Problem Set 4

Problem 1. Prove that every single μ-summable function f is "uniformly integrable", *i.e.* for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |f| \, \mathrm{d}\mu < \epsilon$$

whenever $\mu(A) < \delta$.

Problem 2. Prove that every nonnegative μ-measurable function is μ-integrable.

Problem 3. Prove that the Borel σ-algebra on $\mathbb{R}^n$ is equal to the product of n Borel σ-algebras on $\mathbb{R}$.

Problem 4. Prove that the product measure is (indeed) a measure.

Problem 5. Prove that λ^n is regular for $n \in \mathbb{N}$ (particularly for $n = 1$). Prove that it is Borel, Borel regular and Radon.

Problem 6 (Cavalieri)**.** Let f be a μ-measurable function and let $1 \leq p < +\infty$.

(a) Prove that the function $|f|^p$ is μ-summable iff the function

$$t \mapsto t^{p-1} \mu(\{x : |f(x)| > t\})$$

is λ^1-summable on $[0, +\infty)$.

(b) Prove that

$$\int_{\mathbb{X}} |f|^p \, \mathrm{d}\mu = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) \, \mathrm{d}\lambda^1(t).$$

Hint: Fubini.

Problem 7. Let μ be a Borel measure on $\mathbb{R}^2$ and let A be a Borel set with $\mu(A) < +\infty$. Prove that

$$\int_{A \times A} (x - y)^2 \, \mathrm{d}(\mu \times \mu) < +\infty \quad \Rightarrow \quad \int_A |x| \, \mathrm{d}\mu < +\infty.$$