On Euler—Jaczewski sequence and Remmert—Van de Ven problem for toric varieties

Dedicated to memory of Krzysztof Jaczewski

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Introduction.

In [RV] Remmert and Van de Ven posed a problem concerning holomorphic maps from a complex projective space onto a smooth variety of the same dimension: the question was whether the target variety has to be the projective space as well. The problem has a positive answer provided by Lazarsfeld, [La].

Theorem. [Lazarsfeld] *Suppose that Y is a smooth projective variety of positive dimension. If $\varphi : \mathbf{P}^n \longrightarrow Y$ is a surjective morphism then $Y \simeq \mathbf{P}^n$.*

Lazarsfeld's proof depends on a (somewhat technical) characterisation of the projective space by Mori which was obtained as a by-product of his proof of Frankel-Hartshorne conjecture, [Mo]. We will explain the result in the section on rational curves.

Following Lazarsfeld, questions were raised concerning possible extensions of Remmert – Van de Ven problem for a broader class of varieties. That is, given an X from a class of varieties and a morphism $\varphi : X \longrightarrow Y$ onto a smooth projective variety (with possibly additional assumptions on φ and Y, like $\rho(Y) = 1$, where ρ denotes the Picard number, or the rank of Neron-Severi group of the variety), one would like to deduce the structure of Y, preferably to claim that $Y \simeq \mathbf{P}^n$, unless φ is an isomorphism. In particular, the following cases has been considered: X is a smooth quadric, see [PS] and [CS], X is an irreducible symmetric Hermitian space, [Ts], X is rational homogeneous with $\rho(X) = 1$, [HM] and X is a Fano 3-fold, [Am].

The main result of the present note is the following.

Theorem 1. *Suppose that X is a complete toric variety and Y a smooth projective variety with $\rho(Y) = 1$. If $\varphi : X \longrightarrow Y$ is a surjective morphism then $Y \simeq \mathbf{P}^n$.*

The proof of the main result follows the line of Lazarsfeld's argument, in particular we apply Mori's ideas of considering families of rational curves. A new ingredient of the proof is Euler-Jaczewski sequence for toric varieties which we explain in the subsequent section. As an application we obtain a result on varieties admitting two projective bundle structures (Theorem 2 in the last section of the paper).

In the present paper all varieties are defined over an algebraically closed field of characteristic zero.

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Toric varieties.

For generalities on toric varities we refer the reader to [Fu] or [Od]. Let $X = X_\Sigma$ be a toric variety defined by a fan Σ in the space $\mathbf{N}_\mathbf{R}$, with $\mathbf{N}$ denoting the lattice of 1-parameter subgroups of the big torus $\mathbf{T} \simeq (\mathbf{C}^*)^n$ and $\mathbf{M}$ denoting its dual. By $\Sigma^1 = \{\rho\}$ let us denote the set of rays (1-dimensional cones) in the fan Σ. The equivariant divisor in X (the closure of a codimension 1 orbit of the action of $\mathbf{T}$) associated to a ray $\rho \in \Sigma^1$ we shall denote by D_ρ. The variety X is decomposed into the union of the open orbit of $\mathbf{T}$ and the divisor $\bigcup_{\rho\in\Sigma^1} D_\rho$.

Let us recall the following general fact due to Blanchard, [Bl].

Theorem. [Blanchard] *Let X be a normal complete variety and G be a connected algebraic group acting on X such that the induced action on $Pic(X)$ is trivial (assume, for example, that $H^1(X, \mathcal{O}_X) = 0$). Suppose that $\varphi : X \to Y$ is a morphism to a projective normal variety with connected fibers, so that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y$. Then there exists an action of G on Y such that φ is equivariant.*

Proof. Let $L = \varphi^* L_Y$, where L_Y is an ample line bundle over Y.
We have $X = Proj_X(\bigoplus_m L^{\otimes m})$ and $Y = Proj(\bigoplus_m H^0(X, L^{\otimes m}))$ and $\varphi : X \to Y$ is induced by the evaluation morphism $H^0(X, L^{\otimes m}) \longrightarrow L^{\otimes m}$. The natural action of G on the graded ring of sections $\bigoplus_m H^0(X, L^{\otimes m})$ is clearly compatible with the evaluation.

Corollary 1. *Let X be a toric variety, $\varphi : X \to Y$ a morphism to a projective variety Y and let $X \xrightarrow{f_0} X' \xrightarrow{f_1} Y$ be the Stein factorization of φ. Then X' is a toric variety.*

Proof. By Blanchard's theorem the big torus of X acts on Y with an open orbit. Thus the quotient of the big torus of X by the isotropy of a general point of Y is a torus and it acts on Y with an open orbit, hence Y is toric by [Od], Theorem 1.5.

Euler-Jaczewski sequence. [Ja] Let X be a complete algebraic variety, let $H = H^1(X, \Omega_X)$ and $\mathcal{H}$ be the sheaf $H \otimes \mathcal{O}_X$.

Definition. The short exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{P}_X^\vee \longrightarrow \mathcal{H} \longrightarrow 0$$

corresponding to $Id_H \in \mathrm{Hom}(H, H) = \mathrm{Ext}^1(\mathcal{H}, \Omega_X)$ will be called the Euler-Jaczewski sequence of the variety X whereas the sheaf $\mathcal{P}_X$ will be called the potential sheaf of X.

We have the following characterization of toric varieties in terms of the Euler-Jaczewski sequence, [Ja, 3.1]:

Theorem. [Jaczewski] *A smooth complete and connected variety X is a toric variety if and only if there exists an effective divisor $D = \bigcup i \in ID_i$ with normal crossing such that*

$$\mathcal{P}_X = \bigoplus_{i\in I} \mathcal{O}_X(D_i)$$

where D_i are the irreducible components of the divisor D. The divisors D_i are then the closures of the codimension one orbits of the torus.

On a smooth toric variety we have a short exact sequence:

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathrm{Div}^T X \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

where $\mathrm{Div}^T X$ denotes the group of $\mathbf{T}$ equivariant divisors $\sum_\rho a_\rho D_\rho$. The first map associates to a character its divisor of poles and zeroes while the second map associates to a $\mathbf{T}$-equivariant divisor its class in $\mathrm{Pic}(X)$.
Dualizing the above sequence and tensoring it with $\mathbf{C}$ we obtain

$$0 \longrightarrow N_1(X)_{\mathbf{C}} \longrightarrow \bigoplus_{\rho \in \Sigma^1} \mathbf{C}[\rho] \longrightarrow \mathbf{N_C} \longrightarrow 0$$

where $N_1(X)_{\mathbf{C}}$ is the space of 1-cycles on X.
The maps in the sequence are defined as follows: $N_1(X)_{\mathbf{C}} \ni Z \longrightarrow \sum_{\rho\in\Sigma^1}(Z \cdot D_\rho)\cdot[\rho]$ and $\sum_\rho a_\rho \cdot [\rho] \longrightarrow \sum_\rho a_\rho \cdot e_\rho$, where e_ρ is the generator of the semigroup $\mathbf{N} \cap \rho$.

Lemma 1. *Let $X = X_\Sigma$ be a toric variety as above. Consider $\varphi : X \to Y$ a surjective and generically finite morphism. By $J(\varphi)$ let us denote the subset of Σ^1 corresponding to divisors D_ρ which are mapped to divisors in Y, that is $J(\varphi) = \{\rho \in \Sigma^1 |\ \varphi_* D_\rho \neq 0\}$. Then the restriction of the above map $\underset{\rho\in J}{\oplus} \mathbf{C}[\rho] \longrightarrow \mathbf{N_C}$ is surjective.*

Proof. Consider the Stein factorization of $\varphi : X \to Y$. By corollary 1 the connected part of this factorization is a birational toric morphism $\varphi_0 : X_\Sigma \to X_{\Sigma'}$, where the fan Σ is a subdivision of a fan Σ' and the rays of Σ' are exactly the rays corresponding to the divisors D_ρ, with $\rho \in J(\varphi)$. Since the fan Σ' is complete, its rays span $\mathbf{N_C}$ and we are done.

Note that, for a toric variety, we have $H^1(X, \Omega_X) = Pic(X)_{\mathbf{C}}$. In particular $N_1(X)_{\mathbf{C}} \otimes \mathcal{O}_X = \mathcal{H}^\vee$; denoting by $\mathcal{O}_X[\rho]$ the sheaf $\mathbf{C}[\rho] \otimes \mathcal{O}_X$ and by $\mathcal{N}$ the sheaf $\mathbf{N_C} \otimes \mathcal{O}_X$ we have a commutative diagram of sheaves over X with exact rows and columns, which contains both the Euler-Jaczewski sequence and the sequence we have just discussed (see [Ja], diagram (7) on page 238):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{H}^\vee \arrow[r] \arrow[d, equal] & \bigoplus \mathcal{O}_X[\rho] \arrow[r] \arrow[d, "s"] & \mathcal{N} \arrow[r] \arrow[d, "ev"] & 0 & \\
0 \arrow[r] & \mathcal{H}^\vee \arrow[r] \arrow[d] & \bigoplus \mathcal{O}_X(D_\rho) \arrow[r] \arrow[d] & TX \arrow[r] \arrow[d] & 0 & (*) \\
 & 0 \arrow[r] & \bigoplus \mathcal{O}_{D_\rho}(D_\rho) \arrow[r, equal] \arrow[d] & \bigoplus \mathcal{O}_{D_\rho}(D_\rho) \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Here $s = (s_\rho)$, where s_ρ is the section of $\mathcal{O}_X(D_\rho)$ vanishing along D_ρ, and ev associates to a 1-parameter group $\gamma \in \mathbf{N}$ its tangent field.

Families of curves.

Let Y be a smooth projective variety, and $y \in Y$ a point. We consider schemes $\mathrm{Hom}(\mathbf{P}^1, Y)$, parametrizing morphisms from $\mathbf{P}^1$ to Y, and $\mathrm{Hom}(\mathbf{P}^1, Y; 0 \to y)$, parametrizing morphisms sending $0 \in \mathbf{P}^1$ to $y \in Y$. Let $V \subset \mathrm{Hom}(\mathbf{P}^1, Y)$ be a closed irreducible subvariety; we will call V, by abuse, a family of rational curves on Y and we will denote by V_y the variety $V \cap \mathrm{Hom}(\mathbf{P}^1, Y; 0 \to y)$. If the evaluation $F : \mathbf{P}^1 \times \mathrm{Hom}(\mathbf{P}^1, Y) \to Y$ is a dominating morphism then we will call V a dominating family of rational curves. Suppose that $\rho(Y) = 1$. Then among all dominating families of rational curves on Y (if there exists any) we can choose a family $\mathcal{V}$ parametrizing curves of minimal degree with respect to a chosen ample divisor on Y; we will call such a family a minimal dominating family of rational curves on Y.

Let us note that in case of Theorem 1, that is when Y is dominated by a toric variety and $\rho(Y) = 1$, there exists a minimal dominating family $\mathcal{V}$ of rational curves for Y. For this family we will use the following version of Mori's result, [Mo].

Theorem. [Mori] *Assume that Y is a smooth projective variety such that $\rho(Y) = 1$. Let $\mathcal{V} \subset Hom(\mathbf{P}^1, Y)$ be a minimal dominating family of rational curves on Y. If for a general point $y \in Y$ and for any $f \in \mathcal{V}_y$ the pull-back f^*TY is ample then $Y \simeq \mathbf{P}^n$.*

For the proof of Theorem 1 we will need to understand properties of the minimal dominating family of rational curves on Y from the point of view of properties of the dominating variety $X \longrightarrow Y$. The key is the following technical observation which we prove in a more general setup:

Lemma 2. *Let $\varphi : X \to Y$ be a surjective morphism of smooth irreducible projective varieties. Assume that $\mathcal{M}$ is a dominating family of curves of Y, that is, there exists a variety $\mathcal{C}_Y$ with morphisms $p : \mathcal{C}_Y \longrightarrow \mathbf{M}$ and $q : \mathcal{C}_Y \longrightarrow Y$, the latter morphism is dominating, such that all fibers of p are 1-dimensional and mapped via q to curves in Y. Let $\mathcal{C}_X$ be an irreducible component of the fiber product $X \times_Y \mathcal{C}_Y$ which dominates $\mathcal{C}_Y$; by $\bar{q}$ and $\bar{\varphi}$ let us denote morphisms of $\mathcal{C}_X$ to X and $\mathcal{C}_Y$, respectively. Suppose that D is an irreducible effective Cartier divisor on X such that $\varphi_* D$ is an ample Cartier divisor on Y. Then, for every $m \in \mathcal{M}$, the pull-back $\bar{p}^* D$ is non zero on every connected component of $(p \circ \bar{\varphi})^{-1}(m)$.*

Proof. Let $\mathcal{C}_X \xrightarrow{p_0} \mathcal{M}_0 \xrightarrow{p_1} \mathcal{M}$ be the Stein factorization of the map $p \circ \bar{\varphi} : \mathcal{C}_X \longrightarrow \mathcal{M}$, so that we the following commutative diagram

$$\begin{tikzcd} & & \mathcal{M}_0 \arrow[dr, "p_1"] & \\ \mathcal{C}_X \arrow[urr, "p_0"] \arrow[r, "\bar\varphi"'] \arrow[d, "\bar q"'] & \mathcal{C}_Y \arrow[rr, "p"'] \arrow[d, "q"'] & & \mathcal{M} \\ X \arrow[r, "\varphi"'] & Y & & \end{tikzcd}$$

By our assumptions $p_1 \circ p_0 : \bar{q}^* D \to \mathcal{M}$ dominates $\mathcal{M}$, so its image in $\mathcal{M}_0$ is an irreducible subvariety of maximal dimension, hence, since $\mathcal{M}_0$ is irreducible, it coincides with $\mathcal{M}_0$.

Proof of Theorem 1.

First of all, by toric Chow's Lemma we can assume that X is projective. Then, in view of Corollary 1, considering the Stein factorization of φ, we can assume that X and Y are of the same dimension. Finally, by taking a desingularization of X, we can assume that X is smooth.

Let $\mathcal{V}$ be a minimal dominating family of rational curves on Y. Let $y \in Y$ be a general point which is not contained in the branch locus of φ nor it is contained in the image of $\varphi(\bigcup D_\rho)$. Let $f : \mathbf{P}^1 \to Y$ be a curve in $\mathcal{V}$ passing through y; the pullback of the tangent bundle splits into a sum of line bundles: $f^*TY \simeq \bigoplus \mathcal{O}_{\mathbf{P}^1}(a_l)$. By Mori's theorem explained above we shall be done if all numbers a_l are positive.

Suppose, by contradiction that $a_{\bar{l}} \leq 0$ for some $\bar{l}$; in this case we have a surjection $f^*TY \longrightarrow \mathcal{O}_{\mathbf{P}^1}(a_{\bar{l}})$ and hence a surjection $f^*TY \longrightarrow \mathcal{O}_{\mathbf{P}^1}$. By Lemma 1 and the commutativity of diagram $(*)$ the map $\bigoplus_{\rho\in J} \mathcal{O}_X(D_\rho) \longrightarrow TX$ is generically surjective. Since we assume that φ is generically finite, we get a generically surjective map $\bigoplus_{\rho\in J} \mathcal{O}(D_\rho) \to \varphi^*TY$ which, by the choice of y, is surjective over $\varphi^{-1}(y)$.

Let us take $\mathcal{C}_Y = \mathcal{V} \times \mathbf{P}^1$ with $p : \mathcal{C}_Y \longrightarrow \mathcal{V}$ the projection and $q : \mathcal{C}_Y \longrightarrow Y$ the evaluation. Now consider the situation discussed in Lemma 2

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "\bar\varphi"'] \arrow[d, "\bar q"'] & \mathcal{C}_Y \arrow[r, "p"'] \arrow[d, "q"'] & \mathcal{V} \\ X \arrow[r, "\varphi"'] & Y & \end{tikzcd}$$

Let X_f be a connected component of $(p \circ \bar\varphi)^{-1}(f)$. We obtain a generically surjective map $\bar{q}^*(\bigoplus_{\rho\in J} \mathcal{O}(D_\rho))_{|X_f} \longrightarrow ((\bar{q}^* \circ \varphi^*)TY)_{|X_f} = (\bar\varphi^*(f^*TY))_{|X_f}$ and therefore we have a generically surjective map $\bar{q}^*(\bigoplus_{\rho\in J} \mathcal{O}(D_\rho))_{|X_f} \longrightarrow \mathcal{O}_{X_f}$.
Thus there exists a non-zero section in

$$H^0(X_f, \bar{q}^*(\bigoplus_{\rho\in J} \mathcal{O}(-D_\rho))) = \bigoplus_{\rho\in J} H^0(X_f, \bar{q}^*\mathcal{O}(-D_\rho)),$$

but this is impossible, since for X_f is connected and for any $\rho \in J$ the divisor $\bar{q}^* D_\rho$ is effective non-zero on X_f by Lemma 2.

An application.

Let X be a smooth variety endowed with two different $\mathbf{P}$-bundle structures $\varphi : X \to Y$ and $\psi : X \to Z$. Since fibers of different extremal ray contractions can meet only in points we have $\dim X \geq \dim Y + \dim Z$; an easy corollary of Lazarsfeld's theorem is that we have equality if and only if $X = \mathbf{P}^r \times \mathbf{P}^s$.
Using Theorem 1, we are able to describe the next case; we have the following

Theorem 2. *Let X be a smooth projective variety of dimension n, endowed with two different $\mathbf{P}$-bundle structures $\varphi : X \to Y$ and $\psi : X \to Z$ such that $\dim Y + \dim Z = n+1$. Then either $n = 2m-1$, $Y = Z = \mathbf{P}^m$ and $X = \mathbf{P}(T\mathbf{P}^m)$ or Y and Z have a $\mathbf{P}$-bundle structure over a smooth curve C and $X = Y \times_C Z$.*

Proof. Let $[l_\varphi]$ and $[l_\psi]$ be the numerical equivalence classes of lines in the fibers of φ and ψ. Let x be a point of X and let B_x be the set of points of X that can be joined to x by a chain of rational curves whose numerical class is either $[l_\varphi]$ or $[l_\psi]$.

If there exists a point x such that $B_x = X$ then, by [Ko, IV.3.13.3] the Picard number of X is two, hence Y and Z are Fano varieties of Picard number one. We also note that every pair of points of Z is connected by a chain of rational curves whose members are images of rational curves in X whose numerical class is $[l_\varphi]$.
Let F_t and F_w, with $t,\ w \in Z$, be two fibers of ψ such that $\varphi(F_t) \neq \varphi(F_w)$. Since points t and w are joined by a chain of rational curves as above thus there exists a rational curve Γ on Z such that $\varphi : \psi^{-1}(\Gamma) \longrightarrow Y$ is dominant.
Let $\nu : \mathbf{P}^1 \to Z$ be the normalization of Γ and let $X_\Gamma \longrightarrow \mathbf{P}^1$ be the pull-back of the projective bundle, that is $X_\Gamma = \mathbf{P}^1 \times_\Gamma X$. The composition of the induced map $\bar{\nu} : X_\Gamma \longrightarrow X$ with φ is a surjective morphism $\varphi \circ \bar{\nu} : X_\Gamma \longrightarrow Y$. The variety X_Γ is a $\mathbf{P}$-bundle on $\mathbf{P}^1$, hence a toric variety, and Y is a smooth variety with $\rho(Y) = 1$, hence Theorem 1 applies to give $Y \simeq \mathbf{P}^{\dim Y}$; in the same way we also get $Z \simeq \mathbf{P}^{\dim Z}$.
We conclude this case by the main theorem of [Sa], that is we get $\dim Y = \dim Z = m$ and $X \simeq \mathbf{P}(T\mathbf{P}^m)$.

If B_x is a divisor for every $x \in X$ then, for general (x_1, x_2) in $X \times X$, the divisors B_{x_1} and B_{x_2} are disjoint and because they are numerically equivalent we have $B_x^2 \equiv 0$. We claim that there exists a regular morphism $p : X \longrightarrow C$ with connected fibers, onto a smooth curve C, which contracts all divisors B_x. If $H^1(X, \mathcal{O}_X) \neq 0$ then p is obtained from the Albanese map of X (note that B_x are rationally connected hence contracted by Albanese). If $H^1(X, \mathcal{O}_X) = 0$ then divisors B_x are linearly equivalent, the linear system $|B_x|$ is base point free and defines p. By construction p factors through φ and ψ to produce $p_Y : Y \longrightarrow C$ and $p_Z : Z \longrightarrow C$. Moreover, by construction $\rho(X/C) = 2$ hence $\rho(Y/C) = 1$ and $\rho(Z/C) = 1$.
A general B_x is smooth with two projective bundle structures hence, by what we have said at the beginning of this section, it is a product of projective spaces. Thus a general fiber of p_X as well as p_Z is a projective space and over an open Zariski U subset of C

both morphisms are projective bundles. By taking the closure in Y and Z of a hyperplane section of p_Y and p_Z, respectively, defined over the open set U we get a global relative hyperplane section divisor (we use $\rho(Z/C) = \rho(Y/C) = 1$) hence p_Y and p_Z are projective bundles globally. The conclusion $X = Y \times_C Z$ is immediate.

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