MODULI SPACES OF SHEAVES ON HIGHER DIMENSIONAL VARIETIES

Adrian Langer

Contents

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Lecture 1. Bogomolov's instability and restriction theorems

Let X be a smooth complex projective variety. We want to endow the set of algebraic vector bundles on X with a natural algebraic structure. More precisely, we want to classify holomorphic structures on a given topological vector bundle over X.

First, let us start with a topological classification of vector bundles on a given smooth projective surface. In this case to each vector bundle E one can associate its rank r, the first Chern class $c_1E \in H^2(X,\mathbb{Z})\cap H^{1,1}(X)$ and the second Chern class $c_2E \in H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$. These are all topological invariant and all triples (r, c_1, c_2) with $r \geq 2$ can be realized by holomorphic vector bundles (R. Schwarzenberger).

Now let us fix the topological invariants (r, c_1, c_2). In general, we cannot expect that the set of all vector bundles with fixed invariants have a nice structure of algebraic variety. The necessity of restricting to some subset of the set of all vector bundles can be understood by the following standard example.

Example 1.1.

Consider the set $\{\mathcal{O}(n)\oplus\mathcal{O}(-n)\}$ on $\mathbb{P}^1$. The topological invariants are fixed but we have infinitely many points which cannot form a nice algebraic variety. Moreover, there exists a family of vector bundles $\{E_t\}_{t\in\mathbb{C}}$ such that $E_t \simeq \mathcal{O}\oplus\mathcal{O}$ for $t \neq 0$ but $E_0 \simeq \mathcal{O}(n)\oplus\mathcal{O}(-n)$ (Exercise: construct such a family). So the point corresponding to $\mathcal{O}\oplus\mathcal{O}$ would not be closed in the moduli space.

The natural class of vector bundles which admits a nice moduli space comes from Mumford's Geometric Invariant Theory (GIT). The corresponding vector bundles are called stable (or semistable). However, if we want to get a projective moduli space then we need to add some non-locally free sheaves at the boundary of the moduli space. So we need to define semistabilty and stability in the more general context.

Definition 1.2. *Let X be a smooth n-dimensional projective variety defined over an algebraically closed field k and let H be an ample divisor on X.*

For any rank $r > 0$ torsion free sheaf E we define its slope by

$$\mu(E) = \frac{c_1E\cdot H^{n-1}}{r}.$$

The Hilbert polynomial $P(E)$ is defined by $P(E)(k) = \chi(X, \mathcal{O}_X(kH))$.

(1) E is called slope H-stable if and only if for all subsheaves $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$ we have

$$\mu(F) < \mu(E).$$

(2) E is called Gieseker H-stable if and only if for all proper subsheaves $F \subset E$

$$P(F)/\operatorname{rk} F < P(E)/\operatorname{rk} E$$

(i.e., the inequality holds for large k).

Similarly, one can define (slope or Gieseker) H-semistability by changing the strict inequality sign $<$ to $\leq$.

1.3. Restricting our attention to semistable sheaves is not very restrictive since each (torsion-free) sheaf has a canonical filtration with semistable quotients.

Let us fix a torsion free sheaf E on X. Consider the set $\{\mu(F)\colon F \subset E\}$. This set has a maximal element $\mu_{\max} = \mu_{\max}(E)$ and the set $\{F\colon F \subset E, \mu(F) = \mu_{\max}\}$ contains a sheaf E_1 of largest rank. This element is the largest element in this set with respect to the inclusion relation and it is called the *maximal destabilizing subsheaf* of E.

Now we consider the maximal destabilizing subsheaf E_2' in E/E_1 and set $E_2 = p^{-1}E_2'$ for the natural projection $p\colon E \to E/E_1$. Reiterating this process we get the unique filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that

1. all the quotients $F_i = E_i/E_{i-1}$ are semistable, and
2. $\mu_{\max}(E) = \mu(F_1) > \mu(F_2) > \cdots > \mu(F_m) = \mu_{\min}(E)$.

This filtration is called the *Harder–Narasimhan filtration* of E.

1.4. Properties of slope semistability in characteristic 0*:*

(1) If E_1, E_2 are slope H-semistable torsion free sheaves then $E_1 \otimes E_2/$ torsion is also slope H-semistable.

(2) If $f\colon Y \to X$ is a finite map between smooth projective varieties then a torsion free sheaf F is slope H-semistable if and only if f^*F is slope f^*H-semistable.

Now let us consider the following question: which Chern classes are realized by semistable vector bundles?

The class $\Delta(E) = 2rc_2E - (r-1)c_1^2E$ is called the *discriminant* of E.

A partial answer to the above question is given by the following theorem.

Theorem 1.5. *(Bogomolov) Let X be a smooth complex projective surface. Then for any torsion free (slope) H-semistable sheaf E we have*

$$\Delta(E) \geq 0.$$

Proof. We can assume that E is a locally free sheaf. Indeed, for any torsion free slope H-semistable sheaf E on a smooth surface, the sheaf E^{**} is locally free, slope H-semistable and $\Delta(E^{**}) \leq \Delta(E)$.

For simplicity, we also assume that $c_1E = 0$. The general case can be reduced to this one using either the $\mathbb{Q}$-vector bundle $E(-\frac{1}{r}\det E)$ or the vector bundle $\mathrm{End}E$.

Now let us note that S^nE is slope H-semistable (this follows from 1.4.1). Hence for $C \in |kH|$ and $k \gg 0$ we have

$$h^0(S^nE(-C)) = 0.$$

Therefore from the short exact sequence

$$0 \to O_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \to 0$$

tensored with S^nE we get

$$h^0(S^nE) \leq h^0(S^nE(-C)) + h^0(S^nE|_C) = h^0(S^nE|_C).$$

Considering $Y = \mathbb{P}(E|_C) \to C$ we see that

$$h^0(S^nE|_C) = h^0(Y, \mathcal{O}_{\mathbb{P}(E|_C)}(n))$$

by the projection formula. Since $\dim Y = r$ there exists a constant C such that

$$h^0(S^nE) \le h^0(Y, \mathcal{O}_{\mathbb{P}(E|_C)}(n)) \le C \cdot n^r$$

for all $n > 0$. Similarly, one can see that there exists a constant C' such that

$$h^2(S^nE) = h^0(S^nE^* \otimes K_X) \le C' \cdot n^r$$

for all $n > 0$.

Therefore

$$\chi(X, S^nE) \le h^0(S^nE) + h^2(S^nE) \le (C + C')n^r.$$

But by Exercise 15 we have

$$\chi(X, S^nE) = -\frac{\Delta(E)}{2r}\frac{n^r}{(r+1)!} + \mathrm{O}(n^r),$$

so $\Delta(E) \ge 0$, Q.E.D.

The above proof of Bogomolov's theorem follows quite closely Y. Miyaoka's proof from [Mi].

Proposition 1.6. *([La]) Let X be a smooth complex projective surface and let H be an ample divisor on X. Then for any rank r torsion free sheaf E we have*

$$H^2\Delta(E) + r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}) \ge 0.$$

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the Harder–Narasimhan filtration. Set $F_i = E_i/E_{i-1}$, $r_i = \mathrm{rk}\, F_i$, $\mu_i = \mu(F_i)$. Then by Bogomolov's inequality and the Hodge index theorem

$$\begin{aligned}\frac{\Delta(E)}{r} &= \sum \frac{\Delta(F_i)}{r_i} - \frac{1}{r}\sum_{i<j} r_ir_j\left(\frac{c_1F_i}{r_i} - \frac{c_1F_j}{r_j}\right)^2 \\ &\ge -\frac{1}{rH^2}\sum_{i<j} r_ir_j\left(\left(\frac{c_1F_i}{r_i} - \frac{c_1F_j}{r_j}\right)H\right)^2 = -\frac{1}{rH^2}\sum_{i<j} r_ir_j(\mu_i - \mu_j)^2.\end{aligned}$$

(The Hodge index theorem says that $(DH)^2 \ge D^2 \cdot H^2$ for any divisor D.) Now the proposition follows from the following lemma:

Lemma 1.6.1. *Let r_i be positive real numbers and $\mu_1 > \mu_2 > \cdots > \mu_m$ real numbers. Set $r = \sum r_i$ and $r\mu = \sum r_i\mu_i$. Then*

$$\sum_{i<j} r_ir_j(\mu_i - \mu_j)^2 \le r^2(\mu_1 - \mu)(\mu - \mu_m).$$

Proof. Let us note that

$$\sum_{i<j} r_ir_j(\mu_i - \mu_j)^2 = r\left(\sum_{i=1}^{m-1}\left(\sum_{j\le i} r_j(\mu_j - \mu)\right)(\mu_i - \mu_{i+1})\right).$$

Using $\sum_{j\le i} r_j\mu_j \le (\sum_{j\le i} r_j)\mu_1$ and simplifying yields the required inequality, Q.E.D.

As an application of the above proposition we get the following effective restriction theorem. A weaker version of this theorem was proved by F.A. Bogomolov.

Theorem 1.7. *([La]) Let E be a rank $r \geq 2$ vector bundle on a smooth complex projective surface X. Assume that E is slope H-stable. Let $D \in |kH|$ be a smooth curve. If*

$$k > \left\lfloor \frac{r-1}{r}\Delta(E) + \frac{1}{r(r-1)H^2} \right\rfloor$$

then $E|_D$ is stable.

Proof. Assume that $E|_D$ is not stable. Then there exists a subsheaf $S \subset E|_D$ such that $\mu(S) \geq \mu(E|_D)$. Let us take the maximal subsheaf with this property. In this case the quotient $T = (E|_D)/S$ is a vector bundle on D (it is sufficient to check that T is torsion free; if it has a torsion then the kernel S' of $E|_D \to T/$torsion contains S and $\mu(S') \geq \mu(S)$).

Let G be the kernel of the composition $E \to E|_D \to T$. The sheaf G is called an *elementary transformation* of E along T. Set $\rho = \operatorname{rk} S$. Then $\operatorname{rk} E = \operatorname{rk} G = r$ and $\operatorname{rk} T = r - \rho$ (as a vector bundle on D).

Computing $\Delta(G)$ (use Exercise 16) we get

$$\Delta(G) = \Delta(E) - \rho(r-\rho)D^2 + 2(r \deg_D T - (r-\rho)Dc_1(E)).$$

By assumption $\mu(T) \leq \mu(E|_D)$, so

$$\Delta(G) \leq \Delta(E) - \rho(r-\rho)D^2.$$

Using the stability of E we get

$$\mu_{\max}(G) - \mu(G) = \mu_{\max}(G) - \mu(E) + \frac{r-\rho}{r}DH \leq \frac{r-\rho}{r}kH^2 - \frac{1}{r(r-1)}.$$

Note that we have two short exact sequences:

$$0 \to G \longrightarrow E \longrightarrow T \to 0$$

and

$$0 \to E(-D) \longrightarrow G \longrightarrow S \to 0.$$

In particular $G^* \subset (E(-D))^*$ and

$$\begin{aligned}\mu(G) - \mu_{\min}(G) &= \mu(E(-D)) - \mu_{\min}(G) + \frac{\rho}{r}DH \\ &= \mu_{\max}(G^*) - \mu((E(-D))^*) + \frac{\rho}{r}DH \leq \frac{\rho}{r}kH^2 - \frac{1}{r(r-1)}.\end{aligned}$$

Hence, applying Proposition 1.6 to G we obtain

$$\begin{aligned}0 &\leq H^2\Delta(G) + r^2(\mu_{\max}(G) - \mu(G))(\mu(G) - \mu_{\min}(G)) \\ &\leq H^2\Delta(E) - \rho(r-\rho)(H^2)^2k^2 + r^2\left(\frac{r-\rho}{r}kH^2 - \frac{1}{r(r-1)}\right)\left(\frac{\rho}{r}kH^2 - \frac{1}{r(r-1)}\right).\end{aligned}$$

Therefore

$$\frac{rH^2}{r-1}k \leq H^2\Delta(E) + \frac{1}{(r-1)^2},$$

which contradicts our assumption on k, Q.E.D.

Remarks.

(1) Note that if E is torsion free then the restriction $E|_D$ is also torsion free for a general divisor D in a base point free linear system (see [HL], Corollary 1.1.14 for a precise statement).

(2) If $r > 2$ or $H^2 \geq 2$ it is sufficient to assume that $k > \frac{r-1}{r}\Delta(E)$. If $r = 2$ and $H^2 = 1$ then we need to assume that $k > (\Delta(E)+1)/2$. Looking at the proof one can see that Theorem 1.7 can be further improved at the cost of simplicity.

As a corollary to Theorem 1.7 we get an effective restriction theorem for semistable sheaves.

Corollary 1.8. *Let E be a torsion free sheaf of rank $r \geq 2$. Assume that E is slope H-semistable. Let D be a general curve of a base point free linear system $|kH|$. If*

$$k > \left\lfloor \frac{r-1}{r}\Delta(E) + \frac{1}{H^2 r(r-1)} \right\rfloor$$

then $E|_D$ is semistable.

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the Jordan–Hölder filtration of E (i.e., such a filtration that all the quotients are slope H-stable torsion free sheaves, cf. 2.9). Set $F_i = E_i/E_{i-1}$ and $r_i = \operatorname{rk} F_i$. Let $D \in |kH|$ be any smooth curve such that all the sheaves $F_i|_D$ have no torsion. Then the corollary follows from Theorem and the following inequality

$$\frac{\Delta(E)}{r} \geq \sum \frac{\Delta(F_i)}{r_i}$$

(cf. the proof of Proposition 1.6), Q.E.D.

Some applications of the effective restriction theorem:

Theorem 1.7 implies boundedness of the family of semistable sheaves on complex surfaces. More precisely, there exists a scheme S of finite type over $\mathbb{C}$ and an S-flat sheaf F on $S \times X$ such that the set $\{F_s\}_{s\in S}$, where $F_s = F|_{\{s\}\times X}$, contains all slope H-semistable sheaves with fixed topological invariants.

This implies that the moduli space of Gieseker semistable sheaves on X is a projective scheme of finite type over $\mathbb{C}$.

One can also prove a similar theorem for higher dimensional varieties in any characteristic (see [La]). This implies that the moduli space of Gieseker semistable sheaves on any noetherian scheme over a universally Japanese ring R is a projective scheme of finite type over R (see Theorem 2.6).

Lecture 2

In this lecture we discuss a general problem of representing different functors, we define Quot–schemes, Hilbert schemes, Flag–schemes and we construct moduli spaces of sheaves.

We are interested in providing the set of isomorphism classes of vector bundles on some variety with a natural scheme structure. To say what "natural" means we need a few notions from category theory. They will also be useful in defining the Hilbert scheme, Grothendieck's Quot-scheme and the flag scheme.

Let $\mathcal{C}$ be a category (e.g., of S-schemes) and let $\mathcal{M}\colon \mathcal{C} \to \mathrm{Sets}$ be a functor.

Definition 2.1.

(1) $\mathcal{M}$ is represented *by an object $M \in \mathrm{Ob}\,\mathcal{C}$ if it is isomorphic to the functor h_M defined by $h_M(S) = \mathrm{Hom}_{\mathcal{C}}(S, M)$.*

(2) $\mathcal{M}$ is corepresented *by an object $M \in \mathrm{Ob}\,\mathcal{C}$ if there is a natural transformation $\alpha: \mathcal{M} \to h_M$ such that for any natural transformation $\beta: \mathcal{M} \to h_N$ there exists a unique morphism $\varphi: M \to N$ such that $\beta = h_\varphi \alpha$.*

If $\mathcal{M}$ is a moduli functor (later we will see some examples) and $\mathcal{M}$ is represented by M then we say that M is a *fine moduli space* for the functor $\mathcal{M}$. If the functor $\mathcal{M}$ is corepresented by M then we say that M is a *coarse moduli space* for $\mathcal{M}$.

Let $f: X \to S$ be a projective morphism of Noetherian schemes and let $\mathcal{O}_X(1)$ be an f-ample line bundle on X.

Let E be an S-flat coherent sheaf on X. We want to define a scheme which parametrizes pairs (s, F) consisting of a point $s \in S$ and a coherent quotient sheaf F of E with fixed Hilbert polynomial P. To this end let us first define the functor $\mathfrak{Quot}$: $\{S\text{-schemes}\} \to$ Sets, which to each S-scheme T associates the set of all T-flat quotients

$$E_T = \mathcal{O}_T \otimes E \to F$$

of coherent sheaves on $Y = T \times_S X$ such that for all $t \in T$ the sheaf $F_t = F|_{Y_t}$ on the fiber $Y_t = \mathrm{Spec}\, k(t) \times_T Y$ of $Y \to T$ has Hilbert polynomial P.

Theorem 2.2.
The functor $\mathfrak{Quot}$ is represented by a projective S-scheme $\mathrm{Quot}_{X/S}(E; P) \to S$. This scheme is called Grothendieck's Quot-scheme.

Obviously, the Quot-scheme generalizes the classical Grassmann variety. In the special case when $E = \mathcal{O}_X$ the Quot scheme gives the Hilbert scheme parametrizing S-flat subschemes of the scheme X with given Hilbert polynomial P.

Let E be an S-flat coherent sheaf on X and let P_i, $i = 1, \dots, k$, be some fixed polynomials.

Consider the functor $\mathfrak{Flag}$: $\{S\text{-schemes}\} \to$ Sets, which to each S-scheme T associates the set of all flags

$$0 \subset F_1 \subset \cdots \subset F_k = E_T = \mathcal{O}_T \otimes E$$

of coherent subsheaves on $Y = T \times_S X$ such that
1. the factors $\mathrm{gr}_i = F_i / F_{i-1}$ of this filtration are T-flat, and
2. for all $t \in T$ the sheaf $\mathrm{gr}_{i,t} = \mathrm{gr}_i|_{Y_t}$ has Hilbert polynomial P_i.

Theorem 2.3.
The functor $\mathfrak{Flag}$ is represented by a projective S-scheme $\mathrm{Flag}_{X/S}(E; P_1, \dots, P_k) \to S$.

2.4. Example. (A categorical quotient)

Let a group scheme G acts on a variety X. Then for any k-scheme T the T-points of G (i.e., $G(T) = \mathrm{Hom}(T, G)$) also form a group which naturally acts on the T-points of X.

Consider the functor X/G: $\{k\text{-schemes}\} \to$ Sets, which to any k-scheme T associates a set of $G(T)$-orbits of $X(T)$. If this functor is corepresented by a scheme

Y, then Y is called a *categorical quotient* of X by G. By Mumford's Geometric Invariant Theory (GIT) this is true, e.g., if G is a reductive group acting on a G-linearized projective scheme.

For a category $\mathcal{C}$ let $\mathcal{C}^o$ denotes the category with the same objects and reversed arrows.

In these lectures we are interested in the following moduli functor.

Let $(X, \mathcal{O}_X(1))$ be a smooth polarized projective scheme over an algebraically closed field k. For a k-scheme S let p and q denote the projections of $S \times_k X$ to S and X, respectively. Let us define an equivalence relation $\sim$ on S-flat families of sheaves on X by $F_1 \sim F_2$ if and only if there exists a line bundle $\mathcal{L}$ on S such that $F_1 \simeq F_2 \otimes p^*\mathcal{L}$. Let us note that fibrewise the families F_1 and F_2 define the same family of sheaves on X. Therefore it is natural to introduce the following moduli functor.

The functor $\mathcal{M}_P : \{k\text{-schemes}\}^o \to \mathrm{Sets}$ by sends a k-scheme S to the set of isomorphism classes of S-flat families of Gieseker semistable sheaves on X with Hilbert polynomial P modulo the relation $\sim$.

Theorem 2.5. *There exists a coarse moduli scheme M_P for the functor $\mathcal{M}_P$.*

There exists a more general version of this theorem which will also be useful in the following. Before formulation of this more general theorem we need to generalize the notion of Gieseker semistability and stability to not necessarily torsion free sheaves.

A sheaf E is called *pure* if it is torsion free on its scheme-theoretical support. Equivalently, for any subsheaf $F \subset E$ the dimension of the support of F is equal to the dimension of the support of E (this is denoted by $\dim E$).

The Hilbert polynomial of any coherent sheaf E can be written as a sum

$$P(E)(k) = \chi(E(k)) = \sum_{i=0}^{\dim E} \alpha_i(E) \frac{k^i}{i!}.$$

Let $p(E) = \frac{P(E)}{\alpha_{\dim E}(E)}$ be the reduced Hilbert polynomial. Then a pure sheaf E is called *Gieseker semistable* (*stable*) if for any proper subsheaf $F \subset$E we have $p(F) \leq p(E)$ ($p(F) < p(E)$, respectively).

Theorem 2.6. *Let R be a universally Japanese ring. Let $f\colon X \to S$ be a projective morphism of R-schemes of finite type with geometrically connected fibers and let $\mathcal{O}_X(1)$ be an f-ample line bundle. Then for a fixed polynomial P there exists a projective S-scheme $M_{X/S}(P)$ of finite type over S, which universally corepresents the functor*

$$\mathcal{M}_{X/S}(P)\colon \{\textit{schemes over } S\}^o \to \mathrm{Sets}$$

defined by

$$(\mathcal{M}_{X/S}(P))(T) = \left\{ \begin{array}{l} \textit{equivalence classes of families of pure Gieseker} \\ \textit{semistable sheaves on the fibres of } T \times_S X \to T \\ \textit{which are } T\textit{-flat and have Hilbert polynomial } P \end{array} \right\}.$$

Moreover, there is an open scheme $M^s_{X/S} \subset M_{X/S}$ that universally corepresents the subfunctor of families of geometrically Gieseker stable sheaves.

Universally Japanese rings are also known as *Nagata rings*.

2.7. Example.

Let $X \to T$ be a flat projective morphism of locally Noetherian schemes, whose geometric fibres are varieties. The *Picard scheme* $\mathrm{Pic}^P_{X/T}$ parametrizes line bundles with fixed Hilbert polynomial P on fibres of $X \to T$. In general, it is a quasi-projective scheme (Exercise: Prove that it is an open subset of the moduli scheme M_P) but it is not a fine moduli scheme.

However, if $X \to T$ has a section then it is a fine moduli scheme. In this case there exists a universal family of line bundles on $X \times_T \mathrm{Pic}^P_{X/T}$. This family is denoted by $\mathcal{P}$ and it is called the *Poincare line bundle*.

If X is a smooth variety over an algebraically closed field k then $\mathrm{Pic}^P_{X/k}$ is a projective scheme (Exercise: Prove this!). If P is the Hilbert polynomial of $\mathcal{O}_X$ then $\mathrm{Pic}^P_{X/k}$ is a group scheme with $1 = [\mathcal{O}_X]$ and tensor product as a group action. If char $k = 0$ then by Cartier's theorem any group scheme is smooth, so in particular $\mathrm{Pic}^P_{X/k}$ is smooth.

Finally let us note that for any two polynomials P_1 and P_2 representing line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ the corresponding Picard schemes are isomorphic. An isomorphism between $\mathrm{Pic}^{P_1}_X$ and $\mathrm{Pic}^{P_2}_X$ is given, e.g., by tensoring with $\mathcal{L}_2 \otimes \mathcal{L}_1^*$.

2.8. Idea of the construction of the moduli space of semistable sheaves:

1. The family of all Gieseker semistable sheaves with fixed Hilbert polynomial P is bounded, i.e., it is parametrized by finitely many schemes of finite type. This implies that there exists an integer m_0 such that for all $m \geq m_0$ and for all Gieseker semistable sheaves E with Hilbert polynomial P

a) $E(m)$ is globally generated,

b) $H^i(X, E(m)) = 0$ for $i > 0$.

In particular, these conditions imply that $h^0(E(m)) = \chi(X, E(m)) = P(m)$.

2. Set $V = k^{\oplus P(m)}$ and $\mathcal{H} = V \otimes \mathcal{O}_X(-m)$. Consider the Quot-scheme $Q = \mathrm{Quot}(\mathcal{H}; P)$.

Let R be a subset of Q parametrizing all quotients $[\mathcal{H} \to E]$ such that E is semistable and the induced map $V = H^0(X, \mathcal{H}(m)) \to H^0(X, E(m))$ is an isomorphism. Now for any semistable sheaf E with Hilbert polynomial P and any isomorphism $V \to H^0(E(m))$ we get a point in R corresponding to

$$V \otimes \mathcal{O}_X(-m) \longrightarrow H^0(E(m)) \otimes \mathcal{O}_X(-m) \xrightarrow{ev} E.$$

Therefore $\mathrm{GL}(V)$ acts on R and, at least set theoretically, the quotient of R by $\mathrm{GL}(V)$ parametrizes semistable sheaves on X.

To get a scheme structure on this quotient we need to use Mumford's GIT (Geometric Invariant Theory) as explained in P. Newstead's lectures.

To construct GIT quotients one needs to find a $\mathrm{GL}(V)$-linearized polarization $\mathcal{L}$ on R.

There are two different constructions and both of them will be used in describing line bundles on the moduli space. One of them is due to C. Simpson and it was used in P. Newstead's lectures. The other is due to D. Gieseker, and although it is less natural it was constructed much earlier. The problem with Gieseker's construction is that it does not easily generalize to the "pure" case described in Theorem 2.6.

In Simpson's construction we consider

$$\mathcal{L}_l := \det(p_*(\tilde{F} \otimes q^* \mathcal{O}_X(l))),$$

where $\tilde{F}$ is the universal quotient sheaf on the Quot-scheme. For large l this line bundle is very ample on Q. The Quot scheme is constructed as a subscheme of certain Grassmannian and $\mathcal{L}_l$ comes from the Plücker embedding of this Grassmannian into a projective space.

In Gieseker's construction the constructed line bundle is not ample on Q but only on R.

To any family F of sheaves on a smooth projective variety X/k parametrized by S one can associate the family $\det F$ of determinant line bundles. This induces a natural transformation of functors $\mathcal{M}_P$ and $\mathcal{P}\mathrm{ic}_X$. Since the functor $\mathcal{P}\mathrm{ic}_X$ is represented by $\operatorname{Pic} X$ we get a map $\det : S \to \operatorname{Pic} X$ such that $\det F \simeq \det^* \mathcal{P} \otimes p^* \mathcal{L}$ for certain line bundle $\mathcal{L}$ on S.

If we make this construction for the universal quotient $V \otimes q^* \mathcal{O}_X(-m) \to \tilde{F}$ on $Q \times X$ then we get a line bundle $\mathcal{A} = \mathcal{L}|_{\overline{R}}$ on the closure of R in Q.

This line bundle $\mathcal{A}$ is also a quotient of $\det^*(\bigwedge^r V \otimes p^* \mathcal{P}^*(-rm))$ and the quotient map induces a morphism of schemes

$$\zeta : \overline{R} \to \mathbb{P}_{\operatorname{Pic} X}(\det^*(\bigwedge^r V \otimes p^* \mathcal{P}^*(-rm)))$$

over $\operatorname{Pic} X$. One can easily see that $\zeta|_R$ is injective, so $\mathcal{A} = \zeta^* \mathcal{O}_{\mathbb{P}}(1)$ is det-ample.

2.9. Let E be a Gieseker H-semistable torsion free sheaf. Then either it is Gieseker H-stable or there exists a proper subsheaf $E_1 \subset E$ such that $p(E_1) = p(E)$. By passing to a smaller subsheaf if necessary we can assume that E_1 is Gieseker H-stable. Then the quotient E/E_1 is Gieseker H-semistable with $p(E/E_1) = p(E)$.

By induction on the rank we can therefore construct a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ in which all the quotients E_i/E_{i-1} are slope H-stable and $p(E_i/E_{i-1}) = p(E)$. Such a filtration is called a *Jordan–Hölder filtration* of E (but unlike the Harder–Narasimhan filtration it is not unique).

Let us set $\operatorname{gr}_{JH} E = \bigoplus E_i/E_{i-1}$. One can show that $\operatorname{gr}_{JH} E$ does not depend on the choice of Jordan–Hölder filtration.

Definition 2.10. *Two Gieseker H-semistable sheaves E and E' with $p(E) = p(E')$ are called S-equivalent if* $\operatorname{gr}_{JH} E \simeq \operatorname{gr}_{JH} E'$.

Once we constructed a $\mathrm{GL}(V)$-linearized line bundle $\mathcal{L}_l$ on $\overline{R}$ one has to prove that the points of R correspond to semistable points of $\mathrm{GL}(V)$ action on $(\overline{R}, \mathcal{L}_l)$ and the points of R^s (corresponding to those points $[\mathcal{H} \to E] \in R$ for which E is Gieseker H-stable) correspond to stable points of $\mathrm{GL}(V)$ action on $(\overline{R}, \mathcal{L}_l)$.

By GIT there exists a good quotient $\pi : R \to M$ and M is the moduli space of Gieseker H-semistable sheaves. Moreover, the restriction $\pi|_{R^s} : R^s \to M^s$ is a geometric quotient (i.e., $\mathrm{GL}(V)$-orbits of stable sheaves are closed).

2.11. Example.

An extension of a sheaf F by a sheaf G is an exact sequence of sheaves $0 \to G \to E \to F \to 0$. The set of all extensions is parametrized by $\operatorname{Ext}^1(F, G)$.

Moreover, there exists the *universal extension* $\mathcal{E}$, i.e., such a family of sheaves on $\mathrm{Ext}^1(F,G)\times X$ that $E_\eta = E|_{\{\eta\}\times X}$ is the extension defined by $\eta\in\mathcal{E}\mathrm{xt}^1(F,G)$.

2.12. Geometric meaning of points of the moduli space M.

By the properties of a good quotient the points of M correspond to the closed orbits of $\mathrm{GL}(V)$-action on R (see P. Newstead's lectures). Therefore it is sufficient to identify such orbits. Below we use a slightly different method.

From Example 2.11 one can easily see that S-equivalent Gieseker H-semistable sheaves define the same point in the moduli space.

If we have a point $[\sigma\colon \mathcal{H}\to E]\in R$ and $E_1\subsetneq E$ is Gieseker H-semistable with $p(E_1)=p(E)$ then the universal extension induces a morphism $\mathrm{Ext}^1(E/E_1,E_1)\to M$.

On the line $\mathbb{A}^1$ defined by $\eta=[E]\in \mathrm{Ext}^1(E/E_1,E_1)$ we have $\mathcal{E}_{t\eta}\simeq E$ for $t\neq 0$ and $\mathcal{E}_0\simeq E_1\oplus E/E_1$. Therefore the map $\mathbb{A}^1\to M$ is constant (good quotients are separable) which shows that E defines the same point in M as $\mathrm{gr}_{JH}E$.

One can also show that two Gieseker H-semistable sheaves which are not S-equivalent define different points in M.

Corollary 2.13. *M parametrizes S-equivalence classes of Gieseker H-semistable sheaves.*

Lecture 3

3.1. Let X be an n-dimensional Noetherian scheme. The *Grothendieck group* $K(X)$ of coherent sheaves on X is the quotient of the free abelian group generated by the coherent sheaves on X by the subgroup generated by $[F]-[F']-[F'']$, where F,F' and F'' are coherent sheaves in the exact sequence

$$0\to F'\longrightarrow F\longrightarrow F''\to 0.$$

The Grothendieck group $K^0(X)$ of locally free sheaves is defined in the same way but using only locally free sheaves and short exact sequences of locally free sheaves.

If X is smooth then $K(X)\simeq K^0(X)$ (Exercise 18) has structure of a commutative ring with $1=[\mathcal{O}_X]$ and $[F_1]\cdot[F_2]:=[F_1\otimes F_2]$ for locally free sheaves F_1 and F_2.

In this case we can introduce a quadratic form $\chi\colon K(X)\times K(X)\to\mathbb{Z}$. This is defined by

$$(a,b)\to\chi(a\cdot b)=\int_X \mathrm{ch}(a)\,\mathrm{ch}(b)\,\mathrm{td}(X)$$

for $a,b\in K(X)$ (see Appendix).

A projective morphism $f\colon X\to S$ induces a homomorphism $f_!\colon K(X)\to K(S)$ defined by

$$f_!([F])=\sum_{i\geq 0}(-1)^i[R^if_*F].$$

Proposition 3.2. *If $f\colon X\to S$ is a smooth projective morphism, S is a k-scheme of finite type and F is an S-flat coherent sheaf on X then $[F]\in K^0(X)$ and $f_!([F])\in K^0(S)$. More precisely, F has a locally free resolution $F_\bullet$ such that $R^if_*F_j$ are locally free.*

Let p and q denote natural projections of $S\times X$ to S and X, respectively. The above proposition implies that if E is an S-flat family of sheaves on a smooth projective variety X then $[E]\in K^0(S\times X)$ and we have a map $p_!\colon K^0(S\times X)\to K^0(S)$. Therefore the following definition makes sense:

Definition 3.3. *For such a family E we define the homomorphism $\lambda_E\colon K(X) \to \operatorname{Pic} S$ by*

$$\lambda_E(u) = \det p_!(q^*u \cdot [E]).$$

In this way we constructed line bundles on a scheme parametrizing a given family of sheaves on X. We want to make a similar construction to construct line bundles on the moduli space.

If the moduli space M_P is fine then there exists the universal sheaf $\mathcal{E}$ on $M \times X$ and $\lambda_{\mathcal{E}}$ produces line bundles on M_P for each class $u \in K(X)$. In general M_P is not fine and we cannot expect existence of a homomorphism $\lambda\colon K(X) \to \operatorname{Pic} M_P$. However, such a homomorphism exists if we restrict to some subspace of $K(X)$.

Let c be a class in $K(X)$. Then we can define the moduli space $M(c)$ of Gieseker H-semistable sheaves of class c. This is a well defined open and closed subscheme of M_P for P defined by $P(m) = \chi(X, c\cdot[\mathcal{O}_X(mH)])$. We also set $M^s(c) = M^s_P \cap M(c)$. We will produce line bundles on $M(c)$ and $M^s(c)$.

Set

$$K_c = c^{\perp} = \{u \in K(X)\colon \chi(c \cdot u) = 0\}$$

and

$$K_{c,H} = c^{\perp} \cap \{1, h, h^2, \dots, h^{\dim X}\}^{\perp\perp}.$$

Theorem 3.4.

(1) There exists a group homomorphism $\lambda^s\colon K_c \to \operatorname{Pic} M^s(c)$ such that for any S-flat family $\mathcal{E}$ of Gieseker H-stable sheaves of class c on X if $\Phi_{\mathcal{E}}\colon S \to M^s(c)$ denotes the classifying morphism then $\Phi^*_{\mathcal{E}}(\lambda^s(u)) = \lambda_{\mathcal{E}}(u)$ for $u \in K_c$.

(2) There exists a group homomorphism $\lambda\colon K_{c,H} \to \operatorname{Pic} M(c)$ such that for any S-flat family $\mathcal{E}$ of Gieseker H-semistable sheaves of class c on X if $\Phi_{\mathcal{E}}\colon S \to M(c)$ denotes the classifying morphism then $\Phi^*_{\mathcal{E}}(\lambda(u)) = \lambda_{\mathcal{E}}(u)$ for $u \in K_{c,H}$.

(3) For any $u \in K_{c,H}$ the restriction of $\lambda(u)$ to $M^s(c)$ gives $\lambda^s(u)$.

Proof. We will prove only the first part of the theorem. The rest can be proved in a similar way.

Let $R^s(c) \subset R$ be an open subset of R used in the construction of the moduli space, corresponding to stable sheaves of class c. Let $u \in K(X)$ and set $\mathcal{L} = \lambda_{\tilde{F}}(u)$, where $\tilde{F}$ is the universal quotient sheaf on $R^s(c) \times X$. $\mathcal{L}$ has a natural $\mathrm{GL}(V)$ linearization coming from $\tilde{F}$.

We want to check that if $u \in K_c$ then $\mathcal{L}$ descends to a line bundle on $M^s(c)$. To check it, it is sufficient to check that for any point $[\rho\colon \mathcal{H} \to E] \in R^s(c)$ the stabilizer of $\mathrm{GL}(V)$ action at $[\rho]$ acts trivially on the fibre $\mathcal{L}([\rho])$.

The stabilizer of $\mathrm{GL}(V)$ at ρ is equal to the image of a natural homomorphism $\mathrm{Aut}E \to \mathrm{GL}(V)$ sending φ to $H^0(\rho(m))^{-1} \circ H^0(\varphi(m)) \circ H^0(\rho(m))$ (this is well defined since by our assumption $H^0(\rho(m))$ is an isomorhism).

So we need to understand the action of $\mathrm{Aut}E \simeq k^*$ on $\mathcal{L}([\rho])$. Since higher direct images commute with base change we have $\mathcal{L}([\rho]) \simeq \det H^\bullet(X, E \otimes u)$. Therefore $A \in \mathrm{Aut}E$ acts by $A^{\chi(c\cdot u)}$ and this action is trivial if $u \in K_c$, Q.E.D.

Now let us take two classes c and c^* in $K(X)$ such that $c \in K_{c^*,H}$ and $c^* \in K_{c,H}$ (in particular $\chi(c \cdot c^*) = 0$).

Then we can define $\mathcal{D}_{c,c^*} = \lambda_c(c^*)$ and $\mathcal{D}_{c^*,c} = \lambda_{c^*}(c)$. Let p_1 and p_2 denote the projections of $M(c) \times M(c^*)$ onto the first and the second factor, respectively. Let us take a line bundle $\mathcal{D} = p_1^*\mathcal{D}_{c,c^*} \otimes p_2^*\mathcal{D}_{c^*,c}$.

We will assume that X is either a curve or a surface. We will also make some mild assumptions on the classes c and c^*, e.g., we assume that the rank of c is positive and semistable sheaves of class c^* are pure of dimension $\dim X - 1$. Then the line bundle $\mathcal{D}$ has the following universal property:

Proposition 3.5. *(Le Potier) For all S-flat families $\mathcal{F}$ of Gieseker H-semistable sheaves of class c and $\mathcal{G}$ of Gieseker H-semistable sheaves of class c^* let $\Phi_{\mathcal{F}}: S \to M(c)$ and $\Phi_{\mathcal{G}}: S \to M(c^*)$ be the corresponding classifying morphisms. Then for*

$$\Phi = (\Phi_{\mathcal{F}}, \Phi_{\mathcal{G}}): S \to M(c) \times M(c^*)$$

we have $\Phi^*\mathcal{D} = \det(p_!(\mathcal{F} \otimes \mathcal{G}))$.

If we now assume that $H^2(F \otimes G) = 0$ for all Gieseker H-semistable sheaves F of class c and G of class c^* then we can construct a canonical section $\sigma_{c,c^*} \in H^0(M(c) \times M(c^*), \mathcal{D})$ such that its zero set is equal to $\{([F],[G]): H^1(F \otimes G) \neq 0\}$.

This section gives an element of $H^0(M(c), \mathcal{D}_{c,c^*}) \otimes H^0(M(c^*), \mathcal{D}_{c^*,c})$ so we have a linear map

$$D_{c,c^*}: H^0(M(c^*), \mathcal{D}_{c^*,c})^* \otimes H^0(M(c), \mathcal{D}_{c,c^*}),$$

which is called the strange duality map.

Strange Duality Conjecture 3.6. *Whenever defined and non-zero this map is an isomorphism.*

3.7. Geometric interpretation:

We have a rational map $\Psi: M(c) \dashrightarrow \mathbb{P}(H^0(M(c^*), \mathcal{D}_{c^*,c})$ which sends $[F] \in M(c)$ to the divisor $\{[G] \in M(c^*): H^1(F \otimes G) \neq 0\}$. One can check that $\Psi^*\mathcal{O}(1) = \mathcal{D}_{c,c^*}$. The Strange Duality Conjecture asks if Ψ is a morphism and if the image of Ψ is not contained in any hyperplane.

3.8. Example:

Let us consider the projective plane $\mathbb{P}_2$. Let c be the class of a rank 2 vector bundle with $c_1 = 0$ and $c_2 = n$ and let c^* be the class of $\mathcal{O}_L(-1)$ for some line L.

Then $\chi(c \cdot c^*) = 0$ and one can check that all our assumptions used in the definition of the strange duality map are satisfied.

Let us note that $M(c^*)$ is the dual $\mathbb{P}_2$ (it parametrizes lines in $\mathbb{P}_2$) and $M(c)$ parametrizes semistable sheaves of rank 2 on $\mathbb{P}_2$ with $c_1 = 0$ and $c_2 = n$.

By the Grauert–Müllich theorem the restriction of any such semistable sheaf to a general line is trivial. The lines on which the restriction is non-trivial (the so called *jumping lines*) form a curve of degree n in $\mathbb{P}_2^*$ (this fact is called Barth's theorem).

In this case Ψ is defined everywhere on $M(c)$ and it is known as the Barth morphism. However, even in this case the strange duality conjecture is known only for small n.

Theorem 3.8.1. *(G. Danila) The strange duality map is an isomorphism for $2 \leq n \leq 19$.*

3.9. Example:

Let C be a smooth projective curve. We have $K(C)=\mathbb{Z}\oplus\operatorname{Pic} C$, where the first factor corresponds to the rank and the second to the determinant. We also have a numerical K-group $K(C)_{\mathrm{num}}=\mathbb{Z}\oplus\mathbb{Z}$, where the first factor corresponds to the rank and the second to the degree of the determinant.

Let us take $c=[\mathcal{O}_C^r]\in K(C)$ and $c^*=(1,g-1)\in K(C)_{\mathrm{num}}$. One can easily check that $\chi(c\cdot c^*)=0$ and assumptions needed to define the strange duality map are satisfied.

In this case $\mathcal{D}_{c,c^*}$ is the generator of $\operatorname{Pic} M(c)$ and $M(c^*)=J^{g-1}$ is the Jacobian parametrizing line bundles of degree $g-1$.

Since we have the Poincare line bundle $\mathcal{P}$ on J^{g-1} (se 2.7) we get

$$\mathcal{D}_{c^*,c}=\det(\pi_!(\mathcal{P}\otimes c))=r\det(\pi_!\mathcal{P})=r\Theta,$$

where Θ is a theta divisor defined by $\Theta=\{L\in J^{g-1}\colon h^0(C,L)\neq 0\}$.

The strange duality map $\Psi\colon M\dashrightarrow\mathbb{P}(H^0(J^{g-1},r\Theta))$ sends $[E]$ to $\{L\in J^{g-1}\colon h^0(C,E\otimes L)\neq 0\}$ and the strange duality implies that $h^0(M(c),\mathcal{D}_{c,c^*})=h^0(J^{g-1},r\Theta)$.

This is a special case of the so called *Verlinde formula* computing the number of sections of tensor powers of $\mathcal{D}_{c,c^*}$ on $M(c)$.

Appendix

Let E be a rank r locally free sheaf on a smooth projective variety X over an algebraically closed field k. Let us formally write the total Chern class of E as

$$c(E)=\prod_{i=1}^r(1+a_i),$$

where a_i are degree 1 formal symbols. Then the *Chern character* of E is defined by

$$\operatorname{ch}(E)=\sum_{i=1}^r e^{a_i}.$$

The *Todd class* of E is defined by

$$\operatorname{td}(E)=\prod_{i=1}^r\frac{a_i}{1-e^{-a_i}}.$$

$\operatorname{td}(X)$ denotes the Todd class of the tangent bundle of X.

If X is a surface then $\operatorname{ch}(E)=1+c_1+(\frac12 c_1^2-c_2)$ and $\operatorname{td}(E)=1+\frac12 c_1+\frac1{12}(c_1^2+c_2)$.

Theorem A.1. *(Hirzebruch–Riemann–Roch) We have*

$$\chi(X,E)=\int_X\operatorname{ch}(E)\operatorname{td}(X),$$

where $\int_X c$ denotes the degree of 0-dimensional part of cycle c.

Lecture 4

This lecture starts with deformations of the Quot scheme. This will be used in the local study of moduli spaces.

Theorem 4.1. *Let X be a projective scheme over k, $\mathcal{H}$ a coherent sheaf and P a Hilbert polynomial. Take a k-rational point $[\sigma\colon \mathcal{H} \to F] \in \mathrm{Quot}(\mathcal{H}; P)$ and set $K = \ker \sigma$. Then*

$$\dim \mathrm{Hom}(K, F) - \dim \mathrm{Ext}^1(K, F) \leq \dim_{[\sigma]} \mathrm{Quot}(\mathcal{H}; P) \leq \dim \mathrm{Hom}(K, F).$$

Moreover, if $\mathrm{Ext}^1(K, F) = 0$ then $\mathrm{Quot}(\mathcal{H}; P)$ is smooth at $[\sigma]$.

Corollary 4.2. *Take $[E] \in M_P^s$. If $\mathrm{Ext}^1(E, E) = 0$ then M_P^s is smooth at $[E]$ and has dimension $\dim \mathrm{Ext}^1(E, E)$.*

Proof. Take a point $[\mathcal{H} \to E] \in \mathfrak{Quot}(\mathcal{H}; P)$. From the short exact sequence

$$0 \to K \longrightarrow \mathcal{H} \longrightarrow E \to 0$$

we get

$$\begin{aligned} 0 &\to \mathrm{Hom}(E, E) \to \mathrm{Hom}(\mathcal{H}, E) \to \mathrm{Hom}(K, E) \to \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(\mathcal{H}, E) \to \\ &\to \mathrm{Ext}^1(K, E) \to \mathrm{Ext}^2(E, E) \to \mathrm{Ext}^2(\mathcal{H}, E). \end{aligned}$$

Now $\mathrm{Ext}^i(\mathcal{H}, E) = H^i(E(mH))^{\dim V}$ vanishes for $i > 0$ and has dimension $(\dim V)^2$ for $i = 0$. Since E is stable $\dim \mathrm{Hom}(E, E) = 1$. Therefore $\mathrm{Ext}^1(K, E) = \mathrm{Ext}^2(E, E) = 0$ and $\dim \mathrm{Hom}(K, E) = \dim \mathrm{Ext}^1(E, E) + \dim \mathrm{PGL}(V)$.

From the proof of Theorem 3.4 we know that the stabilizer of the $\mathrm{GL}(V)$-action at every point of R^s is equal to $\mathrm{Aut}E = k^*$. Therefore $R^s \to M^s$ is a $\mathrm{PGL}(V)$-principal bundle.

Now the above dimension count and Theorem 4.1 imply the required assertions, Q.E.D.

In fact one can prove a more precise version of this theorem. Namely, for any stable sheaf F there exists a map

$$\mathrm{Ext}^1(F, F) \supset U \xrightarrow{\Phi} \mathrm{Ext}^2(F, F)$$

well defined on an open neighbourhood of 0 such that the germ of M^s at $[F]$ is the germ of $\Phi^{-1}(0)$ at 0. The map Φ is called *a Kuranishi map* but it is not unique in algebraic category.

Theorem 4.3. *Let X be a smooth projective variety. Then we have a map*

$$\det\colon M \to \mathrm{Pic}\, X$$

sending $[F]$ to $[\det F]$. Then for any stable sheaf F obstructions for $[F]$ to be a smooth point of M map to the obstructions for $[\det F]$ to be a smooth point of $\mathrm{Pic}\, X$ via the trace map

$$\mathrm{Tr}\colon \mathrm{Ext}^2(F, F) \to \mathrm{Ext}^2(\det F, \det F) \simeq H^2(\mathcal{O}_X).$$

Let us denote the kernel of the trace map

$$\mathrm{Tr}\colon \mathrm{Ext}^i(F,F)\to \mathrm{Ext}^i(\det F,\det F)\simeq H^i(\mathcal{O}_X)$$

by $\mathrm{Ext}^i(F,F)_0$.

Since in characteristic 0 any group scheme is smooth, $\mathrm{Pic}\, X$ is smooth. Therefore if F is torsion free stable sheaf then M^s is smooth at $[F]$ if $\mathrm{Ext}^2(F,F)_0=0$.

More precisely, if we set $M(\mathcal{L})=\det^{-1}([\mathcal{L}])$ then the germ of $M^s(\mathcal{L})$ at any point $[F]$ is the germ at 0 of $\Psi^{-1}(0)$ for some map

$$\mathrm{Ext}^1(F,F)_0\supset U\xrightarrow{\Phi}\mathrm{Ext}^2(F,F)_0$$

defined on an open neighbourhood U of 0.

Examples of unobstructed moduli spaces.

Theorem 4.4. *(Maruyama) Let X be a smooth projective surface and let H be an ample divisor such that $HK_X<0$. Then for any slope H-semistable torsion free sheaf E we have $\mathrm{Ext}^2(E,E)=0$. In particular, any moduli space M^s_P is smooth.*

Proof. We will use the following useful version of the Serre duality theorem:

Theorem 4.5. *Let X be a smooth n-dimensional projective variety. Then for any two coherent sheaves F and G we have*

$$\mathrm{Ext}^p(F,G)\simeq(\mathrm{Ext}^{n-p}(G,F\otimes\omega_X))^*.$$

The above theorem implies that $\mathrm{Ext}^2(E,E)\simeq(\mathrm{Hom}(E,E\otimes\omega_X))^*$. Let us assume that $\mathrm{Ext}^2(E,E)\neq 0$ and take a non-zero $\varphi\in\mathrm{Hom}(E,E\otimes\omega_X)$. The determinant $\det\varphi\in\mathrm{Hom}(\det E,\det E\otimes\omega_X^{\otimes r})$ induces a section of $\omega_X^{\otimes r}$. But $H^0(\omega_X^{\otimes r})=0$ since $K_XH<0$. In particular, $\det\varphi=0$ and $\ker\varphi\neq 0$. Since $\ker\varphi\subset E$ and $\mathrm{im}\,\varphi\subset E\otimes\omega_X$ we have

$$\mu(\ker\varphi)\le\mu(E)$$

and

$$\mu(\mathrm{im}\,\varphi)\le\mu(E\otimes\omega_X)=\mu(E)+K_XH.$$

Set $r=\mathrm{rk}\,E$ and $s=\mathrm{rk}\ker\varphi$. Then by the above inequalities

$$r\mu(E)=s\mu(\ker\varphi)+(r-s)\mu(\mathrm{im}\,\varphi)\le r\mu(E)+(r-s)K_XH<r\mu(E),$$

a contradiction, Q.E.D.

4.6. Example:

Let X be a surface with $\omega_X\simeq\mathcal{O}_X$ (e.g., a K3 surface). Then for any Gieseker H-stable sheaf E we have

$$\dim\mathrm{Ext}^2(E,E)_0=\dim\mathrm{Hom}(E,E)_0=0,$$

since the trace map $\mathrm{Tr}\colon\mathrm{Hom}(E,E)\to H^0(\mathcal{O}_X)$ is an isomorphism.

In particular, the moduli space $M^s(r,\mathcal{L},c_2)$ Gieseker H-stable sheaves of rank r with fixed determinant $\mathcal{L}$ and $c_2E=c_2$ is smooth of dimension

$$\dim\mathrm{Ext}^1(E,E)_0=-\chi(E,E)+\chi(\mathcal{O}_X)=\Delta(E)-(r^2-1)\chi(\mathcal{O}_X).$$

Le Potier–Simpson bound on the number of sections of a sheaf.

Lemma 4.7. *Let C be a smooth curve and let E be a semistable vector bundle on C. Then*
$$h^0(E) \le \operatorname{rk} E\,[\mu(E)+1]_+,$$
where $[x]_+ = \max(x,0)$.

Proof. Let us take an effective divisor D of degree $d > \mu(E)$. By assumption $H^0(E(-D)) = 0$. From the short exact sequence
$$0 \to \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \to 0$$
tensored by E, we get
$$h^0(E) \le h^0(E(-D)) + h^0(E|_D) = \operatorname{rk} E \cdot d.$$
If we take a divisor of degree $[\mu(E)+1]_+$ then we get the required inequality, Q.E.D.

Corollary 4.8. *If E is a locally free sheaf on a smooth curve C then*
$$h^0(E) \le \operatorname{rk} E\,[\mu_{\max}(E)+1]_+$$

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$ be the Harder–Narasimhan filtration of E. Then
$$h^0(E) \le \sum h^0(E_i/E_{i-1}) \le \sum \operatorname{rk}(E_i/E_{i-1})[\mu(E_i/E_{i-1})+1]_+ \le r\,[\mu_{\max}(E)]_+,$$
Q.E.D.

From now on let X be a smooth projective surface defined over an algebraically closed field of characteristic 0 and let H be a very ample divisor on X.

Theorem 4.9. *(Grauert–Müllich) Let F be a torsion free sheaf on X. Then for a general curve $C \in |H|$*
$$\mu_{\max}(F|_C) \le \mu_{\max}(F) + \frac{\operatorname{rk} F - 1}{2} H^2$$
and
$$\mu_{\min}(F|_C) \ge \mu_{\min}(F) - \frac{\operatorname{rk} F - 1}{2} H^2.$$

Theorem 4.10. *(Le Potier–Simpson) For any torsion free sheaf F we have*
$$h^0(X,F) \le \frac{\operatorname{rk} F}{2H^2}\left(\left[\mu_{\max}(F) + \frac{\operatorname{rk} F + 3}{2} H^2\right]_+\right)^2.$$

Proof. Let us take a general curve $C \in |H|$. Then $F|_C$ is locally free and we have short exact sequences
$$0 \to F(-((i+1)C)) \longrightarrow F(-iC) \longrightarrow F(-iC)|_C \to 0.$$

Hence

$$h^0(F(-iC)) \le h^0(F(-(i+1)C)) + h^0(F(-iC)|_C).$$

Since $h^0(F(-iC)) = 0$ for large i (e.g., by Serre's theorem) we get

$$h^0(F) \le \sum_{i=0}^{\infty} h^0(F(-iC)|_C).$$

Now we apply Corollary 4.8 and the Grauert–Müllich theorem to get

$$h^0(F) \le \operatorname{rk} F \sum_{i=0}^{\infty} [\mu_{\max}(F) + \frac{\operatorname{rk} F - 1}{2} H^2 - iCH].$$

Summing up we get the required inequality, Q.E.D.

Proposition 4.11. *Let E and F be slope H-semistable sheaves. Then*

$$h^0(\operatorname{Hom}(E, F)) \le \frac{\operatorname{rk} E \operatorname{rk} F}{2H^2} \left(\left[\mu(F) - \mu(E) + \frac{\operatorname{rk} E + \operatorname{rk} F + 1}{2} H^2 \right]_+ \right)^2 .$$

Proof. The proof is almost the same as above except that in the last step we use

$$\mu_{\max}(\operatorname{Hom}(E, F)|_C) = \mu_{\max}(F|_C) - \mu_{\min}(E|_C) \le \mu(F) - \mu(E) + \frac{\operatorname{rk} E + \operatorname{rk} F - 2}{2} H^2.$$

The last inequality follows from the Grauert–Müllich theorem, Q.E.D.

Definition 4.12. *Let $e \ge 0$ be a real number. A torsion free sheaf E is called* e-stable *if for all $E' \subset E$ of rank $0 < \operatorname{rk} E' < \operatorname{rk} E$ we have*

$$\mu(E') < \mu(E) - \frac{e\sqrt{H^2}}{\operatorname{rk} E'}.$$

Note that any e-stable sheaf ($e \ge 0$) is slope H-stable.

If E is a torsion free slope H-semistable sheaf which is not e-stable for some $e \ge 0$ then we have a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

such that all quotients $F_i = E_i/E_{i-1}$ are slope H-semistable,

$$\mu(E_1) > \mu(E) - \frac{e\sqrt{H^2}}{\operatorname{rk} E_1}$$

and

$$\mu(F_2) \ge \cdots \ge \mu(F_k).$$

Such a filtration can be constructed first by taking e-destabilizing subsheaf of E with torsion free quotient E/E_1 and then lifting the Harder–Narasimhan filtration of this quotient to E.

Our aim is to bound the dimesnion of the locus $R(e)$ of e-unstable sheaves in $R_{\mathcal{L}}$ and the corresponding locus in $M(r, \mathcal{L}, c_2)$.

In the following we always fix the determinant $\mathcal{L}$.

Note that for a point $[\rho: \mathcal{H} \to E] \in R(e)$ we have a filtration

$$0 \subset \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_k = \mathcal{H}$$

such that $\mathcal{H}_0 = \ker \rho$ and $\mathcal{H}_i/\mathcal{H}_{i-1} = F_i$ for $i \geq 1$.

Set $P_i = P(F_i)$. Then $[\rho: \mathcal{H} \to E]$ lies in the image of the map

$$Y = \operatorname{Flag}(\mathcal{H}; P_1, \ldots, P_k) \to \operatorname{Quot}(\mathcal{H}; P)$$

obtained by forgetting all of the flag except for $\mathcal{H}_0 \to \mathcal{H}$. There exists only a finite number of such Flag schemes which are non-empty. So in order to bound the dimension of $R(e)$ it is sufficient to bound the dimension of Y.

To do this one can use a version of Theorem 4.1 for Flag schemes. The only thing we need to know about this is that the dimension of the Flag scheme $\operatorname{Flag}(\mathcal{H}; P_1, \ldots, P_k)$ at $[\mathcal{H}_\bullet]$ is less or equal to the dimension of certain Ext-group:

$$\dim \operatorname{Ext}^0_+(\mathcal{H}_\bullet, \mathcal{H}_\bullet) \leq h^0(\operatorname{Hom}(\mathcal{H}, \mathcal{H})) - 1 + \sum_{i \leq j} \dim \operatorname{Ext}^1(F_j, F_i).$$

Now the dimension of $\operatorname{Ext}^1(F_j, F_i)$ can be computed using the Euler characteristic $\chi(F_j, F_i)$ (which can be computed by the Riemann–Roch formula) and the dimensions of $\operatorname{Hom}(F_j, F_i)$ and $\operatorname{Hom}(F_i, F_j \otimes \omega_X) \simeq (\operatorname{Ext}^2(F_j, F_i))^*$. These dimensions can be bounded using Le Potier–Simpson's theorem (see Proposition 4.11). After some computation O'Grady used this idea to prove the following theorem.

Theorem 4.13. *(O'Grady)*

$$\dim R(e) \leq \left(1 - \frac{1}{2r}\right)\Delta + (3r-1)e^2 + \frac{r[K_X H]_+}{2\sqrt{H^2}} e + b(r, X, H) + (h^0(\mathcal{H}))^2 - 1,$$

where $b(r, X, H)$ is some explicit function depending only on r, X and H.

Lecture 5

Let $M(e)$ denote the locus of e-unstable sheaves in the moduli space $M(r, \mathcal{L}, c_2)$. Let us set

$$d(e) = \left(1 - \frac{1}{2r}\right)\Delta + (3r-1)e^2 + \frac{r[K_X H]_+}{2\sqrt{H^2}} e + b(r, X, H).$$

Theorem 5.1.

$$\dim M(e) \leq d(e) + r^2 - 1.$$

Proof. In the previous lecture we showed how to bound the dimension of $R(e)$. By the construction $\pi: R(e) \to M(e) = R(e)/\operatorname{GL}(V)$, so that

$$\dim M(e) \leq \dim R(e) - \dim \pi^{-1}([E])$$

for some $[E] \in M(e)$. But the stabilizer of $\mathrm{GL}(V)$ action at $[\mathcal{H} \to E]$ is equal to $\mathrm{Aut} E$ (cf. the proof of Theorem 3.4), so

$$\dim \pi^{-1}([E]) = \dim \mathrm{GL}(V) - \dim \mathrm{Aut}(E) \geq (h^0(\mathcal{H}))^2 - r^2$$

and the asserted inequality follows from Theorem 4.13, Q.E.D.

Let E be a Gieseker H-stable sheaf of rank r with determinant $\mathcal{L}$ and $c_2 E = c_2$. Set $\Delta = \Delta(E)$. By the remark after Theorem 4.3 we have

$$\dim \mathrm{Ext}^1(E,E)_0 - \dim \mathrm{Ext}^2(E,E)_0 \leq \dim_{[E]} M(r,\mathcal{L},c_2) \leq \dim \mathrm{Ext}^1(E,E)_0.$$

Now we can compute

$$\dim \mathrm{Ext}^1(E,E)_0 - \dim \mathrm{Ext}^2(E,E)_0 = \chi(\mathcal{O}_X) - \chi(E,E) = \Delta(E) - (r^2-1)\chi(\mathcal{O}_X).$$

Using Le Potier–Simpson's theorem (Proposition 4.11) and Serre's duality (Theorem 4.5) we have

$$\begin{aligned}\dim \mathrm{Ext}^2(E,E)_0 &= \dim \mathrm{Ext}^2(E,E) - h^2(\mathcal{O}_X) = \dim \mathrm{Hom}(E, E\otimes \omega_X) - h^2(\mathcal{O}_X)\\ &\leq \frac{r^2}{2H^2}\left(\left[K_X H + \left(r+\frac{1}{2}\right)H^2\right]_+\right)^2.\end{aligned}$$

Therefore we proved

Proposition 5.2. *For any $[E] \in M^s = M^s(r,\mathcal{L},c_2)$ we have*

$$\Delta - (r^2-1)\chi(\mathcal{O}_X) \leq \dim_{[E]} M^s \leq \Delta - (r^2-1)\chi(\mathcal{O}_X) + \beta_\infty,$$

where β_∞ is some explicit function depending only on r, X and H.

Corollary 5.3. *Let $e \geq 0$ be a fixed real number. If $\Delta \gg 0$ then $d(e)+r^2-1 < \Delta - (r^2-1)\chi(\mathcal{O}_X)$. In particular, slope H-unstable sheaves form a closed subset of large codimension in every component of $M(r,\mathcal{L},c_2)$ and the bound on this codimesion can be explicitly computed.*

Let $\mathcal{F}$ be a flat family of rank r torsion free sheaves on X parametrized by S. Then we define the *boundary* ∂S of S by

$$\partial S = \{s \in S \colon \mathcal{F}_s = \mathcal{F}|_{\{s\}\times X} \text{ is not locally free}\}.$$

One can prove the following lemma.

Lemma 5.4. *If $\partial S \neq \emptyset$ then* $\mathrm{codim}(\partial S, S) \leq r-1$.

For a subset $Z \subset M$ we set $\partial Z = \pi(\partial \pi^{-1}(Z))$, where $\pi\colon R \to M = R/\mathrm{GL}(V)$ comes from the construction of the moduli space and as the family parametrized by $\pi^{-1}(Z)$ we consider the universal quotient.

We also set $Z^\mu = \{[F] \in Z \colon F \text{ is slope } H\text{-stable}\}$.

Now we can state the main theorem of this lecture.

Theorem 5.5. *There exist explicit functions $1-\frac{1}{2r}<\lambda_2(r)<1$, $\lambda_1(r,X,H)$ and $\lambda_0(r,X,H)$ such that if $Z\subset M$ is an irreducible closed subset with $\partial Z^{\mu}=\emptyset$ then*

$$\dim Z \le \lambda_2\Delta+\lambda_1\sqrt{\Delta}+\lambda_0.$$

Proof. The proof will be in several steps. We will show that the bound holds under stronger assumption $\partial Z=\emptyset$. The general case can be reduced to this one by using Theorem 4.13 for $e=0$ and Lemma 5.4.

Proposition 5.6. *Let $C\in|nH|$ be a smooth curve. Let M^s_C be the moduli space of stable locally free sheaves on C of rank r and with determinant $\mathcal{L}|_C$. Let $Z\subset M$ be a closed irreducible subset with $\partial Z=\emptyset$. If $\dim Z>\dim M^s_C=(r^2-1)(g(C)-1)$ then there exists $[F]\in Z$ such that $F|_C$ is not stable.*

Idea of the proof.

There exists a determinant line bundle $\mathcal{L}$ on M such that the multiples of $\mathcal{L}$ define rational maps $M \dashrightarrow M_C$. If $\partial Z=\emptyset$ and $F|_C$ is stable for all $[F]\in Z$ then $Z\to M_C$ is a morphism. Take any two distinct points $[F_1],[F_2]\in Z$. Then for large n and $i=0,1$ we have $\mathrm{Ext}^i(F_1,F_2(-nH))=\mathrm{Ext}^i(F_2,F_1(-nH))=0$. Therefore $\mathrm{Hom}(F_1,F_2)=\mathrm{Hom}(F_1|_C,F_2|_C)$ and $F_1|_C$ and $F_2|_C$ give different points in M_C. This means that a certain multiple of $\mathcal{L}$ separates any two points of Z. In particular, $\mathcal{L}|_Z$ is ample and the map $Z\to M_C$ is finite. This contradicts our assumption on $\dim Z$, Q.E.D.

Proposition 5.7. *Let C be a smooth projective curve of genus g. Consider the map*

$$\Phi\colon \mathrm{Flag}(\mathcal{H};P_1,\dots,P_k)\to \mathrm{Quot}(\mathcal{H};P)$$

obtained by forgetting all of the flag except for $\mathcal{H}_0\to\mathcal{H}$. Let r_i and μ_i denote ranks and degrees of sheaves corresponding to P_i. Then the codimension of the image of Φ in $\mathrm{Quot}(\mathcal{H};P)$ is equal to

$$\sum_{i<j} r_ir_j(\mu_i-\mu_j+g-1).$$

The above proposition can be proved using deformation theory for Flag schemes as in the proof of Theorem 4.13.

Proposition 5.8. *Let $\mathcal{F}$ be a flat family of locally free sheaves of rank r on a smooth projective curve C parametrized by S. Set $S^u=\{s\in S\colon \mathcal{F}_s$ is not stable$\}$. If $S^u=\emptyset$ then $\mathrm{codim}(S^u,S)\le\frac{r^2}{4}g(C)$.*

Proof. We can assume that S is a Quot–scheme and $\mathcal{F}$ is the universal bundle. There is no harm in doing so since there exists m such that all $\mathcal{F}_s(m)$ are globally generated and then we use the universal property of Quot–schemes and the corresponding inequality for Quot–schemes to obtain the required inequality for S.

Now S^u can be constructed as an image of some Flag–scheme as in the proof of Theorem 4.13. So the required bound follows from Proposition 5.7, Q.E.D.

Proposition 5.9. *Let $C\in|nH|$ be a smooth curve and $e>0$ a real number. Let $Z\subset M$ be a closed irreducible subset with $\partial Z=\emptyset$. If $\dim Z>\dim M^s_C$ and*

$\dim Z \geq \frac{r^2}{4}g(C) + d(e) + r^2$ then there exists $[F] \in Z$ such that F is e-stable and $F|_C$ is not stable.

Proof. Consider the projection $\pi: R \to M$ used in the construction of M. Let T be an irreducible component of $\pi^{-1}(Z)$ dominating Z. The fibre of π over a point $[F] \in Z$ has dimension $\dim \mathrm{GL}(V) - \dim \mathrm{Aut} F \geq dim\, \mathrm{GL}(V) - r^2$. Therefore

$$\dim T \geq \dim Z + \dim \mathrm{GL}(V) - r^2 \geq d(e) + \dim \mathrm{GL}(V) + \frac{r^2}{4}g(C).$$

By Proposition 5.6 and our assumptions there exists $[F] \in Z$ such that $F|_C$ is not stable. If we consider the family of sheaves on C obtained by restriction of the universal quotient on $R \times X$ to $T \times C$ then Proposition 5.8 and Theorem 4.13 imply that $\dim T^u \geq \dim T - \frac{r^2}{4}g(C) \geq d(e) + \dim \mathrm{GL}(V) > \dim R(e)$. This implies the required assertion, Q.E.D.

Proposition 5.10. *Take $e = n(r-1)\sqrt{H^2}$ and assume that there exists $[F] \in Z$ such that F is e-stable and $F|_C$ is not stable. If*

$$\dim Z > \Delta - (r^2 - 1)\chi(\mathcal{O}_X) + \beta_\infty + \frac{r^2}{4} - \frac{r-1}{2}C(C-K)$$

then $\partial Z \neq \emptyset$.

Proof. Let us assume that $\partial Z = \emptyset$. Since $F|_C$ is not stable there exits a subsheaf $F' \subset F|_C$ such that $F'' = F|_C/F'$ is locally free and $\mu(F') \geq \mu(F|_C) \geq \mu(F'')$.

Consider an elementary transformation E of F along F'' (cf. the proof of Theorem 1.7). We have two short exact sequences

$$0 \to E \longrightarrow F \longrightarrow F'' \to 0$$

and

$$0 \to F \longrightarrow E(C) \longrightarrow F'(C) \to 0.$$

Consider $\Sigma = \mathrm{Quot}_C(E(C)|_C; P(F'(C)))$. For any point $[\sigma: E(C)|_C \to G] \in \Sigma$ we have a short exact sequence

$$0 \to \mathcal{F}_\sigma \longrightarrow E(C) \longrightarrow G \to 0.$$

The sheaves $\{\mathcal{F}_\sigma\}$ fit into a family parametrized by Σ and such that

a) $\mathcal{F}_{[E|_C \to F']} = F$,

b) G has torsion if and only if $\mathcal{F}_\sigma$ is not locally free.

Moreover, all $\mathcal{F}_\sigma$ are slope H-stable. This follows from the fact that if $\mathcal{A} \subset \mathcal{F}_\sigma$ then $\mathcal{A} \subset E(C) \subset F(C)$, so by e-stability of F we have

$$\mu(A) < \mu(F(C)) - \frac{e\sqrt{H^2}}{\mathrm{rk}\,\mathcal{A}} = \mu(F) + CH - \frac{r - \mathrm{rk}\,\mathcal{A} - 1}{\mathrm{rk}\,\mathcal{A}}CH \leq \mu(F) = \mu(\mathcal{F}_\sigma).$$

This shows that we have a classifying morphism $\varphi: \Sigma \to M^\mu$ such that $\varphi([E(C)|_C \to F'(C)]) = [F] \in Z$. Let U be an irreducible component of $\varphi^{-1}(Z)$, which dominates $\varphi(\varphi^{-1}(Z))$. Then

$$(*) \qquad \dim U \geq \dim Z + \dim \Sigma - \dim M.$$

By Theorem 4.1 we have

$$\dim \Sigma \geq \chi(F'', F'(C)) = \operatorname{rk} F' \operatorname{rk} F''(\mu(F') - \mu(F'') + C^2 + 1 - g(C)) \geq \frac{r-1}{2} C(C-K).$$

Substituting this inequality into $(*)$, using our assumption on $\dim Z$ and the inequality $\dim M \leq \Delta - (r^2-1)\chi(\mathcal{O}_X) + \beta_\infty$ from Proposition 5.2, we get $\dim U > \frac{r^2}{4}$.

There exists a projective scheme $\operatorname{Grass}(E(C)|_C, r')$ over C which parametrizes locally free quotients of $E(C)|_C$ of rank r'. The projection $\operatorname{Grass}(E(C)|_C, r') \to C$ is locally trivial with fibres being usual Grassmanians of dimension $r'(r-r') \leq \frac{r^2}{4}$. Since U parametrizes a family of rank $r' = \operatorname{rk} F'$ locally free sheaves on C, we have a map

$$\varphi \colon U \times C \to \operatorname{Grass}(E(C)|_C, r')$$

induced by restriction of the universal quotient on $\Sigma \times C$ to $U \times C$. Since $\dim U > \frac{r^2}{4}$ there exists a curve $D \subset U$ such that for a point $c \in C$ the set $D \times \{c\}$ is contracted by φ to a point. By the rigidity theorem this implies that $D \times C$ is also contracted to a point. But this means that the points of D parametrize the same quotient, which is impossible, Q.E.D.

The above proposition and Proposition 5.9 imply that if $\partial Z = \emptyset$, $\dim Z > (r^2-1)(g(C)-1)$ and $\dim Z \geq \frac{r^2}{4} g(C) + d(e) + r^2$ for $e = n(r-1)\sqrt{H^2}$ then

$$\dim Z \leq \Delta - (r^2-1)\chi(\mathcal{O}_X) + \beta_\infty + \frac{r^2}{4} - \frac{r-1}{2} C(C-K).$$

For large Δ we can suitably choose n so that we get Theorem 5.5. The result is also true for small Δ because we can choose λ_0 greater than possible dimensions of the moduli spaces (cf. Proposition 5.2), Q.E.D.

Lecture 6

As before we consider a smooth projective surface X defined over an algebraically closed field of characteristic 0.

This lecture is devoted to proofs of good properties of the moduli space $M = M(\Delta)$ of rank r Gieseker H-semistable sheaves with fixed determinant and large discriminant Δ.

Theorem 6.1. *There exists an explicit function $a_2 = a_2(r, X, H)$ such that if $\Delta \geq a_2$ then every irreducible component of $M(\Delta)$ is generically smooth of dimension $\Delta - (r^2-1)\chi(\mathcal{O}_X)$.*

This theorem is due to Donaldson (also Friedman and Zuo) in the rank 2 case and due to O'Grady and, independently, Gieseker and Li in the higher dimensional case. In the proof we follow O'Grady's approach.

Let us set

$$W = \{[F] \in M(\Delta) \colon \operatorname{Ext}^2(F, F) \neq 0\}.$$

For a subset $Z \subset M$ we set

$$\beta(Z) = \min\{\dim \operatorname{Hom}(E, E \otimes \omega_X)_0 \colon [E] \in Z\}.$$

We also use the notation from Theorem 5.4.

Theorem 6.2. *There exist an explicit function $\lambda_0'(r,X,H)$ such that*

$$\dim W \le \lambda_2\Delta + \lambda_1\sqrt{\Delta} + \lambda_0'.$$

Proof of Theorem 6.1.

The above theorem and Proposition 5.2 imply that for large Δ and for every $[F]\in M^s$ the dimension of M^s at $[F]$ is much larger then the dimension of W. On the other hand, $\operatorname{codim}(M-M^s, M)$ is also large for large Δ (see Corollary 5.3), so almost all points $[F]$ in M correspond to Gieseker H-stable sheaves with vanishing $\operatorname{Ext}^2(F,F)_0$. Such sheaves define smooth points of M, which proves Theorem 6.1, Q.E.D.

Proof of Theorem 6.2.

Let Z be an irreducible component of W of maximal dimension. Assume that

$$\dim Z \le \lambda_2\Delta + \lambda_1\sqrt{\Delta} + \lambda_0.$$

Then $\partial Z^{\mu} \ne \emptyset$ by Theorem 5.5.

Any subset S in M has a natural stratification by locally closed subsets $S_i = \{[F]\in S\colon h^0(F^{**}/F)=i\}$.

Let Y be an irreducible component of the open stratum of ∂Z^{μ} and let l be the constant value of $h^0(F^{**}/F)$ for $[F]\in Y$. Then we have a map $\pi\colon Y\to M(\Delta_1)$, where $\Delta_1 = \Delta - 2rl$, which sends $[F]$ to F^{**}. Let Z_1 be the closure of the image of this map.

If $\partial Z_1 \ne \emptyset$ then we continue with this procedure. At some point $\partial Z_n = \emptyset$ because $\Delta > \Delta_1 > \cdots > \Delta_n \ge 0$ by Bogomolov's inequality (Theorem 1.5).

Consider a general point $[E]\in Z_1$. By the construction E is locally free, so we can consider a morphism $\psi\colon \operatorname{Quot}(E;l)\to M(\Delta)$ which to a point $[\sigma\colon E\to T]$ associates $\ker\sigma$. Since E is slope H-stable, $\ker\sigma$ is also slope H-stable so ψ is well defined. By the construction, $\pi^{-1}([E])$ is contained in the image ψ. The scheme $\operatorname{Quot}(E;l)$ is irreducible of dimension $l(r+1)$, so $\dim\pi^{-1}([E]) \le l(r+1)$.

By Lemma 5.4

$$\dim Z_1 \ge \dim Y - l(r+1) \ge \dim Z - (r-1) - l(r+1) \ge \dim Z - (2r-1)l - 1.$$

Let us set $\epsilon_1 = \dim Z_1 - (\dim Z - (2r-1)l - 1)$.

We make a similar computation for Z_i and eventually we get

$$\dim Z_n = \dim Z - (2r-1)\sum l_i - n + \sum \epsilon_i.$$

Lemma 6.3. *(1) $\beta(Z)\le\beta(Z_1)\le\cdots\le\beta(Z_n)$*

(2) If $\epsilon_i = 0$ then $\beta(Z_{i-1}) < \beta(Z_i)$.

Proof. Let $[E]\in Z_i$ be such a sheaf that $\dim\operatorname{Hom}(E, E\otimes\omega_X)_0$ attains minimum on Z_i. By the construction there exists $[F]\in\partial Z_{i-1}$ such that $F\subset E$. Then $\operatorname{Hom}(F,F\otimes\omega_X)_0 \subset \operatorname{Hom}(E,E\otimes\omega_X)_0$, which proves the first part.

To simplify notation we assume that $i=1$. If $\epsilon_1 = 0$ then $\dim Z_1 = \dim Y - l(r+1)$. Hence $\pi^{-1}([E])$ has dimension $l(r+1)$ and it must be equal to the image of $\psi\colon\operatorname{Quot}(E;l)\to M$.

So it is sufficient to prove that for general $[\sigma: E \to T] \in \mathrm{Quot}(E; l)$ we have $\mathrm{Hom}(F, F \otimes \omega_X)_0 \neq \mathrm{Hom}(E, E \otimes \omega_X)_0$ for the kernel F of σ.

For any non-zero $f \in \mathrm{Hom}(E, E \otimes \omega_X)_0$ there exist l points x_i such that f is not a multiple of identity on $E(x_i)$. Then the kernel of $[\sigma: E \to \bigoplus \mathcal{O}_{x_i}]$ is not preserved by f, so $f \ni \mathrm{Hom}(F, F \otimes \omega_X)_0$, Q.E.D.

Now we can return to the proof of Theorem 6.2.

Let us recall that $\beta(Z_n) \leq \beta_\infty$, where β_∞ was defined using Le Potier–Simpson's bound (see Proposition 5.2). Therefore Lemma 6.3 implies that $\sum \epsilon_i \geq n - \beta_\infty$. Since $\Delta = \Delta_n + 2r \sum l_i$ we have

$$\dim Z_n \geq \dim Z - \left(1 - \frac{1}{2r}\right)(\Delta - \Delta_n) - \beta_\infty.$$

Hence

$$\begin{aligned} \dim Z_n - \left(1 - \frac{1}{2r}\right)\Delta_n &\geq \dim Z - \left(1 - \frac{1}{2r}\right)\Delta - \beta_\infty \\ &\geq \left(\lambda_2(r) - \left(1 - \frac{1}{2r}\right)\right)\Delta + \lambda_1\sqrt{\Delta} + \lambda_0' - \beta_\infty. \end{aligned}$$

On the other hand, $\partial Z_n = \emptyset$ so

$$\dim Z_n - \left(1 - \frac{1}{2r}\right)\Delta_n \leq \left(\lambda_2(r) - \left(1 - \frac{1}{2r}\right)\right)\Delta_n + \lambda_1\sqrt{\Delta_n} + \lambda_0$$

by Theorem 5.5. If we choose $\lambda_0' > \lambda_0 + \beta_\infty$ then for large Δ we get a contradiction, Q.E.D.

Theorem 6.4. *(Gieseker–Li, O'Grady) There exists some number n_0 such that for all $\Delta \geq n_0$ the moduli space $M(\Delta)$ is normal, irreducible variety of dimension $\Delta - (r^2 - 1)\chi(\mathcal{O}_X)$. Furthermore, increasing n_0, we can also assume that $M(\Delta)$ is $\mathbb{Q}$-factorial.*

Proof. We will sketch the proof of irreducibility (which is the most difficult part). The first step is to prove that if $\Delta \gg 0$ then each irreducible component of $M(\Delta)$ contains a point $[F]$ such that

(1) F is not locally free,
(2) $h^0(F^{**}/F) = 1$,
(3) $\mathrm{Ext}^2(F^{**}, F^{**})_0 = 0$.

This can be achieved in much the same way as in the proof of Theorem 6.2.

Let Σ_Δ denote the set of irreducible components of $M(\Delta)$. For each irreducible component N of $M(\Delta)$ let us choose a point $[F]$ in this component such that F is slope H-stable, locally free and $\mathrm{Ext}^2(F, F)_0 = 0$. Take any point $x \in X$. Then we define a map $\rho: \Sigma_\Delta \to \Sigma_{\Delta+2r}$ by sending N to the component containing $\ker(F \to \mathcal{O}_x)$. One can check that this map is well defined. Moreover, by the first step of the proof this map is surjective.

We want to show that for any two locally free slope H-stable sheaves E_1 and E_2 with $\mathrm{Ext}^2(E_i, E_i)_0 = 0$ there exists an integer l such that the irreducible components containing $[E_i]$ are mapped by ρ^l to the same component of $M(\Delta+2rl)$. Then for $\Delta' \gg \Delta$ the set $\Sigma_{\Delta'}$ consists from one point, which is equivalent to irreducibility of $M(\Delta')$.

We can choose m such that $E_i(m)$ are globally generated. Then by Serre's lemma (see Exercise 23) E_i fit into short exact sequences

$$0 \to \mathcal{O}_X(-m)^{r-1} \longrightarrow E_i \longrightarrow I_{Z_i} \cdot \mathcal{M} \to 0,$$

where $\mathcal{M} = \det E_i \otimes \mathcal{O}((r-1)m)$ and I_{Z_i} are ideals of some zero-dimensional schemes Z_i. Set $I_Z = I_{Z_1} \cap I_{Z_2}$. Then the pull back of $E_i \in \operatorname{Ext}^1(I_{Z_i} \cdot \mathcal{M}, \mathcal{O}_X(-m)^{r-1})$ to $\operatorname{Ext}^1(I_Z \cdot \mathcal{M}, \mathcal{O}_X(-m)^{r-1})$ defines a slope H-stable sheaf F_i, corresponding to a point in $M(\Delta + 2r(\deg Z_1 - \deg Z))$. Moreover, F_1 and F_2 belong to the universal family of extensions parametrized by $\operatorname{Ext}^1(I_Z \cdot \mathcal{M}, \mathcal{O}_X(-m)^{r-1})$ (see Example 2.11). Therefore $\rho^{\deg Z_1 - \deg Z}$ maps the components containing $[E_i]$ to the component containg both $[F_1]$ and $[F_2]$, Q.E.D.

Exercises:

(1) Let E and F be non-isomorphic Gieseker H-stable sheaves. Assume that $p(E) \geq p(F)$. Show that $\operatorname{Hom}(E, F) = 0$.

(2) Prove that any Gieseker H-stable sheaf E is simple (i.e., $\operatorname{Hom}(E, E) \simeq k$).

(3) Find an automorphism group of a direct sum of Gieseker H-stable sheaves with the same normalized Hilbert polynomial (such a sum is called a *polystable sheaf*).

(4) Prove that any vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles.

(5) Prove that the tangent bundle $T_{\mathbb{P}^2}$ of $\mathbb{P}^2$ is stable in any characteristic.

(6) Show an example of a finite morphism $f: C \to D$ of smooth projective curves and a stable vector bundle E on C such that f_*E is not even semistable.

(7) Use Bogomolov's theorem to prove the Kodaira vanishing theorem on surfaces: if L is an ample line bundle on a smooth complex projective surface then $H^1(X, K_X + L) = 0$.

(8) M. Raynaud constructed a smooth projective surface X over an algebraically closed field of characteristic p and an ample line bundle $\mathcal{L}$ on X such that $H^1(X, \mathcal{L}^{-1}) \neq 0$. Use this and Exercise 7 to construct a counterexample to Bogomolov's instability theorem in characteristic p.

(9) Let X be a smooth projective surface X over an algebraically closed field of characteristic p. One can show that although Bogomolov's inequality fails there exists some $\alpha(r, X, H)$ depending only on r, X and H such that $\Delta(E) \geq \alpha(r, X, H)$ any slope H-semistable torsion free sheaf E.

Use this and Exercise 8 to construct a counterexample to 1.4.2 in characteristic p.

(10) Show that $\Delta(E) \geq 0$ for any slope semistable torsion free sheaf E on $\mathbb{P}^2$ in any characteristic. Show that in this case Theorem 1.7 also holds.

(11) Show that $\Delta(T_{\mathbb{P}^2}) = 3$ and for all smooth curves $C \in |\mathcal{O}_{\mathbb{P}^2}(k)|$ with $k \leq 2$, the restriction $E|_C$ is not stable. Theorem 1.7 ensures that $E|_C$ is stable for any smooth curve $C \in |\mathcal{O}_{\mathbb{P}^2}(k)|$ if $k \geq 3$ (by Exercise 10 this holds in any characteristic).

(12) Let $F\colon X \to X$ be the absolute Frobenius morphism. Use the previous exercise and the fact that $F^*E \subset S^pE$ for any vector bundle E to show that 1.4.1 also fails in characteristic p.

(13) Show that 1.4.1 and 1.4.2 fail if slope semistability is replaced by Gieseker semistability.

(14) Show that 1.4.1 and 1.4.2 fail if slope semistability is replaced by slope stability.

Solution: 1.4.1 fails:

If E is slope H-stable with degree zero then E^* is also slope H-stable. If 1 holds for stability then $E \otimes E^*$ is slope stable. But $\mathcal{O}_X$ is a direct summand in $\operatorname{End} E = E \otimes E^*$ (char $k = 0$), a contradiction.

(15) Let E be a vector bundle on a smooth surface X. Use the Leray–Hirsch theorem and the Riemann–Roch formula on $\mathbb{P}(E)$ to prove that

$$\chi(X, S^nE) = -\frac{\Delta(E)}{2r}\frac{n^r}{(r+1)!} + \mathrm{O}(n^r).$$

(16) Let E be a rank r vector bundle on a smooth surface X. Let G be an elementary transformation of E along a vector bundle T on a smooth curve D. Show that G is a rank r vector bundle on X and we have

$$c_1G = c_1E - \operatorname{rk} T \cdot D$$

and

$$c_2G = c_2E + \deg_D T - \operatorname{rk} T(D \cdot c_1E) + \frac{\operatorname{rk} T(\operatorname{rk} T - 1)}{2}D^2.$$

Solution:

The first equality follows since T is trivial outside finitely many points of D, $c_1T = c_1(\mathcal{O}_D^{\operatorname{rk} T})$ and from

$$0 \to \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \to 0.$$

The second equality follows from the Riemann–Roch formula.

(17) Construct the flag scheme $\operatorname{Flag}(E/S; P_1, \dots, P_k) \to S$ using existence of the Quot-schemes (i.e., prove Theorem 2.3 using Theorem 2.2).

(18) Let X be a smooth variety over an algebraically closed field k. Then there exists an isomorphism $K^0(X) \simeq K(X)$ (cf. [Ha], Chapter 3, Exercise 6.9).

(19) In the above notation show that for any coherent sheaves F_1 and F_2 we have

$$[F_1] \cdot [F_2] = \sum_i (-1)^i [\mathcal{T}\mathrm{or}_i(F_1, F_2)]$$

(20) Let us set $[F_2]^* = \sum (-1)^i [\mathcal{E}\mathrm{xt}^i(F_2, \mathcal{O}_X)]$. Show that

$$[F_1] \cdot [F_2]^* = \sum (-1)^i [\mathcal{E}\mathrm{xt}^i(F_2, F_1)].$$

(21) For any two coherent sheaves F_1 and F_2 let us set

$$\chi(F_1, F_2) = \sum (-1)^i \dim \operatorname{Ext}^i(F_1, F_2).$$

Show that $\chi(F_1, F_2) = \chi([F_2] \cdot [F_1]^*)$.

(22) Let E and F be torsion free sheaves on a smooth surface X. Set $r_E = \operatorname{rk} E$, $r_F = \operatorname{rk}_F$ and $\xi_{E,F} = \frac{c_1 F}{r_F} - \frac{c_1 E}{r_E}$. Show that

$$\chi(E, F) = -\left(r_E \frac{\Delta_F}{2r_F} + r_F \frac{\Delta_E}{2r_E} \right) + r_E r_F \left(\frac{1}{2}\xi_{E,F}^2 - \frac{1}{2}\xi_{E,F} K_X + \chi(\mathcal{O}_X) \right).$$

(23) *Serre's Lemma.*

Let E be a globally generated rank r vector bundle on a smooth projective variety X. Show that there exists a rank $(r-1)$ trivial subsheaf F of E such that the quotient E/F is torsion free.

Below we give a very incomplete bibliography. We restricted only to a few new papers (apart from [Mi], which usually seems to be forgotten while talking about Bogomolov's inequality) and a few books in which one can find a more complete list of references (see [HL] and [LP1]).

References

[Da] G. Danila, *Formule de Verlinde et dualité étrange sur le plan projectif*, Thése, Paris 7 (1999).

[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, 1977.

[HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics **31**, 1997.

[La] A. Langer, *Semistable sheaves in positive characteristic*, preprint (2001).

[LP1] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Mathematics **54**, 1997.

[LP2] ———, *Dualité étrange sur les surfaces*, preprint (2001).

[Mi] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math. **10** (1987), 449–476.

Adrian Langer Instytut Matematyki UW, ul. Banacha 2, 02–097 Warszawa, Poland
E-mail address: `alan@mimuw.edu.pl`