb{P}_\theta (1 - \epsilon < \min(X_1, \ldots, X_n) \leq \max(X_1, \ldots, X_n) < 1 + \epsilon) = 1,$$

so that $\mathcal{R}_{\text{limit}} = \bigcap_{\theta > 1} \mathcal{R}_\theta = \{(1, \ldots, 1)\}$

and

$$\mathbb{P}_\theta((X_1, \ldots, X_n) \in \mathcal{R}_{\text{limit}}) = 0.$$

3. The samples $X_{11}, \ldots, X_{n1}$ and $X_{12}, \ldots, X_{n2}$ are independent. The log likelihood is:

$$\log L(\mu_1, \mu_2, \sigma_1, \sigma_2) = \text{const} - n \log \sigma_1 - n \log \sigma_2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^{n} (x_{i1} - \mu_1)^2 - \frac{1}{2\sigma_2} \sum_{i=1} (x_{i2} - \mu_2)^2$$

For constrained problem (under H_0) the max. likelihood of $\mu = \mu_1 = \mu_2$ is:

$$\widehat{\mu} = \frac{1}{2n} (\sum_{i=1}^{n} x_{i1} + \sum_{i=1}^{n} x_{i2}) = \overline{x}_{..}$$

and the max. likelihood for $\sigma^2 = \sigma_1^2 = \sigma_2^2$ is:

$$\widehat{\sigma}^2 = \frac{1}{2n} \left(\sum_{i=1}^{n} (x_{i1} - \overline{x}_{..})^2 + \sum_{i=1}^{n} (x_{i2} - \overline{x}_{..})^2 \right).$$

For the unconstrained problem,

$$\widehat{\mu}_1 = \overline{x}_{.1} \qquad \widehat{\mu}_2 = \overline{x}_{.2}$$

while

$$\widehat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_{i1} - \overline{X}_{.1})^2, \qquad \widehat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_{i2} - \overline{X}_{.2})^2.$$

Then, for the constrained problem,

$$L(\widehat{\mu}, \widehat{\sigma}) = \frac{1}{(2\pi)^n \widehat{\sigma}^{2n}} \exp\{-n\}$$

and, for the unconstrained problem,

$$L(\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\sigma}_1, \widehat{\sigma}_2) = \frac{1}{(2\pi)^n \widehat{\sigma}_1^n \widehat{\sigma}_2^n} \exp\{-n\},$$

so that the LRT is:

$$\lambda(x) = \frac{\widehat{\sigma}_1^n \widehat{\sigma}_2^n}{\widetilde{\sigma}^{2n}}.$$

4. (a) $\sum_{j=1}^{n_1} X_j = n_1\overline{X} \sim \Gamma(n_1, \theta)$ so that $2n_1\theta\overline{X} \sim \Gamma(n_1, \frac{1}{2}) = \chi^2_{2n_1}$. Similarly, $2\theta n_2\overline{Y} \sim \chi^2_{2n_2}$. It follows that $\frac{\theta\overline{X}}{\lambda\overline{Y}} = \Delta\frac{\overline{X}}{\overline{Y}} \sim F_{2n_1,2n_2}$. Hence the $1-\alpha$ symmetric confidence interval is $f\left(1-\frac{\alpha}{2}\right) \le \Delta\frac{\overline{X}}{\overline{Y}} \le f(\frac{\alpha}{2})$ giving a confidence interval of

$$\Delta \in \left[\frac{\overline{Y}}{\overline{X}} f(1-\frac{\alpha}{2}), \frac{\overline{Y}}{\overline{X}} f(\frac{\alpha}{2})\right]$$

as required.

(b) Under $H_0 : \Delta = 1$, $F := \frac{\overline{X}}{\overline{Y}} \sim F_{2n_1,2n_2}$. Reject for observed value of F greater than $f(\frac{\alpha}{2})$ or less than $f(1-\frac{\alpha}{2})$. Hence acceptance region is $[f(1-\frac{\alpha}{2}), f(\frac{\alpha}{2})]$.

5. (a) $T := 2\lambda\sum_{j=1}^{n} X_j \sim \chi^2_{2n}$. For $\lambda = \frac{1}{\mu_0}$,

$$\mathbb{P}_{\mu_0}(\overline{X} \ge \mu_0\frac{k_{2n,\alpha}}{2n}) = \mathbb{P}(T > k_{2n,\alpha}) = \alpha.$$

Clearly the power function is increasing in μ, so this is a size α test.

(b)

$$\beta(\mu) = \mathbb{P}_\mu\left(\overline{X} \ge \mu_0\frac{k_{2n,\alpha}}{2n}\right) = \mathbb{P}(T \ge \frac{\mu_0}{\mu}k_{2n,\alpha}) = 1 - F\left(\frac{\mu_0}{\mu}k_{2n,\alpha}\right)$$

where F is the c.d.f. for the χ^2_{2n}.

(c) From CLT, approximately $\overline{X} \sim N(\mu, \frac{\mu^2}{n})$ so that

$$\beta(\mu) \simeq 1 - \Phi\left(\frac{\mu_0\frac{k_{2n,\alpha}}{2n} - \mu}{\mu/\sqrt{n}}\right) = \Phi\left(\frac{\sqrt{n}(\mu-\mu_0)}{\mu} + \frac{\sqrt{n}\mu_0}{\mu}(1 - \frac{k_{2n,\alpha}}{2n})\right)$$

Now, by CLT, if $V \sim \chi^2_{2n}$ then V is approximately $N(2n, 4n)$, so

$$\alpha = \mathbb{P}(W > k_{2n,\alpha}) \simeq \mathbb{P}(Z > \frac{k_{2n,\alpha} - 2n}{2\sqrt{n}})$$

so $z_\alpha \simeq \frac{k_{2n,\alpha}-2n}{2\sqrt{n}}$ and hence

$$\beta(\mu) \simeq \Phi\left(\frac{\sqrt{n}(\mu-\mu_0)}{\mu} - \frac{\mu_0}{\mu} z_\alpha\right)$$

as required.

6. (a) Construct UMP by Karlin-Rubin Theorem. Let $T = \sum_i X_i$, then T is sufficient for θ. Also, $T \sim \text{Poiss}(n\theta)$. To show that test with critical region $T \in [t_0, +\infty)$ is UMP, it is sufficient by K-R to show MLR. For $\theta_1 < \theta_2$,

$$\lambda(\theta_1, \theta_2; t) = \frac{L(\theta_1, t)}{L(\theta_0, t)} = \left(\frac{\theta_1}{\theta_0}\right)^t e^{-n(\theta_1-\theta_0)}$$

This is clearly monotone in t, hence likelihood ratio satisfies MLR property, hence UMP level α test is: reject H_0 for $\sum_j X_j > k$ for an integer k, chosen as the smallest value such that

$$\mathbb{P}_{\theta_0}(\sum_j X_j > k) \leq \alpha.$$

(b) For $S \sim \text{Poiss}(n\theta)$ when the parameter value is θ,

$$\beta(\theta) = \mathbb{P}_\theta\left(S > k\right)$$

Use S is the number of events by time 1 in a Poisson process with parameter $n\theta$. Then $\{S > k\} = \{T < 1\}$ where T is the time until the kth event. $T \sim \Gamma(k, n\theta)$ so that $2n\theta T \sim \Gamma(k, \frac{1}{2}) \sim \chi^2_{2k}$. It is clear that $\mathbb{P}_\theta(T < 1)$ is increasing in θ.

(c) From part (b), the chi squared distribution.

(d) By CLT: $\sum_j X_j \sim N(n\theta, n\theta)$ approximately; reject H_0 if $\frac{\sum_j X_j - n\theta_0}{\sqrt{n\theta_0}} = \frac{k-n\theta_0}{\sqrt{n\theta_0}} > z_\alpha$; k is the lowest integer greater than $n\theta_0 + \sqrt{n}\sqrt{\theta_0} z_\alpha$.

7. (a) First part is trivial: since $D_n \geq |\widehat{F}_n(x) - F_0(x)|$ for each x, it follows that

$$\mathbb{P}_F(|\widehat{F}_n(x) - F_0(x)| \geq k_\alpha) \leq \mathbb{P}_F(D_n \geq k_\alpha)$$

for eacn $x \in \mathbb{R}$ and hence

$$\sup_{x\in\mathbb{R}} \mathbb{P}_F(|\widehat{F}_n(x) - F_0(x)| \geq k_\alpha) \leq \mathbb{P}_F(D_n \geq k_\alpha)$$

(b) Approximation for k_α:

$$\alpha > \mathbb{P}_F(|\widehat{F}_n(x) - F(x)| > k_\alpha)$$

so

$$\alpha > \mathbb{P}_F(|n\widehat{F}_n(x) - nF(x)| > nk_\alpha)$$

Consider $X_1, \ldots, X_n$ i.i.d. $U(0,1)$ variables, then $F(x) = x$ and $n\widehat{F}_n(x) \sim Bi(n,x) \sim N(nx, nx(1-x))$ (central limit approximation). Let $Y \sim N(nx, nx(1-x))$ then

$$\alpha = \sup_{x \in \mathbb{R}} \mathbb{P}(|Z| > \frac{\sqrt{n}\widetilde{k}_\alpha}{\sqrt{x(1-x)}})$$

Recall that $\alpha = 0.1$. The supremum occurs at $x = \frac{1}{2}$. With $n = 80$,

$$0.1 = \mathbb{P}(|Z| > 17.89\widetilde{k}_{0.1}) \Rightarrow 17.89\widetilde{k}_{0.1} = 1.65 \Rightarrow \widetilde{k}_{0.1} \simeq 0.09$$

(value for the Kolmogorov Smirnov statistic: $k_\alpha = 0.136$).

(c) Note that $80 \times 0.136 \simeq 11$. For F_0 the $N(0,1)$ c.d.f., $80 \times F_0(1.5) = 74.4$, so look for:

$$\mathbb{P}_F(|n\widehat{F}_n(1.5) - 74.4| \geq 11)$$

where $n\widehat{F}_n(1) = \text{Binomial}(80, \mathbb{P}_F(X \leq 1.5))$. Using $\tau = \frac{\sqrt{3}}{\pi}$, it follows that

$$\mathbb{P}_F(X \leq 1.5) = \frac{1}{1 + e^{-1.5\pi/\sqrt{3}}} \simeq 0.94.$$

The answer is therefore

$$\mathbb{P}(Y \leq 56) + \mathbb{P}(Y \geq 79) \qquad Y \sim \text{Binomial}(80, 0.94) \simeq N(75.2, 4.5).$$

which is:

$$(1 - \Phi(9.1)) + (1 - \Phi(1.65)) \simeq 0.05$$

In other words, the test is catastrophically awful. The null hypothesis is wrong, nevertheless, we only reject it with probability 0.05.

(d) Choose a point x such that $0 < F(x) < 1$, $0 < F_0(x) < 1$ and $F(x) \neq F_0(x)$ and let $c_{0.1} \simeq 1.22$ the value such that

$$\lim_{n \to +\infty} \mathbb{P}_{F_0}(\sqrt{n}D_n \geq c_{0.1}) = 0.1.$$

Using $Y := n\widehat{F}_n(x) \sim \text{Binomial}(nF(x), nF(x)(1 - F(x))$, it follows from the central limit theorem that the power $\beta_n(F)$ (based on sample size n) satisfies:

$$\begin{aligned} \lim_{n \to +\infty} \beta_n(F) \quad \geq \quad & \Phi\left(\frac{\sqrt{n}(F_0(x) - F(x))}{\sqrt{F(x)(1-F(x))}} - \frac{c_\alpha}{\sqrt{F(x)(1-F(x))}}\right) \\ & + \Phi\left(\frac{\sqrt{n}(F(x) - F_0(x))}{\sqrt{F(x)(1-F(x))}} - \frac{c_\alpha}{\sqrt{F(x)(1-F(x))}}\right) \to 1 \end{aligned}$$

from which the result follows; if $F_0(x) > F(x)$ then the first term converges to 1 and the second to 0; if $F(x) > F_0(x)$ then the first term converges to 0 and the second to 1.

8. The first and second are similar to arguments given before (earlier tutorial exercises). The third and fourth are similar; here is the argument for the fourth.

$$\mathbb{P}(V_{\psi,\alpha} > v) = \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\psi\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{(-\infty,X_k]}(X_j)\right)\left|\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{(-\infty,X_k]}(X_j) - F_0(X_k)\right|^{\alpha} > v\right)$$

Now let $Y_k = F_0(X_k)$ so that $Y_1, \dots, Y_n$ are i.i.d. $U(0,1)$, then

$$\mathbb{P}(V_{\psi,\alpha} > v) = \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\psi\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{(-\infty,Y_k]}(Y_j)\right)\left|\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{(-\infty,Y_k]}(Y_j) - Y_k\right|^{\alpha} > v\right)$$

which does not depend on F_0.