Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$

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1 Introduction

The Nagata Conjecture has attracted a lot of attention recently [1], [3], [4]. It seems to be understood by now, mostly thanks to the works of Biran [1], [2] that the conjecture, though originally formulated for the projective plane $\mathbb{P}^2$ should hold in a much broader context of arbitrary algebraic surfaces (or even varieties of arbitrary dimension, though it is not clear what the right bound on the number of points in higher dimensions should be). A convenient way to treat the Nagata Conjecture is to use the language of Seshadri constants. We adopt this point of view in this paper.
Even if the conjecture itself seems to demand for new methods, it is reasonable to ask if there are some obvious counterexamples for the small number of points on the given surface. In the classical case of $\mathbb{P}^2$ the existence of counterexamples for $r \leq 9$ points follows from the Riemann-Roch theorem. In this paper we show that somewhat unexpectedly this need not to be the case, even on such a simple variety as $\mathbb{P}^1 \times \mathbb{P}^1$.

Notation

For simplicity we denote the Néron-Severi group of a variety X tensored by $\mathbb{Q}$ by $NS(X)$. By a polarization of type (a, b) or by a curve of type (a, b) in the product $\mathbb{P}^1 \times \mathbb{P}^1$ we mean a curve of bidegree a, b. We work throughout the paper over the field $\mathbb{C}$ of complex numbers.

2 Seshadri constants and the Nagata-Biran Conjecture

Recall that a polarized variety is a pair (X, L) consisting of a smooth variety X and an ample line bundle L on X.
We assume that X is a smooth projective variety, L is a nef line bundle on X (that is for all curves $C \subset X$ we have $L.C \geq 0$) and $x_1, \ldots, x_r \in X$ are fixed points.

Definition 1 *The Seshadri constant of L at points $x_1, \ldots, x_r$ is the real number:*

$$\varepsilon(L; x_1, \ldots, x_r) = \inf_{C \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L.C}{\sum_{i=1}^r mult_{x_i} C},$$

where the infimum is taken over all (irreducible) curves passing through at least one of the points $x_1, \ldots, x_r$.

It follows from Kleiman's nefness criterion that $\varepsilon(L; x_1, \ldots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}$, where $n = \dim X$. If $\varepsilon(L; x_1, \ldots, x_r) = \sqrt[n]{\frac{L^n}{r}}$ then we say that the Seshadri constant is maximal, otherwise $\left(\text{i.e. } \varepsilon(L; x_1, \ldots, x_r) < \sqrt[n]{\frac{L^n}{r}}\right)$ - the Seshadri constant is submaximal.

Definition 2 *We say that a curve C computes the Seshadri constant if*

$$\varepsilon(L; x_1, \ldots, x_r) = \frac{L.C}{\sum_{i=1}^r mult_{x_i} C}.$$

Note that if C computes $\varepsilon(L; x_1, \ldots, x_r)$ then necessarily

$$\frac{L.C}{\sum mult_{x_i} C} \leq \sqrt[n]{\frac{L^n}{r}}$$

by the above upper bound. This justifies the following

Definition 3 *We say that a curve $C \subset X$ is L-submaximal at the points $x_1, \ldots, x_r$ (or simply submaximal) if:*

$$\frac{L.C}{\sum_{i=1}^r mult_{x_i} C} < \sqrt[n]{\frac{L^n}{r}},$$

Let $C \subset X$ be a curve passing through the points $x_1, \ldots, x_r$ with multiplicities $m_1, \ldots, m_r$ respectively. To a curve C we assign its multiplicity vector $M_C = (m_1, \ldots, m_r) \in \mathbb{Z}^r$.

Definition 4 *Let $C \subset X$ be a curve with the multiplicity vector $M_C = (m_1, \ldots, m_r)$. We say that C is Riemann-Roch expected (in short R-R expeted) if*

$$h^0(\mathcal{O}_X(C)) - \sum_{i=1}^{r} \binom{m_i + 1}{2} > 0.$$

This simply means that a curve C is R-R expected if its existence follows from the naive dimension count, note that $\binom{m+1}{2}$ is the maximal number of independent conditions imposed on the linear system $|C|$ in order to obtain a divisor with multiplicity $\geq m$ at a fixed point.
Now we are in the position to formulate the

Nagata-Biran conjecture *Let (X, L) be a polarized surface. Let k_0 be the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve and let $N_0 = k_0^2 L^2$. With the above assumptions*

$$\varepsilon(L; x_1, \ldots, x_r) = \sqrt{\frac{L^2}{r}}$$

for $x_1, \ldots, x_r \in X$ very general and $r \geq N_0$.

Remark 1 (1) On $(\mathbb{P}^2, \mathcal{O}(1))$ $N_0 = 9$ and the curves computing the Seshadri constant for $r \leq N_0$ points are R-R expected.
(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with the polarization $(1,1)$ $N_0 = 8$ and again all curves computing the Seshadri constant for at most 8 points are R-R expected.

3 Submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$

In this paper we show that on $\mathbb{P}^1 \times \mathbb{P}^1$ with the polarization $(2,1)$ the curves computing the Seshadri constant for $r \leq 9$ points are R-R expected and submaximal unless $r = 4$ or 9. We don't know if for $r = 10, \ldots, 16$ submaximal curves exist, but if they do, we show that they are not R-R expected.

Let (X, L) be a polarized surface with the Picard number ρ. Let $L_1, \ldots, L_r$ be a fixed basis of the Néron-Severi group $NS(X)$ and let $x_1, \ldots, x_r$ be fixed points on X. To a curve $C \subset X$ we assign a vector

$$v_C = (l_1, \ldots, l_\rho, m_1, \ldots, m_r) \in NS(X) \times \mathbb{Q}^r$$

such that $C \equiv l_1 L_1 + \ldots + l_\rho L_\rho$ and $M_C = (m_1, \ldots, m_r)$.
The following lemma extends Proposition 1.8 and Proposition 4.5 of [4].

Proposition 1 *Let (X, L) be a polarized surface with the Picard number ρ. Let $x_1, \ldots, x_r \in X$ be points such that the Seshadri constant $\varepsilon(L; x_1, \ldots, x_r)$ is submaximal. Then there exist at most $\rho + r$ irreducible and reduced curves computing $\varepsilon(L; x_1, \ldots, x_r)$.*

Proof.
Let $C_1, \ldots, C_s$ be curves computing the Seshadri constant. Each of the curves has a vector $v_{C_i} = v_i = (l_1^{(i)}, \ldots, l_\rho^{(i)}, m_1^{(i)}, \ldots, m_r^{(i)}) \in NS(X) \times \mathbb{Q}^r$ for $i = 1, \ldots, s$.
If $s > \rho + r$ then the equation:

$$\sum_{i=1}^{s} \lambda_i v_i = 0 \qquad \text{where} \quad \lambda_i \in \mathbb{Q} \tag{1}$$

has a non-trivial solution. We may in fact assume that $\lambda_i \in \mathbb{Z}$ (because we can multiply both sides of this equation by the common denominator).
Now we define curves C_+ and C_- in the following way:

$$C_+ := \sum_{i=1}^{s} \beta_i C_i, \quad \text{where} \quad \beta_i = \begin{cases} \lambda_i, & \text{if} \quad \lambda_i \geq 0 \\ 0, & \text{if} \quad \lambda_i < 0 \end{cases}$$

$$C_- := \sum_{i=1}^{s} \gamma_i C_i, \quad \text{where} \quad \gamma_i = \begin{cases} 0, & \text{if} \quad \lambda_i \geq 0 \\ -\lambda_i, & \text{if} \quad \lambda_i < 0 \end{cases} .$$

Then of course

$$C_+ \overset{(1)}{\equiv} C_-. \tag{2}$$

and in particular the vectors of multiplicity

$$M_+ = (m_1^+, \ldots, m_r^+),\ M_- = (m_1^-, \ldots, m_r^-)$$

are equal.
Let $M = (m_1, \ldots, m_r)$ be the vector of multiplicity at points $x_1, \ldots, x_r$ of both curves. Curves C_+ and C_- are submaximal (as a combination with nonnegative integer coefficients of submaximal curves). Hence

$$\frac{L.C_+}{\sum_{i=1}^r m_i} < \sqrt{\frac{L^2}{r}} \tag{3}$$

and

$$\frac{L.C_-}{\sum_{i=1}^r m_i} < \sqrt{\frac{L^2}{r}} \tag{4}$$

The way in which we defined these curves implies that C_+ and C_- have no common components, thus:

$$C_-^2 \overset{(2)}{=} C_+.C_- \overset{\text{Bezout th.}}{\geq} \sum_{i=1}^r m_i^2 \geq \frac{1}{r}\left(\sum_{i=1}^r m_i\right)^2 = \frac{1}{\sqrt{r}}\sum_{i=1}^r m_i \cdot \frac{1}{\sqrt{r}}\sum_{i=1}^r m_i$$

$$\overset{(3),(4)}{>} \frac{L.C_+}{\sqrt{L^2}} \cdot \frac{L.C_-}{\sqrt{L^2}} \overset{(2)}{=} \frac{(L.C_-)^2}{L^2} \overset{\text{Hodge index th.}}{\geq} C_-^2$$

which obviously is a contradiction, so s can be equal at most $\rho + r$. $\square$

Before we procced, we need some more notation.
For a vector $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ we define
$|M| := \sum_{i=1}^r m_i$,
$\alpha(M) := \max\{|m_i - m_j| :\ i, j = 1, \ldots, r\}$,
$l(M)$ - the maximal number of independent conditions imposed by singularities of multiplicity $m_1, \ldots, m_r$ on a linear system on a surface.

Lemma 1 *If $M_1, M_2 \in \mathbb{Z}^r$ are of the form $M_1 = (m, \ldots, m, m, m+2)$ and $M_2 = (m, \ldots, m, m+1, m+1)$ then $l(M_2) < l(M_1)$.*

Proof. The proof is purely arithmetic

$$l(M_2) - l(M_1) = 2 \cdot \binom{m+2}{2} - \left[\binom{m+3}{2} + \binom{m+1}{2}\right] =$$
$$(m+2)(m+1) - \frac{1}{2}[(m+3)(m+2) + (m+1)m] =$$
$$m^2 + 3m + 2 - m^2 - 3m - 3 = -1$$

$\square$

An obvious consequence of this lemma is

Corollary 1 *Let $\mathcal{M}_p = \{M \in \mathbb{Z}^r : |M| = p\}$. If not all multiplicities are equal then vectors which impose the least number of conditions on a curve C are the ones with $\alpha(M) = 1$.*

Hence if M is such a vector, then up to permutation

$$M = \; (\underbrace{m, \ldots, m}_{i}, \underbrace{m+\delta, \ldots, m+\delta}_{r-i}) \;,$$

where $\delta \in \{-1, 1\}$. Obviously $l(M)$ is independent of the way the multiplicities are ordered i.e. $l((m_1, \ldots, m_r)) = l((m_{\sigma(1)}, \ldots, m_{\sigma(r)}))$ for any permutation $\sigma \in S_r$.

Corollary 2 $\#\{M = \; (\underbrace{m, \ldots, m}_{i}, \underbrace{m+\delta, \ldots, m+\delta}_{r-i}) \; \in \mathbb{Z}^r : \; |M| = p$ *and* $\alpha(M) = 1\} = \binom{r}{i}$.

We proved above that the number of curves computing the maximal Seshadri constant is equal at most $\rho + r$. Then analysing the inequality $\binom{r}{i} \leq \rho + r$ for $\rho = 2$ and $i = 0, \ldots, 7$ (for $i = 8, \ldots, 16$ we have the same inequalities as for $i = 0, \ldots, 7$) we obtain the following conclusion:

Corollary 3 *Let (X, L) be a polarized surface with the Picard number $\rho = 2$ and let $x_1, \ldots, x_r \in X$ be fixed points. If $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ is the multiplicity vector of a submaximal curve C at points $x_1, \ldots, x_r$, then M is almost homogeneous i.e. up to permutation M is of the form: $M = (m, \ldots, m, m + \delta)$ with $\delta \in \{-1, 0, 1\}$.*

Now we can formulate the main result of this paper:

Proposition 2 *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. If L is a polarization of type $(2, 1)$ then for $10 \leq r \leq 15$ points there are no R-R expected submaximal curves on X.*

Proof. Let r be fixed and suppose to the contrary that $C \subset X$ of type (a, b) is R-R expected and submaximal. Without loss of generality we can assume that the vector of multiplicity of C is of the form $M = (m, \ldots, m, m + \delta)$, where $\delta \in \{-1, 0, 1\}$, $m \in \mathbb{Z}$ (from corollary 3). Hence a number of conditions imposed by M is equal:

$$l(M) = (r-1)\left(\begin{array}{c} m+1 \\ 2 \end{array}\right) + \left(\begin{array}{c} m+\delta+1 \\ 2 \end{array}\right) =$$

$$\frac{1}{2}\left[rm^2 + rm + 2m\delta + \delta^2 + \delta\right].$$

Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)) = ab + a + b + 1$, C is R-R expected and by Proposition 1 there is no continuous family of submaximal curves we must have

$$ab + a + b = \frac{1}{2}\left[rm^2 + rm + 2m\delta + \delta^2 + \delta\right]$$

or equivalently

$$2b = \frac{rm^2 + rm + 2m\delta + \delta^2 + \delta - 2a}{a+1}. \tag{5}$$

The submaximality of C means that

$$\frac{a + 2b}{\sum_{i=1}^{r} m_i} < \frac{2}{\sqrt{r}}. \tag{6}$$

Conditions (5) and (6) give the inequality:

$$\sqrt{r}a^2 - (\sqrt{r} + 2rm + 2\delta)a + \sqrt{r}(rm^2 + rm + 2m\delta + \delta^2 + \delta) - 2rm - 2\delta < 0.$$

We view the above inequality as an inequality on the variable a. We know that the set of solutions is non-empty, hence

$$r + 12r\sqrt{r}m + 12\sqrt{r}\delta + 4\delta^2 - 4r^2m - 4r\delta^2 - 4r\delta > 0.$$

Substituting $\sqrt{r} = t \overset{t \geq 0}{\longleftrightarrow} r = t^2$ we obtain

$$-4t^4m + 12t^3m + (1 - 4\delta^2 - 4\delta)t^2 + 12\delta t + 4\delta^2 > 0 \tag{7}$$

This inequality has the simplest form for $\delta = 0$. In this case we have

$$-4t^4m + 12t^3m + t^2 > 0$$

equivalently we get

$$4tm(3 - t) + 1 > 0,$$

which of course implies that $t < 3$. So we obtained a condition: if $\delta = 0$ then $r \in (0, 9) \cap \mathbb{Z}$.

All that is left for us to do now is to look up how the inequality (7) behaves if $\delta \in \{-1, 1\}$. More precisely, we try to estimate m. Obviously

$$m(-4t^4 + 12t^3) > (4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2. \tag{8}$$

We observe that if $t \in (0, 3)$ then

$$m > \frac{(4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2}{-4t^4 + 12t^3}$$

which for $\delta = 1$ gives a lower bound for m:

$$m > \frac{7t^2 - 12t - 4}{-4t^4 + 12t^3}. \tag{9}$$

However for $\delta = -1$ we obtain:

$$m > \frac{-t^2 + 12t - 4}{-4t^4 + 12t^3}. \tag{10}$$

Analyzing the sign of the numerator in the inequalities (9) and (10) it is not difficult to check that, we can take m and t such that for individual values of δ the inequality becomes true. Then the existence of R-R expected submaximal curves with vectors of multiplicity $M = \widetilde{M}_\sigma$ (where $\widetilde{M} = (m, \ldots, m, m +$

$\delta), \sigma \in S_r, m \in \mathbb{Z}_{\geq 1}, \delta \in \{-1, 0, 1\})$ follows for $r \in \{1, \ldots, 8\}$.
Now we show that for $t \in (3, +\infty)$ the condition (8) cannot be satisfied. Obviously, if $t \in (3, +\infty)$ then the inequality (8) has the form:

$$m < \frac{(4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2}{-4t^4 + 12t^3}.$$

Substituting specific values for δ, we obtain
a) for $\delta = 1$

$$m < \frac{7t^2 - 12t - 4}{-4t^4 + 12t^3} < 0.$$

a contradiction.
b) for $\delta = -1$

$$m < \frac{t^2 - 12t + 4}{4t^4 - 12t^3} < \frac{1}{4t^2 - 12t} < 1.$$

Again there is no m such that (8) holds. □

It might seem that on rational surfaces the lower bound for the number of points for which the Nagata-Biran Conjecture holds is $N = 9$. This is not the case as the following example shows.

Example 1 *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the polarization L of type $(k, 1)$. Then for $r = 2k + 4$ points there exists a R-R expected submaximal curve C of type $(k^2 + k, k + 1)$ with the vector of multiplicity $M_C = (k, \ldots, k, k + 1)$. This is an easy computation.*

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References

[1] Biran P., Constructing new ample divisors out of old ones. Duke Math. J. 98 (1999), 113-135

[2] Biran P., A stability property of symplectic packing. Invent. Math. 136 (1999), 123-155

[3] Harbourne B., Seshadri constants and very ample divisors on algebraic surfaces. preprint alg-geom 0103029

[4] Szemberg T., Global and local positivity of line bundles. Habilitationsschrift Essen 2001

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