On manifolds whose tangent bundle is big and 1-ample

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Abstract: We prove that if X is a complex projective manifold of dimension ≥ 4 whose tangent bundle is big and 1-ample then X is either the projective space $\mathbb{P}^n$ or the smooth quadric $\mathbb{Q}^n$.
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Contents

1 Introduction.

The purpose of this note is to prove the following result.

Theorem 1.1. *Let X be a complex projective manifold of dimension n. Suppose that the tangent bundle T_X is big and 1-ample. Then X is isomorphic either to the projective space $\mathbb{P}^n$ or to the smooth quadric $\mathbb{Q}^n$, or if $n = 3$ to complete flags $F(1,2) \cong \mathbb{P}(T_{\mathbb{P}^2})$ on $\mathbb{P}^2$.*

The definitions of *big* and *1-ample* as well and other notions discussed in the introduction will be explained in the subsequent section.

Since Mori proved in [Mo] that the projective space is the only manifold whose tangent bundle is ample it became apparent that positivity properties of the tangent bundle define rather restricted classes of manifolds. There are several ways one can define the notion of positivity of the tangent bundle and, accordingly, several classification results related to these notions. We refer the reader to Peternell's article [Pe] which reviews these matters.

Our motivation to study these problems comes from a conjecture by Campana and Peternell related to the notion of *nefness* which is a natural algebro-geometric extension of the notion of *ampleness*. Namely, in [CP] it is conjectured that the only Fano manifolds with nef tangent bundle are rational homogeneous spaces. We note that 1.1 follows easily from this conjecture.

The proof of 1.1 splits into two cases, of $\rho(X) = 1$ and $\rho(X) > 1$, respectively. In the former case we study rational curves on X and use Wierzba's result from [Wie] about the existence of a symplectic form on the space parameterizing 1-dimensional fibers of a symplectic contraction. This allows us to use a result about contractions of contact manifolds from [KPSW] and subsequently yields enough information about the rational curves on X so that we can conclude the argument. The case $\rho(X) > 0$ requires a theorem from [DPS] about smoothness of Fano-Mori contractions of manifolds with T_X nef and Mori's original result from [Mo].

In the appendix we reprove the two results mentioned above from [Wie] and [DPS] since their original proofs are not quite correct.

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2 Preliminaries.

2.1 Basic definitions and notation.

In this section we review some basic notions concerning vector bundles, Minimal Model Program and parameterizing schemes. For more information one may look into [KM] and [Ko].

Firstly, we recall that any vector bundle over the projective line $\mathbb{P}^1$ is decomposable into line bundles. Given any non-increasing sequence of integral numbers $a_1 \geq a_2 \geq \cdots \geq a_r$ by $E(a_1, \ldots, a_r)$ we will denote the bundle $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ over $\mathbb{P}^1$ and by $\mathbb{P}(a_1, \ldots, a_r)$ the associated projective bundle. Alternatively, $E(a_1^{k_1}, \ldots, a_r^{k_r})$ will denote $\mathcal{O}(a_1)^{\oplus k_1} \oplus \cdots \mathcal{O}(a_r)^{\oplus k_r}$, where now $a_1 > \cdots > a_r$.

Let X be a projective variety of dimension n over the field of the complex numbers and let L be a line bundle over X. We say that L is *nef* if $L.C \geq 0$ for every irreducible curve $C \subset X$. If L is nef we say that it is *big* if $L^n > 0$. We say that L is *semiample* if $L^{\otimes m}$ is generated by global sections for $m \gg 0$. If L is semiample then the graded ring of its sections $R(X, L) := \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$ is finitely generated $\mathbb{C}$-algebra and $Y_L := \mathrm{Proj}(R(X, L))$ is a complex projective variety. The evaluation of sections $H^0(X, L^{\otimes m}) \otimes \mathcal{O}_X \to L^{\otimes m}$ yields a proper surjective morphism of projective schemes:

$$\varepsilon : X = \mathrm{Proj}_X(\bigoplus_{m \geq 0} L^{\otimes m}) \longrightarrow Y_L = \mathrm{Proj}(\bigoplus_{m \geq 0} H^0(X, L^{\otimes m}))$$

which we will call the evaluation morphism or the contraction of X associated to (or supported by) L.

In the above situation the target variety Y is normal, the morphism $\varepsilon : X \to Y$ is proper, surjective and its fibers are connected, that is $\varepsilon_* \mathcal{O}_X = \mathcal{O}_Y$. More generally, the notion of contraction will apply to any morphism which has these properties (however, when talking about contractions we will assume that the morphism in question is not an isomorphism). In particular, X does not have to be projective. A contraction ε is called elementary if all curves contracted by ε are numerically proportional, or equivalently, the relative Picard number $\rho(X/Y)$ is 1. The contraction ε is called Fano-Mori, respectively, crepant, if $-K_X$ is ε ample, respectively, $K_X = \varepsilon^* K_Y$.

All of the above notions introduced for a line bundle L extend to a vector bundle of arbitrary rank. Namely, let E be a vector bundle of rank r over a projective variety X. Then the projectivisation of E, defined as $\mathbb{P}(E) := \mathrm{Proj}_X(\bigoplus_{m\geq 0} S^m(E))$ admits a relative hyperplane section bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Then we say that E has one of the properties discussed above for L if and only if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ has such a property. Properties of nef vector bundles are discussed in [CP]. If E is semiample then, by definition, $\mathcal{O}_{\mathbb{P}(E)}(m)$ is generated by global sections for $m \gg 0$ and we have the evaluation map

$$\varepsilon : \mathbb{P}(E) = \mathrm{Proj}_X \bigoplus_{m\geq 0} (S^m E) \longrightarrow \mathrm{Proj}(\bigoplus_{m\geq 0} H^0(X, S^m E))$$

The following definition is probably not as commonly used as these before; it is due to Sommese.

Definition 1. *Let E be a semi-ample vector bundle over a projective variety X. We say that E is k-ample if the evaluation map has fibers of dimension smaller or equal than k.*

Example 1. Let $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ be a smooth quadric. Then the surjection of twisted forms $(\Omega_{\mathbb{P}^{n+1}})_{|\mathbb{Q}^n} \otimes \mathcal{O}(2) \to \Omega_{\mathbb{Q}^n}(2)$ induces the inclusion $\mathbb{P}(\Omega_{\mathbb{Q}^n}(2)) \subset \mathbb{P}(\Omega_{\mathbb{P}^{n+1}}(2))$. On the other hand $\mathbb{P}(\Omega_{\mathbb{P}^{n+1}}(2))$ can be identified with flags of points and lines on $\mathbb{P}^{n+1}$ and the evaluation morphism contracts flags to the Grassmanian of lines $Gr(1, \mathbb{P}^{n+1})$. The induced surjective morphism $\mathbb{P}(\Omega_{\mathbb{Q}^n}(2)) \to Gr(1, \mathbb{P}^{n+1})$ is generically two-to-one and its exceptional locus are the lines in $\mathbb{P}^{n+1}$ which are contained in the quadric $\mathbb{Q}^n$. Thus $\Omega_{\mathbb{Q}^n}(2)$ is big and 1-ample. Since we have $\Omega_{\mathbb{Q}^n}(2) \cong T_{\mathbb{Q}^n}$, the same concerns $T_{\mathbb{Q}^n}$.

Example 2. Let V be a 3-dimensional vector space. We consider triples: (λ, π, σ) where λ and π are linear subspaces of V of dimension 1 and 2, respectively, and $\sigma \in \mathfrak{sl}(V)$, subject to the following conditions:

$$\sigma^3 = 0, \ \mathrm{im}(\sigma^2) \subset \lambda \subset \ker(\sigma), \ \lambda \subset \pi, \ \mathrm{im}(\sigma) \subset \pi \subset \ker(\sigma^2)$$

The subset consisting of classes of such triples in $F(1,2,V) \times \mathbb{P}(\mathfrak{sl}(V))$ can be identified with the projectivisation of the tangent bundle to flags $F(1,2,V)$, or $\mathbb{P}(T_{\mathbb{P}(T_{\mathbb{P}^2})})$, and the projection to $\mathfrak{sl}(V)$ is the evaluation map onto $\{[\sigma] \in \mathfrak{sl}(V) : \mathrm{rk}\,\sigma^3 \leq 2\}$. The map is birational and its exceptional locus is over $\{\sigma : \mathrm{rk}\,\sigma = 1\}$ where the fibers are 1-dimensional. Thus $T_{\mathbb{P}(T_{\mathbb{P}^2})}$ is big and 1-ample.

Let us note that the above two examples, presented here ad hoc, are special cases of contractions of projectivisations of tangent bundles of homogeneous varieties G/P, where G is simple linear group and $P \subset G$ is its parabolic subgroup. Such contractions come naturally from the canonical map to the Lie algebra of G, see e.g. [F].

Given a projective variety X we will consider the scheme $\mathrm{Hom}(\mathbb{P}^1, X)$ parameterizing morphisms from $\mathbb{P}^1$ to X. In order to parametrize rational curves on X we will consider the scheme $\mathrm{RatCurves}^n(X)$, as defined in [Ko, II.2] to which we refer for details. Given an irreducible component $\widehat{\mathcal{M}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ containing morphisms parameterizing rational curves in X (i.e. all morphism in $\widehat{\mathcal{M}}$ are birational onto their image) the associated component $\mathcal{M}$ of $\mathrm{RatCurves}^n(X)$ is the quotient of (the normalization of) $\mathcal{M}$ by the natural right $Aut(\mathbb{P}^1)$ action. Over $\mathcal{M}$ we have the universal $\mathbb{P}^1$-bundle $u : \mathcal{U} \to \mathcal{M}$ which is the quotient of (the normalization of) $\mathbb{P}^1 \times \widehat{\mathcal{M}}/Aut(\mathbb{P}^1)$ by the action $g(p, f) = (g^{-1}(p), f \circ g)$, c.f. [Ko, II.2.15].

There is a natural map $\widehat{e} : \mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, X) \to X$ such that $\widehat{e}(p, f) = f(p)$; it descends to $e : \mathcal{U} \to X$. We call each of these two maps evaluation.

Suppose that X is smooth, then $\mathrm{Hom}(\mathbb{P}^1, X)$ is smooth at f if $H^1(\mathbb{P}^1, f^*T_X) = 0$. If in addition X is symplectic (see definition 3) then $\mathrm{Hom}(\mathbb{P}^1, X)$ is smooth at f if $h^1(\mathbb{P}^1, f^*T_X) \leq 1$, see e.g. [Ran], [Ka]. In such a case the tangent space $T_f \mathrm{Hom}(\mathbb{P}^1, X)$ can be identified with $H^0(\mathbb{P}^1, f^*T_X)$ and the derivative of the evaluation map is related to the evaluation of sections of f^*T_X. That is, $T\widehat{e}_{(p,f)} = Tf_{(p)} + ev_p$ where $ev_p : H^0(\mathbb{P}^1, f^*T_X) \to f^*T_X \otimes k(p) = T_{f(p)}X$ is the evaluation of sections at the point p, [Ko, II.3.4].

2.2 Contact and symplectic manifolds.

Definition 2. *Let M be a compact complex manifold of dimension $2n - 1$ with a line bundle L. A twisted holomorphic form $\theta \in H^0(M, \Omega_M \otimes L)$ is called a contact form if $\theta \wedge (d\theta)^{\wedge(n-1)}$ is everywhere non-zero.*

One can check that although $d\theta$ is not defined as a twisted holomorphic form its restriction to the kernel of θ makes sense so that $\theta \wedge (d\theta)^{\wedge(n-1)}$ is a good form in $H^0(M, \Omega_M^{\wedge(2n-1)} \otimes L^n)$. The kernel of θ is called *contact distribution* and a manifold admitting a contact form is called a *contact manifold*. Contact manifolds are discussed in [LB] and [KPSW], where we refer the reader for details.

Definition 3. *Let N be a complex manifold of dimension $2n$. A closed 2-form $\sigma \in H^0(N, \Omega_N)$ is called symplectic if $\sigma^{\wedge n}$ is everywhere non-zero.*

A manifold N admitting a symplectic form is called a *symplectic manifold*.

Given a contact manifold M we can produce a symplectic structure on the $\mathbb{C}^*$ bundle associated with L, c.f. [Be, 4.2]. Namely, we let $N = \mathrm{Spec}_M(\bigoplus_{m\in\mathbb{Z}} L^m)$ be the principal $\mathbb{C}^*$ bundle associated to L, with the projection $\pi : N \to M$. If L is given on a covering $(U_i)_{i\in I}$ by transition functions g_{ij} then in fibers of π over $\pi^{-1}(U_i)$ we have local coordinates z_i satisfying relations $z_i = g_{ij}z_j$ over $\pi^{-1}(U_i \cap U_j)$. The action of $\mathbb{C}^*$ is described by multiplying the coordinate z_i, that is for $t \in \mathbb{C}^*$ we set $t(z_i) = t \cdot z_i$.

Given a twisted form $\theta \in H^0(M, \Omega_M \otimes L)$, trivialized on the covering $(U_i)_{i\in I}$ by forms θ_i satisfying relations $\theta_i = g_{ij}\theta_j$ on $U_i \cap U_j$, we can define a closed 2-form on N by setting on $\pi^{-1}(U_i)$

$$\sigma = d\left(\frac{\theta_i}{z_i}\right) = \frac{d\theta_i + \theta_i \wedge dz_i/z_i}{z_i}$$

Since $\theta_i/z_i = \theta_j/z_j$ on $\pi^{-1}(U_i \cap U_j)$ so the local data $(\theta_i/z_i)_{i\in I}$ defines a global 1-form on N of weight (-1) with respect to the $\mathbb{C}^*$ action. The 2-form σ is therefore closed and of weight (-1) too. Let us note that the formula given in [KPSW, 2.2] is not quite correct since the denominator is missing; however the conclusion is right: being of weight (-1) the form σ defines a section of $\Lambda^2\mathcal{L} \otimes L$, where the sheaf $\mathcal{L}$ is defined in [KPSW, 2.1].

The form σ will be called the *symplectisation* of θ, c.f. [Be, 4.2].

We note that the above construction is preserved by the pull-backs. That is, given a morphism $\varphi : M' \to M$ and the pull-back of the $\mathbb{C}^*$ bundle $\pi : N \to M$ to $\pi' : N' \to M'$ with the $\mathbb{C}^*$

equivariant morphism $\widehat{\varphi}: N' \to N$ we have $\sigma' = \widehat{\varphi}^*(\sigma)$, where the form σ' is constructed as above from $\varphi^*(\theta)$.

Lemma 2.1. *The form θ is contact if and only if the form σ is symplectic.*

Proof. It is enough to check locally that

$$\sigma_i^{\wedge n} = \frac{n \cdot (d\theta_i)^{\wedge(n-1)} \wedge \theta_i \wedge dz_i}{z_i^{n+1}}$$

$\square$

The following example is classical, see [Be, 4.2].

Example 3. Let X be a compact complex manifold of dimension n. We consider the projectivisation of its tangent bundle $\mathcal{X} := \mathbb{P}(T_X) = \mathrm{Proj}_X \bigoplus_{m\geq 0}(S^m T_X)$ with the relative hyperplane section bundle $L = \mathcal{O}(1)$. If we remove the zero section from the total space of the cotangent bundle $T_X^* = \mathrm{Spec}_X \bigoplus_{m\geq 0} S^m T_X$, call the result $\widehat{\mathcal{X}}$, and consider the natural projection on $\mathcal{X}$ then $\pi : \widehat{\mathcal{X}} \to \mathcal{X}$ is the principal $\mathbb{C}^*$ bundle associated to L^{-1}.

Over $\mathcal{X}$ we have a canonical contact structure defined by a form in $H^0(\mathcal{X}, \Omega_{\mathcal{X}} \otimes L) \cong H^0(X, End(T_X))$ associated to the identity. If (x_k) are local analytic coordinates on $U \subset X$ and $\zeta_k = \partial/\partial x_k$ associated differentiations (vector fields) then on $\mathcal{X}$ the differentiations ζ_k define sections of L and on $\mathcal{X}$ over U we take $\theta = \sum_k \zeta_k dx_k$, see the discussion before Proposition 2.14 in [KPSW].

In the affine covering $U_i = \{\zeta_i \neq 0\}$ of $\mathcal{X}$ over U the bundle L is defined by transition functions $g_{ij} = \zeta_j/\zeta_i$ while over U_i the form θ can be written as $\theta_i = \sum_k (\zeta_k/\zeta_i) \cdot dx_k$. On $\widehat{\mathcal{X}}$, over U_i, the function $z_i = 1/\zeta_i$ is the coordinate in the $\mathbb{C}^*$ bundle associated to L. Now we use the previous construction

$$\sigma_i := d\left(\frac{\theta_i}{z_i}\right) = d(\zeta_i \cdot \theta_i) = \sum_k d\zeta_k \wedge dx_k$$

and we get the standard symplectic form on the cotangent bundle T^*X.

3 Proof of the main theorem.

In the course of the proof of 1.1 we will assume that T_X is 1-ample but not ample since if T_X is ample then $X \cong \mathbb{P}^n$ by Mori, [Mo].

3.1 Case $\rho(X) = 1$

In this section we prove the following

Proposition 3.1. *Let assume that X is a complex projective manifold of dimension n whose tangent bundle is big and 1-ample but not ample, and moreover $\rho(X) = 1$. Then X is a smooth quadric $\mathbb{Q}^n$, and $n \geq 3$.*

The proof will take the rest of this section.

We define the following varieties

$$\mathcal{X} := \mathbb{P}(T_X) = \mathrm{Proj}_X(\bigoplus_{m\geq 0} S^m T_X) \qquad \widehat{\mathcal{X}} := \mathrm{Spec}_X(\bigoplus_{m\in\mathbb{Z}} S^m T_X)$$
$$\mathcal{Y} := \mathrm{Proj}(\bigoplus_{m\geq 0} H^0(X, S^m T_X)) \qquad \widehat{\mathcal{Y}} := \mathrm{Spec}(\bigoplus_{m\in\mathbb{Z}} H^0(X, S^m T_X))$$

By $p : \mathcal{X} \to X$ we denote the projection.

Let $\mathcal{L}_\mathcal{X}$ and $\mathcal{L}_\mathcal{Y}$ denote the $\mathcal{O}(1)$ bundles on the respective Proj-schemes. We have projections $\pi_\mathcal{X} : \widehat{\mathcal{X}} \to \mathcal{X}$ and $\pi_\mathcal{Y} : \widehat{\mathcal{Y}} \to \mathcal{Y}$ which are principal $\mathbb{C}^*$ bundles associated to $\mathcal{L}_\mathcal{X}$ and $\mathcal{L}_\mathcal{Y}$, respectively. If by ε and $\widehat{\varepsilon}$ we denote the respective evaluation morphisms, as in section 2.1, then the following diagram commutes and makes $\widehat{\mathcal{X}}$ a fiber product of $\mathcal{X}$ and $\widehat{\mathcal{Y}}$ over $\mathcal{Y}$.

$$\begin{tikzcd} \widehat{\mathcal{X}} \arrow[r, "\widehat{\varepsilon}"] \arrow[d, "\pi_\mathcal{X}"'] & \widehat{\mathcal{Y}} \arrow[d, "\pi_\mathcal{Y}"] \\ \mathcal{X} \arrow[r, "\varepsilon"] & \mathcal{Y} \end{tikzcd}$$

By E and $\widehat{E}$ let us denote the exceptional locus of ε and $\widehat{\varepsilon}$, respectively.

Lemma 3.2. *The morphisms ε and $\widehat{\varepsilon}$ are elementary crepant contractions and moreover $\widehat{\varepsilon}$ is a symplectic contraction. The exceptional loci of both of them are irreducible divisors and all positive dimensional fibers are either $\mathbb{P}^1$ or two $\mathbb{P}^1$'s meeting at a node.*

Proof. For $\widehat{\varepsilon}$ these are properties of symplectic contractions proved in [Wie, Ch. 5], see also 4.1, and then by $\mathbb{C}^*$ action we descend them to the respective properties of ε. □

Let $\mathcal{M} \subset \mathrm{RatCurves}^n(\mathcal{X})$ and, respectively, $\widehat{\mathcal{M}} \subset \mathrm{RatCurves}^n(\widehat{\mathcal{X}})$ be a subset parameterizing rational curves contracted be ε and $\widehat{\varepsilon}$, respectively. By $\mathcal{U}$ and $\widehat{\mathcal{U}}$ we denote the universal families over $\mathcal{M}$ and $\widehat{\mathcal{M}}$, respectively. Since $\widehat{\varepsilon}$ is a pull-back map of $\mathbb{C}^*$ bundles over the morphism ε the natural projection $\pi_\mathcal{M} : \widehat{\mathcal{M}} \to \mathcal{M}$ is a $\mathbb{C}^*$ bundle too and it extends to the map of universal families $\pi_\mathcal{U} : \widehat{\mathcal{U}} \to \mathcal{U}$ which commutes with their evaluation.

Lemma 3.3. *The variety $\mathcal{M}$ is a connected projective contact manifold of dimension $2n-3$.*

Proof. We have the following commuting diagram

$$\begin{tikzcd} \widehat{\mathcal{X}} \arrow[d, "\pi_\mathcal{X}"'] & \widehat{\mathcal{U}} \arrow[l, "\widehat{e}"'] \arrow[r, "\widehat{u}"] \arrow[d, "\pi_\mathcal{U}"] & \widehat{\mathcal{M}} \arrow[d, "\pi_\mathcal{M}"] \\ \mathcal{X} & \mathcal{U} \arrow[l, "e"'] \arrow[r, "u"] & \mathcal{M} \end{tikzcd}$$

where e and $\widehat{e}$ are evaluation morphisms, u and $\widehat{u}$ are analytic or étale $\mathbb{P}^1$-bundles, and vertical arrows are $\mathbb{C}^*$ bundles. According to section 5 of [Wie] the variety $\widehat{\mathcal{M}}$ is smooth and connected of dimension $2n-2$ so $\mathcal{M}$ is connected, smooth of dimension $2n-3$ and moreover it is projective because $\mathcal{X}$ is projective and the family of curves parametrized by $\mathcal{M}$ is unsplit in the sense of [Ko].

Since the pullback $\mathcal{L}_\mathcal{U} = e^*\mathcal{L}_\mathcal{X}$ is trivial along the fibers of u thus it descends to a line bundle $\mathcal{L}_\mathcal{M}$ over $\mathcal{M}$ such that $u^*\mathcal{L}_\mathcal{M} = \mathcal{L}_\mathcal{U}$. Let θ and σ be the forms over $\mathcal{X}$ and $\widehat{\mathcal{X}}$, respectively, constructed like in Example 3. We can pull-back the form θ to $\theta_{\mathcal{U}} \in H^0(\mathcal{U}, \Omega_{\mathcal{U}} \otimes \mathcal{L}_{\mathcal{U}})$. Since the form $\theta_{\mathcal{U}}$ is trivial along fibers of u, it descends to a form $\theta_{\mathcal{M}} \in H^0(\mathcal{M}, \Omega_{\mathcal{M}} \otimes \mathcal{L}_{\mathcal{M}})$ such that $e^*\theta_{\mathcal{M}} = \theta_{\mathcal{U}}$. By the similar reasons there exists a 2-form $\sigma_{\mathcal{M}}$ over $\widehat{\mathcal{M}}$ such that $\widehat{e}^*\sigma = \widehat{u}^*\sigma_{\mathcal{M}}$, see the arguments preceding 4.2.

Now, by Wierzba's result, 4.2, the form $\sigma_{\mathcal{M}}$ is symplectic and by the construction the form $\sigma_{\mathcal{M}}$ is a symplectisation of $\theta_{\mathcal{M}}$, so by 2.1 the form $\theta_{\mathcal{M}}$ is contact. □

Let $\mathcal{M}' \subset \mathrm{Hom}(\mathbb{P}^1, \mathcal{X})$ be the component dominating $\mathcal{M}$. The morphism $p : \mathcal{X} \to X$ induces a morphism $p_* : \mathrm{Hom}(\mathbb{P}^1, \mathcal{X}) \to \mathrm{Hom}(\mathbb{P}^1, X)$ such that $p_*(f) = p \circ f$. Let $M' = p_*(\mathcal{M}')$. We note that, being a projective bundle, p maps any rational curve which is a component of a fiber of ε birationally onto its image in X. Thus we can descend p_* to a morphism $\mathcal{M} \to \mathrm{RatCurves}^n(X)$.

Lemma 3.4. *The set M' is an irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ parameterizing rational curves on X of minimal degree with respect to $-K_X$. In particular any $f \in M'$ is birational onto its image and moreover $f^*T_X \cong E(2, 1^{n-2}, 0)$. The induced morphism $\mathcal{M} \to M := M'/Aut(\mathbb{P}^1) \subset \mathrm{RatCurves}^n(X)$ is an isomorphism.*

Proof. Let $C \subset X$ be a rational curve such that $-K_X \cdot C$ is minimal. Let $f_C : \mathbb{P}^1 \to C \subset X$ be its normalization. By [Ke] we may assume that $-K_X \cdot C \leq n$ and therefore $f_C^*T_X$ is not ample and it admits a surjective morphism $f_C^*T_X \to \mathcal{O}$. The section of $\mathbb{P}(f_C^*T_X)$ associated to $f_C^*T_X \to \mathcal{O}$ is contracted by $\mathbb{P}(f_C^*T_X) \to \mathbb{P}(T_X) = \mathcal{X} \to \mathcal{Y}$ hence C is dominated by a fiber of ε and $f_C = p_*(f)$ for some $f \in \mathcal{M}'$. This proves the first part of the lemma and shows that p_* descends to $\mathcal{M} \to M$.

If $f_C^*T_X \cong E(a_1, \ldots, a_{n-r}, 0^r)$ for $a_1 \geq \cdots \geq a_{n-r} > 0$ then the possible surjections $f^*T_X \to \mathcal{O}$ (modulo homotheties) are parametrized by $\mathbb{P}^{r-1}$, hence $p_*^{-1}(f_C) \cong \mathbb{P}^{r-1}$. Thus $\mathcal{M} \to M$ is a morphism of smooth projective varieties with connected fibers and it is an isomorphism if and only if $f_C^*T_X \cong E(2, 1^{n-2}, 0)$ for every $f_C \in M'$. Suppose that this is not the case.

We claim that then $\mathcal{M} \to M$ factors through a Fano-Mori contraction of $\mathcal{M}$. Indeed, this follows from general properties of morphisms of smooth varieties: either $\mathcal{M} \to M$ is a birational morphism of smooth varieties and as such factors through a Fano-Mori contraction (actually, it is enough that M has terminal singularities) or a general fiber of the fiber space $\mathcal{M} \to M$ is a positive dimensional projective space and by adjunction $K_{\mathcal{M}/M}$ is not nef. We note however that in the case considered we have even more: $\mathcal{L}_{\mathcal{X}}$ is p-ample so a similar feature holds for $\mathcal{L}_{\mathcal{M}}$ with respect to $\mathcal{M} \to M$, but then since $K_{\mathcal{M}} = -n \cdot \mathcal{L}_{\mathcal{M}}$ so in fact $\mathcal{M} \to M$ is a Fano-Mori contraction itself.

Thus, by [KPSW, 2.12], $\dim M \leq n-1$ and looking at the splitting type we see that for every $f_C \in M'$ we have $f_C^*T_X \cong E(2, 0^{n-1})$. Then, if $U \to M$ is the universal family of rational curves then the evaluation $U \to X$ is étale, [Ko, II.2 and II.3.4]. Since, being Fano X is simply connected, it implies that $U \cong X$ which contradicts the assumption $\rho(X) = 1$. □

Once, for any minimal rational curve, the isomorphism $(T_X)_{|C} \cong E(2, 1^{n-2}, 0)$ is determined, the proposition 3.1 can be concluded in several ways, c.f. [CS] and [Wie2]. We propose yet another way, by reducing the problem to a result from [Wiś]. We need the following general observation.

Lemma 3.5. *Let E be a vector bundle over a smooth curve C of rank n and degree $d = \deg E$. Let us consider the projectivisation $\pi : \mathbb{P}(E) \to C$ with $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and F denoting the numerical equivalence class of the fiber of π. Assume that $D \subset \mathbb{P}(E)$ is an irreducible effective divisor which is numerically equivalent to $aF + bL$. If the restriction morphism $\pi_D : D \to C$ is smooth then either $b = 1$ (i.e. D is a projective sub-bundle) or $a/b = -d/n$.*

Proof. The differentials of functions defining locally D give a section of $\Omega_{\mathbb{P}(E)/C} \otimes \mathcal{O}_D(D)$ and D is smooth over C if and only if this section has no zeroes, see [Ha, II.8.12 and III.10.4]. Thus the necessary condition for the smoothness of D over C is the vanishing of $c_{n-1}(\Omega_{\mathbb{P}(E)/C} \otimes \mathcal{O}_D(D))$ so the proof boils down to computing of this class.

To this end let us set an auxiliary $\mathbb{Q}$ numerical class $L_0 = L - (d/n)F$; then $D \equiv bL_0 + a_0F$ where $a_0 = a + (bd)/n$. Now using the relative Euler sequence [Ha, II.8.13] and the formula for the top Chern class of the twisted bundle [Fu, 3.2.3] we get the following formula (the classes of 0 dimensional cycles we identify naturally with integers)

$$c_{n-1}(\Omega_{\mathbb{P}(E)/C} \otimes \mathcal{O}_D(D)) \equiv \sum_{k=0}^{n-1} D^{n-k} \cdot \binom{n}{k} \cdot (-L_0)^k = a_0 \cdot \sum_{k=0}^{n-1} \left[\binom{n}{k} \cdot (n-k) \cdot b^{n-k-1} \cdot (-1)^k \right]$$

where the last equality is obtained by using the following relations in $\mathbb{P}(E)$: $F^2 \equiv 0$, $L_0^n = 0$ and $F \cdot L_0^{n-1} = 1$. The sum in the last formula is just the expansion of the derivative $\partial(b-1)^n/\partial b = n \cdot (b-1)^{n-1}$ so that we get $c_{n-1}(\Omega_{\mathbb{P}(E)/C} \otimes \mathcal{O}_D(D)) = a_0 \cdot n \cdot (b-1)^{n-1}$ and the lemma follows. $\square$

Conclusion of proof of 3.1. According to 3.2 (see also 4.1) a general fiber of $\varepsilon_{|E}$ is either $\mathbb{P}^1$ or two $\mathbb{P}^1$'s meeting at a node. If C denotes an irreducible component of the fiber then $E \cdot C = -2$ and $E \cdot C = -1$, respectively. Since $Pic\mathcal{X} = p^*PicX \oplus \mathbb{Z} \cdot \mathcal{L}_X$ and $\mathcal{L}_X \cdot C = 0$ it follows that in the latter case there exists $H \in PicX$ such that $H \cdot p(C) = 1$ hence, by 3.4, $-K_X = nH$ and $X \cong \mathbb{Q}^n$ by a well known result of Kobayashi and Ochiai.

If fibers of $E \to \varepsilon(E)$ are irreducible then the evaluation morphism $e : \mathcal{U} \to E \subset \mathcal{X}$ is a bijective closed immersion (because of 4.2) hence we have an isomorphism $\mathcal{U} \cong E$. Since, moreover, by 3.4, $\mathcal{M} \cong M$, then E can be identified with the universal family of rational curves parametrized by M and the map $p_{|E} : E = \mathcal{U} \to X$ can be identified with the evaluation of curves parametrized by M. Thus, by [Ko, II.3.5] $E \to X$ is smooth and we can use 3.5. If $E \subset \mathbb{P}(T_X)$ is a projective sub-bundle then it is associated to a section of $T_X \otimes \mathcal{O}(-H)$ for some ample H, and by [Wa], $X \cong \mathbb{P}^n$ and this case can be eliminated.

Thus, by 3.5 we have $E \equiv d \cdot \mathcal{L}_X + (d/n) \cdot p^*K_X$ and since $E \cdot C = -2$ and $K_X \cdot p(C) = -n$ we get $d = 2$. But then E defines a symmetric nowhere degenerate form in $H^0(X, S^2T_X \otimes \mathcal{O}(2K_X/n))$ and we are in the situation of [Wiś, Thm. 2] or [Ye], hence $X \cong \mathbb{Q}^n$.

This concludes the proof of 3.1.

3.2 Case $\rho(X) > 1$.

Let us recall that a Fano-Mori contraction of a normal variety X is a morphism $\psi : X \to Y$ onto a normal variety, with connected fibers, such that $-K_X$ is ψ ample (by definition a contraction is not an isomorphism). If X is a Fano manifold with $\rho(X) > 1$ then it admits non-trivial Fano-Mori contractions onto positive dimensional varieties. If T_X is nef then by a result of Demailly, Peternell and Schneider, [DPS, Thm. 5.2], the contractions and their targets are smooth, see 4.4. We begin this section by proving the following somewhat more general result.

Theorem 3.6. *Let X be a projective manifold whose tangent bundle is k-ample. Suppose that $\psi : X \to Y$ is a Fano-Mori contraction of X onto a positive dimensional variety. If $k_Y = \dim X - \dim Y$ then $k_Y \leq k$ and T_Y is $(k - k_Y)$-ample.*

Proof. By 4.4 the contraction ψ is a smooth morphism and Y is a smooth variety. The bundle $\psi^* T_Y$ is the quotient of T_X and therefore it is k ample. On the other hand $\mathbb{P}(\psi^* T_Y) = X \times_Y \mathbb{P}(T_Y)$ and the evaluation morphism of $\mathbb{P}(\psi^* T_Y)$ is the composition of projection $X \times_Y \mathbb{P}(T_Y) \to \mathbb{P}(T_Y)$ and the evaluation on $\mathbb{P}(T_Y)$. Now comparing the dimensions of the respective fibers we get the theorem. □

Now we can deal with the case of manifolds with 1-ample tangent bundle.

Proposition 3.7. *Let X be a Fano manifold with 1-ample tangent bundle. If $\rho(X) > 1$ then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong \mathbb{P}(T_{\mathbb{P}^2})$.*

Proof. The manifold X has at least two different nontrivial Fano-Mori contractions $\psi_i : X \to Y_i$, $i = 1, 2$, onto smooth varieties Y_i, by 4.4. By 3.6, $\dim Y_i = n - 1$ and T_{Y_i} is ample. So by [Mo] $Y_i \cong \mathbb{P}^{n-1}$ and by 4.4 the contraction ψ_i is a $\mathbb{P}^1$ bundle. Now the proposition follows by Sato's classification of manifolds which have two projective bundle structures [Sa], or by classification of Fano bundles over the projective space, [SW], [APW]. □

This concludes the proof of 1.1.

4 Appendix.

In the remaining part of this paper we consider two results which are essential for our proof of 1.1. We reprove these results since the existent proofs have — in our opinion — gaps.

4.1 A symplectic form on a parameterizing scheme.

In this section we discuss the following result of Wierzba from [Wie].

Theorem 4.1. *Let $\varphi : X \to Y$ be an elementary contraction of a symplectic manifold X with $\dim X = 2n$. Assume that the exceptional locus of φ is an irreducible divisor E and $\varphi(E) = Z \subset Y$. Suppose that all fibers of φ are of dimension ≤ 1. Then Z is a symplectic manifold of dimension $2n - 2$ and moreover either every fiber of φ over Z is $\mathbb{P}^1$ or every fiber of φ over Z is union of two $\mathbb{P}^1$'s meeting at a node.*

The above theorem is proved in 5 of [Wie], the two cases appearing in the theorem are called A_1 and A_2, respectively, depending on the type of singularity the variety Y has along Z. The A_2 case is dealt in codimension ≤ 3. The Wierzba's argument in this case. however, seems to contain a statement concerning crepant contractions of threefolds which sounds as follows: *if a divisorial crepant elementary contraction of a smooth 3-fold has a general fiber consisting of two $\mathbb{P}^1$ meeting at a node then all fibers are of the same type*, [Wie, proof of 5.9], see also [W1, Erratum]. One can check that the blow-up of hypersurface $x^3 + t \cdot y^2 + z^2 = 0$ along a line $x = y = z = 0$ will produce an example of a crepant contraction of a smooth variety which contradicts this claim, c.f. [W2, Prop. 3.2].

Wierzba's argument can be improved by using the following result which we formulate in terms of the families of rational curves. Namely, any irreducible component of a 1-dimensional fiber of φ is a smooth rational curve, [Wie, 4.1]. Let $\mathcal{M} \subset \mathrm{RatCurves}^n(X)$ be a subset parameterizing rational curves contracted by φ. Over $\mathcal{M}$ we have the universal family $u : \mathcal{U} \to \mathcal{M}$ which is an analytic or étale $\mathbb{P}^1$-bundle and which has the canonical evaluation map $e : \mathcal{U} \to X$. Equivalently, $\mathcal{M}$ is a component of $\mathrm{Hilb}(X)$ which parametrizes irreducible components of 1-dimensional fibers of φ and $\mathcal{U} \subset X \times \mathcal{M}$ is the universal family over $\mathcal{M}$.

We can transport the form σ to a 2-form $\sigma_{\mathcal{M}} \in H^0(\mathcal{M}, \Omega^2_{\mathcal{M}})$ such that $u^*\sigma_{\mathcal{M}} = e^*\sigma$. Indeed, $e^*\sigma$ gives a form $\sigma_{\mathcal{U}}$ in $\Omega^2_{\mathcal{U}}$ and from the exact sequence

$$0 \longrightarrow u^*\Omega^2_{\mathcal{M}} \longrightarrow \Omega^2_{\mathcal{U}} \longrightarrow u^*\Omega_{\mathcal{M}} \otimes \Omega_{\mathcal{U}/\mathcal{M}} \longrightarrow 0$$

and the fact that $u_*(u^*\Omega_{\mathcal{M}} \otimes \Omega_{\mathcal{U}/\mathcal{M}}) = 0$ we get $u_*(\Omega^2_{\mathcal{U}}) = \Omega^2_{\mathcal{M}}$ hence $\sigma_{\mathcal{U}}$ descends to a 2-form on $\mathcal{M}$.

The following is the main result of the present section.

Theorem 4.2. *In the notation introduced above, $\mathcal{M}$ is a smooth irreducible variety of dimension $2n-2$ and moreover the following three properties are satisfied:*

(i) the derivative of the map e is of rank $2n-1$, that is, e is an immersion;

(ii) for every rational curve C contracted by φ we have $(T_X)_{|C} \cong E(2, 0^{2n-2}, -2)$;

(iii) the form $\sigma_{\mathcal{M}}$ is non-degenerate and closed, hence a symplectic form on $\mathcal{M}$.

Let us indicate how 4.2 implies 4.1. This is an argument from [Wie, Sect. 5]. By looking at the surface case we know that a general fiber of $\varphi_{|E}$ is either $\mathbb{P}^1$ or union of two $\mathbb{P}^1$ (reducible conic). The variety $\mathcal{M}$ parametrizes components of fibers of φ so we have a morphism $\mathcal{M} \to Z = \varphi(E)$ which is generically one-to-one or two-to-one, respectively. The evaluation morphism $e : \mathcal{U} \to E$ is generically bijective and $\mathcal{U}$ is smooth, hence $e : \mathcal{U} \to E$ is the normalization, and by 4.2 it is of maximal rank everywhere (closed immersion). Let $\Delta \subset \mathcal{U}$ be the conductor of the map e which is the set where e is not bijective; by adjunction $\Delta = -K_{\mathcal{U}} + e^*(E)$, [It, Thm. 5.6]. Then, since fibers of u are mapped bijectively to components of fibers of φ, it follows that Δ contains no fiber of u. Thus either Δ is empty and all fibers are smooth $\mathbb{P}^1$ (this is the A_1 case) or Δ is a section of $u : \mathcal{U} \to \mathcal{M}$. Let us deal with this latter case.

The map $e_{|\Delta}$ is an immersion and two-to-one everywhere. Indeed, suppose that this is not the case: then we have more than two local analytic branches of E, let us call them E_i, meeting at a point. We note that then E_i's meet in codimension 2 and since their intersection does not contain a fiber of φ we have $\varphi(E_1 \cap E_i) = \varphi(E_1)$ for all i. Thus all branches dominate a local component of $\varphi(E)$ which contradicts that over a general point of $\varphi(E)$ we have two branches at most. Hence the map $\Delta \to \varphi(E)$ is two-to-one everywhere and all fibers are line pairs meeting at nodes.

Now let us deal with the proof of theorem 4.2: its first part is proved in [Wie] so that we will concentrate on proving the properties *(i)* —*(iii)*, the proof will take the rest of this section. At this point we note that $\sigma_{\mathcal{M}}$ is closed because $u^*(d\sigma_{\mathcal{M}}) = d(u^*\sigma_{\mathcal{M}}) = d(e^*\sigma) = e^*(d\sigma) = 0$ and the derivative of u is injective (since u is a submersion) so that $d\sigma_{\mathcal{M}} = 0$. We also note that since $\mathcal{M}$ parametrizes components of fibers of φ the evaluation map e is finite-to-one.

Lemma 4.3. *The conditions (i) — (iii) in 4.2 are equivalent.*

Proof. Let C be a rational curve contracted by φ which is represented by a point $[C] \in \mathcal{M}$; we will write also $u^{-1}([C]) = C$. Let us consider the following commutative diagram in which the vertical arrows are defined by the respective 2-forms, while the horizontal arrows come from derivatives of the maps e and u.

$$
\begin{tikzcd}
e^*(T_X)_{|C} \arrow[d, "e^*(\sigma)"] & (T_{\mathcal{U}})_{|C} \arrow[l, "Te"'] \arrow[r, "Tu"] \arrow[d, "\sigma_{\mathcal{U}}"] & u^*(T_{\mathcal{M}})_{|C} \arrow[d, "u^*(\sigma_{\mathcal{M}})"] \\
e^*(\Omega_X)_{|C} \arrow[r, "De"] & (\Omega_{\mathcal{U}})_{|C} & u^*(\Omega_{\mathcal{M}})_{|C} \arrow[l, "Du"']
\end{tikzcd}
$$

In this diagram $e^*(T_{\mathcal{M}})_{|C} = T_{[C]}\mathcal{M} \otimes \mathcal{O}_C = H^0(C, N_{C/X}) \otimes \mathcal{O}_C$ and the map Te fits into the following map of sequences

$$
\begin{tikzcd}
0 \arrow[r] & T_C \arrow[r] \arrow[d, equal] & (T\mathcal{M})_{|C} \arrow[r] \arrow[d, "Te"] & N_{C/\mathcal{M}} \cong H^0(C, N_{C/X}) \otimes \mathcal{O}_C \arrow[r] \arrow[d, "ev"'] & 0 \\
0 \arrow[r] & T_C \arrow[r] & (T_X)_{|C} \arrow[r] & N_{C/X} \arrow[r] & 0
\end{tikzcd}
$$

where $ev : H^0(C, N_{C/X}) \otimes \mathcal{O}_C \to N_{C/X}$ is the evaluation of sections, [Ko, II.3.4, II.2.15].

By [Wie, 5.1] it is known that $N_{C/X} \cong E(1^a, 0^{2n-2a-2}, -1^a, -2)$ for some non-negative $a \leq n-1$, so the evaluation is injective at every point of C if and only if $a = 0$. This means that *(i)* and *(ii)* are equivalent.

The maps Tu and Du are surjective and injective everywhere, respectively. Therefore at every point $p \in C$ we have $\operatorname{rk} \sigma_{\mathcal{U}} = \operatorname{rk} \sigma_{\mathcal{M}}$, and clearly the kernel of $\sigma_{\mathcal{U}}$ contains $T_C \subset (T_{\mathcal{U}})_{|C}$. On the other hand $\sigma_{\mathcal{U}}$ factors through T_e and this means that $\operatorname{rk}(ev) \geq \operatorname{rk}(\sigma_{\mathcal{U}})$, hence *(iii)* implies *(i)*.

If $(T_X)_{|C} \cong E(2, 0^{2n-2}, -2)$ then we consider the composition

$$
(T_{\mathcal{U}})_{|C} \cong E(2, 0^{2n-2}) \xrightarrow{Te} E(2, 0^{2n-2}, -2) \xrightarrow{\sigma} E(2, 0^{2n-2}, -2) \xrightarrow{De} (\Omega_{\mathcal{U}})_{|C} \cong E(0^{2n-2}, -2)
$$

Since σ is an isomorphism and Te and De are injection and projection, respectively, we see that $\operatorname{rk} \sigma_{\mathcal{U}} = 2n-2$ hence *(ii)* implies *(iii)*. □

Conclusion of proof of 4.2. The condition *(ii)* is true for a general rational curve contracted by φ, [Wie, 5.1]. Let $\Sigma := \{\det \sigma_{\mathcal{M}} = 0\} \subset \mathcal{M}$ be the locus where the form $\sigma_{\mathcal{M}}$ is degenerate; in order to arrive to the contradiction we assume that Σ is non-empty; hence it is a divisor in $\mathcal{M}$. If $\mathcal{U}_\Sigma = u^{-1}(\Sigma)$ then by 4.3 the map e is not immersion exactly at $\mathcal{U}_\Sigma$. Since e is finite-to-one the rank of e on $\mathcal{U}_\Sigma$ is generically $2n-2$, [Ha, III.10.6], so for a generic C in any component of Σ we have $N_{C/X} \cong E(1, 0^{2n-4}, -1, -2)$. If $[C] \in \Sigma$ is a smooth point then we can write:

$$
\begin{tikzcd}
0 \arrow[r] & T_C \arrow[r] \arrow[d, equal] & (T_{\mathcal{U}_\Sigma})_{|C} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus(2n-3)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_C \arrow[r] \arrow[d, equal] & (T_{\mathcal{U}})_{|C} \arrow[r] \arrow[d, "Te"] & \mathcal{O}^{\oplus(2n-2)} \arrow[r] \arrow[d, "ev"] & 0 \\
0 \arrow[r] & T_C \arrow[r] & (T_X)_{|C} \arrow[r] & E(1, 0^{2n-4}, -1, -2) \arrow[r] & 0
\end{tikzcd}
$$

The composition of the right-hand-side vertical arrows $\mathcal{O}^{\oplus(2n-3)} \to E(1, 0^{2n-4}, -1, -2)$ is generically of rank $2n-3$ except the point $p = \operatorname{supp}(\mathit{coker}(\mathcal{O}^{\oplus(2n-3)} \to E(1, 0^{2n-4})))$ on C

where the rank drops to $2n-4$. Since the map φ contracts C it follows that at p the map $T(\varphi\circ e)$ has rank $\leq 2n-4$. Let us define $\widehat{\Sigma}\subset\mathcal{U}_\Sigma$ as the locus of points where $\mathrm{rk}(T(\varphi\circ e))\leq 2n-4$. Then $\widehat{\Sigma}$ dominates Σ via u and in fact the map $\widehat{\Sigma}\to\Sigma$ is generically one-to-one. Since points in $\mathcal{M}$, hence in Σ, parametrize components of fibers of φ it follows that $(\varphi\circ e)_{|\widehat{\Sigma}}$ is generically finite-to-one, so that $\dim(\varphi(e(\widehat{\Sigma})))=2n-3$ which contradicts [Ha, III.10.6].

This concludes the proof of 4.2.

4.2 Smoothness of a contraction.

The present section is devoted to reproving the following result by Demailly, Peternell and Schneider, [DPS, Thm. 5.2].

Theorem 4.4. *Let X be a smooth variety over the field of complex numbers. Suppose that ψ : $X\longrightarrow Y$ is a proper surjective morphism with connected fibers onto a normal variety Y such that*
(i) $-K_X$ is ψ-ample (that is ψ is a Fano-Mori contraction),
(ii) T_X is ψ-nef.
Then the variety Y as well as the morphism ψ are smooth.

The proof of the intermediate Pop. 5.1 in the proof of this theorem in [ibid] is not correct. Namely, let us take an elliptic curve E with the abelian group structure and an element $\varepsilon\in E$ such that $\varepsilon+\varepsilon=0$. On the product $E\times\mathbb{C}^2$ we define $\mathbb{Z}_2$ action by the involution $(e,z)\to(e+\varepsilon,-z)$ and take the quotient $X=(E\times\mathbb{C}^2)/\mathbb{Z}_2$. Then X admits a contraction to $Y=\mathbb{C}^2/\mathbb{Z}_2$ with all fibers being elliptic curves and the central fiber being of multiplicity 2 so that the pullback of any divisor from Y is singular along this fiber and thus the inductive argument from the proof of Prop. 5.1 from [ibid] fails.

Our proof of 4.4 will be performed in a sequence of lemmas and follows some ideas from [DPS] (see the proof of lemma 4.12); in addition however we use arguments about smoothing chains of rational curves, 4.8 (see proofs of 4.10 and 4.11). We note that Thomas Peternell has presented us recently with another proof of 4.4 related closely to the original argument from [DPS].

We begin with two lemmas concerning nef bundles on $\mathbb{P}^1$.

Lemma 4.5. *Let E be a nef bundle over $\mathbb{P}^1$. By $r^+(E)$ let us denote the rank of the maximal ample sub-bundle of E. If $E'$is a small deformation of E then $r^+(E')\geq r^+(E)$.*

Proof. We note that $r^+(E)=\mathrm{rk}(E)-h^0(E^*)$ so the claim follows by the semicontinuity of cohomology. ☐

Lemma 4.6. *Let $\mathbb{P}(E)$ be a projective bundle over $\mathbb{P}^1$, with E^* nef of rank n, Let us consider $B=D_1\cap\cdots\cap D_s$, where $D_i\in|\mathcal{O}_{\mathbb{P}(E)}(r_i)|$, for $i=1,\ldots,s$. Then the intersection $B_x:=B\cap\mathbb{P}(E)_x$ does not depend on the point $x\in\mathbb{P}^1$. In particular, if for some $x\in\mathbb{P}^1$ the intersection $B\cap\mathbb{P}(E)_x$ is, as a set, a linear subspace of $\mathbb{P}(E)_x$ of dimension $m-1$ then set theoretically $B=\mathbb{P}(E')\subset\mathbb{P}(E)$ where the bundle E' is a quotient of E by a trivial sub-bundle of rank $n-m$.*

Proof. First of all, let us consider a decomposition $E\cong\mathcal{O}_{\mathbb{P}^1}^{\oplus k}\oplus E_0$ where the dual E_0^* is ample. The two natural projections define two disjoint projective sub-bundles of $\mathbb{P}(E)$, that is $\mathbb{P}(E_0)$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k})$, where the latter sub-bundle is not unique. Obviously, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is negative over $\mathbb{P}(E_0)$ so $\mathbb{P}(E_0)\subset D_i$ for all i, so that over every point, $D_i\cap\mathbb{P}(E)_x$ is defined as the cone with vertex

$\mathbb{P}(E_0)_x$ over the divisor $D_i \cap \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k})_x$. Therefore it suffices to prove the lemma for the trivial bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k})$.

But over a trivial bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k}) = \mathbb{P}^1 \times \mathbb{P}^{k-1}$ the line bundle $\mathcal{O}(r_i)$ is just the pull-back of $\mathcal{O}_{\mathbb{P}^{k-1}}(r_i)$ via the second projection $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k}) \longrightarrow \mathbb{P}^{k-1}$. This implies that $D_i \cap \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus k})$ is constant on fibers over $\mathbb{P}^1$. The second part of the lemma follows immediately. □

Lemma 4.7. *For the proof of 4.4 one may assume that Y is affine and ψ is elementary, that is $\rho(X/Y) = 1$.*

Proof. The first part is clear because the conclusion of 4.4 concerns local behavior of both Y and ψ. Suppose now that 4.4 is true with the additional assumption that ψ is elementary. Take now an arbitrary $\psi : X \to Y$. By the cone theorem, [KM, 3.25], it factors into $\psi = \psi_2 \circ \psi_1$, where $\psi_1 : X \to X_1$ is elementary. Thus we can apply 4.4 for the elementary contraction ψ_1 and get X_1 and ψ_1 smooth. The tangent bundle T_{X_1} is ψ_2 nef because its pullback $\psi_1^* T_{X_1}$ is a quotient of T_X via the tangent map $T\psi_1 : T_X \to \psi^* T_{X_1}$. Moreover by [KMM] the divisor $-K_{X_1}$ is ψ_2 ample. Therefore $\psi_2 : X_1 \to Y$ satisfies conditions of 4.4 and $\rho(X_1/Y) < \rho(X/Y)$ so the second part of the lemma follows by the induction with respect to ρ. □

Thus from now on we assume that we are in the situation of 4.4 and moreover Y is affine and ψ is elementary. We recall that a closed set $Z \subset X$ is called rationally connected and, respectively, rationally chain connected if two generic points of Z can be joined by a rational curve or connected chain of rational curves, respectively, see [Ko, IV.3.2] for precise definition.

Lemma 4.8. *In the situation of 4.4 let $C = \bigcup_i C_i \subset \psi^{-1}(y)$ be a chain of rational curves contained in a (set-theoretic) fiber of ψ. Then there exists a smoothing of C with a (smooth) point of C fixed. In particular any closed subset of the fiber is rationally chain connected if and only if it is rationally connected.*

Proof. This is just [Ko, II.7.6.1]. □

Lemma 4.9. *In the situation of 4.4 let $C \subset \psi^{-1}(y)$ be a rational curve with the normalization $f : \mathbb{P}^1 \to C \subset X$. If C is a generic member of a family of rational curves which joins two generic points of an irreducible subset $Z \subset \psi^{-1}(y)$ then $r^+(f^* T_X) \geq \dim Z$.*

Proof. The property follows from looking at the derivative of the evaluation map of the Hom schemes, as in [Ko, II.3.5]: the rank of the derivative of the evaluation map of a family of curves with a (generic) point fixed is equal to $r^+(f^* T_X)$ and it has to be not smaller then the dimension of the target Z. □

Lemma 4.10. *In the situation of 4.4 the morphism ψ is equidimensional.*

Proof. The proof is similar to an argument from the first part of the proof of 2.12 from [KPSW]. Namely, we take a general fiber X_g of the morphism ψ and consider a component M of Chow(X) which contains the class of X_g. Let $u : U \to M$ be the universal family of cycles over M with the evaluation $e : U \to X$. We normalize M and pull-back U, if necessary, and call the resulting varieties and morphisms by the same names. The morphism e is birational and there exists a birational morphism $\kappa : M \to Y$ such that $\psi \circ e = \kappa \circ u$. If e is an isomorphism then κ is an isomorphism as well, because ψ is elementary; the morphism u is equidimensional so we are done in this case.

Now assume the contrary and suppose that e is not an isomorphism, which means that it admits a positive dimensional fiber, say over $x \in X$. Each fiber of u is rationally chain connected, [Ko, IV.3.5.2], and therefore the variety $Z(x) := e(u^{-1}(u(e^{-1}(x))))$ is rationally chain connected as well. Moreover $\dim Z(x) > \dim X_g = \dim X - \dim Y$ because otherwise all the cycles parametrized by $u(e^{-1}(x))$ would coincide. By 4.8 the set $Z(x)$ is actually rationally connected and by 4.9 it contains a rational curve C with the normalization $f : \mathbb{P}^1 \to C$ such that $r^+(f^*T_X) \geq \dim Z(x) > \dim X_g$. By 4.5 for any small deformation f_t of f the inequality $r^+(f_t^*T_X) > \dim X_g$ holds as well. On the other hand, since deformations of f dominate X so the image of some small deformation f_t is contained in a general fiber and therefore it f^*T_x has a trivial quotient of rank equal to $\dim Y$ (because the normal of a general fiber is trivial) so that $\dim X_g \geq r^+(f_t^*T_X)$, a contradiction. □

Lemma 4.11. *In the situation of 4.4 all fibers of the morphism ψ are irreducible.*

Proof. Since ψ is equidimensional so every fiber is rationally chain connected, [Ko, IV.3.5.2], so by 4.8 it is rationally connected, hence irreducible. □

Lemma 4.12. *In the situation of 4.4 every fiber of ψ with the reduced structure is a smooth Fano variety with a trivial normal bundle.*

Proof. We assume that Y is affine with the coordinate ring $A(Y) = H^0(X, \mathcal{O}_X)$. Let $y \in Y$ be a closed point with the maximal ideal $\mathfrak{m}_y \subset A(Y)$ generated by functions $g_1, \ldots, g_s$. Let F' be a scheme-theoretic fiber of ψ over y, with ideal sheaf $\mathcal{J}' := \psi^{-1}(\mathfrak{m}_y) \cdot \mathcal{O}_X$; that is, $\mathcal{J}'$ is generated by functions $g_i \in H^0(X, \mathcal{O}_X) = A(Y)$.

Let us consider $F = \mathrm{red}(F')$, the reduced structure of F', with ideal sheaf $\mathcal{J}$. We have an exact sequence:

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_X \otimes \mathcal{O}_F \longrightarrow \Omega_F \longrightarrow 0$$

If functions $f_1, \ldots, f_m \in \mathcal{O}_{X,x}$ generate the ideal $\mathcal{J}$ at a point $x \in F$ then their differentials $df_1, \ldots, df_m$ define linear divisors in $\mathbb{P}(\Omega_X \otimes k(x))$ and $\mathbb{P}(\Omega_F \otimes k(x)) \subset \mathbb{P}(\Omega_X \otimes k(x))$ is the intersection of such divisors.

Similarly, for any $g \in \mathcal{J}$ we can define a divisor in $\mathbb{P}(\Omega_X \otimes \mathcal{O}_F)$ is the following way: Let $r = r(g)$ be a maximal integer such that $g \in \mathcal{J}^r$. Then $D(g) \subset \mathbb{P}(\Omega_X \otimes \mathcal{O}_F)$ is defined by the product of differentials. More precisely, we have the morphism of $\mathcal{O}_F$ modules

$$\delta^r : \mathcal{J}^r/\mathcal{J}^{r+1} \longrightarrow S^r\Omega_X \otimes \mathcal{O}_F$$

such that for $f_i \in \mathcal{J}$ we set $\delta^r(f_1 \cdots f_r) = df_1 \cdots df_r \otimes 1$; because of the Leibniz rule the map is well defined and $\mathcal{O}_F$ linear, c.f. [Ei, Prop. 16.3]. Now, if locally $g = \sum f_{i_1} \cdots f_{i_r}$ then the divisor $D(g)$ is defined by the equation $\sum df_{i_1} \cdots df_{i_r} = 0$.

Let us set

$$Bs(\mathcal{J}') := \bigcap_{g \in \mathcal{J}'} D(g) = D(g_1) \cap \cdots \cap D(g_s) \subset \mathbb{P}(\Omega_X \otimes \mathcal{O}_F)$$

Then we have the immediate inclusion of sets $Bs(\mathcal{J}') \supset \mathbb{P}(\Omega_F)$ and moreover over a general point $x \in F$ these two sets coincide. Indeed, if we blow-up a general smooth point $x \in F \subset X$ and call the exceptional divisor E_x then we have set-theoretical equalities

$$Bs(\mathcal{J}')_x := Bs(\mathcal{J}') \cap \mathbb{P}(\Omega_X \otimes k(x)) = E_x \cap \widehat{F'} = E_x \cap \widehat{F} = \mathbb{P}(\Omega_F \otimes k(x))$$

where $\widehat{F}$ is the strict transform of F, and similarly for F'.

Let $F^0 \subset F$ denote the smooth locus of F over which $Bs(\mathcal{J}')_x = \mathbb{P}(\Omega_F \otimes k(x))$. We claim that $F^0 = F$. Since in F any two point of F can be joined by a chain or rational curves [Ko, IV.3.5.1], then it is enough to prove that given a rational curve which meets F^0 it has to be contained in F^0.

Let $f : \mathbb{P}^1 \longrightarrow X$ be such that $f(\mathbb{P}^1) \subset F$ and $f(\mathbb{P}^1) \cap F^0 \neq \emptyset$. If we set $E = f^*\Omega_X$ and $D_i = f^*D(g_i)$ then we are in the situation of 4.6. Thus $f^{-1}(Bs(\mathcal{J}'))$ is constant along $\mathbb{P}^1$ and a projective bundle. Since moreover it contains $\mathbb{P}(f^*\Omega_{F_i})$ and coincides with it at a point then $f^{-1}(Bs(\mathcal{J}')) = \mathbb{P}(f^*\Omega_F)$ and in addition $f^*\Omega_F$ is a sheaf of constant rank along $f(\mathbb{P}^1)$. Thus, by [Ha, II.8.15] F_i is smooth along $f(\mathbb{P}^1)$ and also $f(\mathbb{P}^1) \subset F_i^0$.

Having proved that F is smooth it remains to conclude that it is Fano and its normal bundle is trivial. To this end, by 4.6 the bundle $f^*N_{F/X}$ is trivial of rank $c = \operatorname{codim}_X F = \dim Y$ for any $f : \mathbb{P}^1 \to C \subset F$. Since the group of 1-cycles on F is generated by rational curves [Ko, IV.3.13] it follows that $c_1(N_{F/X})$ is trivial hence, by adjunction, $-K_F = (-K_X)_{|F}$ is ample. Also, by adjunction, we compute the anti-canonical divisor of the projectivisation $-K_{\mathbb{P}(N_{F/X})} = \mathcal{O}_{\mathbb{P}}(c) - K_F$. Since $N_{F/X}$ is nef, as the quotient of $T_X \otimes \mathcal{O}_F$ and $-K_F$ is ample, it follows that $\mathbb{P}(N_{F/X})$ is Fano and by Kawamata-Shokurov base-point-freeness the bundle $N_{F/X}$ is semi-ample. Now we conclude, like in the proof of [AW, 1.2], that $N_{F/X}$ is trivial. □

Now the theorem 4.4 can be concluded either as in [DPS], or by the following, somewhat more general result.

Lemma 4.13. *Let $\psi : X \to Y$ be a Fano-Mori contraction. Suppose that the reduced fiber over $y \in Y$ is smooth and its normal bundle is trivial. Then Y is smooth at y and the morphism ψ is smooth along F.*

Proof. Again, we may assume that Y is affine with coordinate ring $A(Y)$, so that $H^0(X, \mathcal{O}_X) = A(Y)$. By c we denote the codimension of $F = \operatorname{red}\varphi^{-1}(y)$ in X. Let $\beta : X' \to X$ be the blow-up of F, with the exceptional divisor E, and consider the composition $\psi' = \psi \circ \beta$. The relative hyperplane line bundle $\mathcal{O}_E(1) = -E_{|E}$ is spanned by c global sections and moreover $-K_{X'} - 2E = \mathcal{O}_E(c+1) - \beta^*K_X$ is ample on E hence it is ψ' ample. Thus, by Kawamata-Viehweg vanishing, $H^1(X, -2E) = 0$ and, by standard cohomology sequence, any section of $-E_{|E}$ extends to X' and descends to a function on Y vanishing at y. In other words, using the notation from the proof of 4.12, we have functions $g_1, \ldots, g_c \in \mathfrak{m}_y$ whose pullbacks generate ideal sheaf $\mathcal{J}$ of F at every point. Since $\mathcal{J} \supset \mathcal{J}' = \psi^{-1}(\mathfrak{m}_y) \cdot \mathcal{O}_X$ we have equality $\mathcal{J}' = \mathcal{J}$ and $g_1, \ldots, g_c$ generate $\mathfrak{m}_y$. By the semicontinuity $c \leq \dim Y$ hence $c = \dim Y$ and $g_1, \ldots, g_c$ are regular generators of $\mathfrak{m}_y$. Thus Y is smooth at y and moreover the morphism ψ is smooth by [Ha, III.10.4]. □

This concludes the proof of Theorem 4.4.

References

[APW] Ancona, V., Peternell, T., Wiśniewski, J.A., Fano bundles and splitting theorems on projective spaces and quadrics, *Pacific J. Math.* **163** (1994), 17–42.

[AW] Andreatta, M., Wiśniewski, J.A.: On manifolds whose tangent bundle contains an ample subbundle, Inv. Math.

[Be] Beauville, A., Riemannian Holonomy and Algebraic Geometry. preprint math.AG/9902110.

[CP] Campana, F., Peternell, T.: Projective manifolds whose tangent bundles are numerically effective, *Math. Ann.* **289** (1991), 169–229.

[CS] Cho, K and Sato, E: Smooth projective varieties with the ample vector bundle $\bigwedge^2 T_X$ in any characteristic, *J. Math. Kyoto Univ.* **35**, 1, 1-33, 1995.

[DPS] Demailly, J.P.; Peternell, T. and Schneider, M.: Compact complex manifolds with numerically effective tangent bundles, *J. Alg. Geom.* **3**, 295-345, 1994.

[Ei] Eisenbud, D., *Commutative Algebra with a view towards Algebraic Geometry*, Springer Verlag 1995.

[F] Fu, B., Symplectic resolutions of nilpotnent orbits. preprint, math.AG/0205048.

[Fu] Fulton, W., *Intersection Theory*, Springer Verlag 1984.

[Ha] Hartshorne, R.: *Algebraic Geometry*, Springer Verlag 1977.

[It] Iitaka, *Algebraic Geometry*, Springer Verlag.

[Ka] Kaledin, D.: Symplectic resolutions: deformations and birational maps, *preprint*, 2000.

[Ke] Kebekus, S.:Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron, *Complex Geometry (Göttingen, 2000)*, 147-155, Springer, Berlin, 2002.

[KPSW] Kebekus, S.; Peternell, T.; Sommese, A.J. and Wiśniewski, J.A.: Projective contact manifolds, *Invent. Math.* **142**, 1-15, 2000.

[Ko] Kollár, J.: *Rational curves on algebraic varieties*, Springer-Verlag 1996.

[KMM] Kollár, J., Miyaoka, Y., Mori, S., Rational connectedness and boundness of Fano manifolds, *J. Diff. Geom.* **36** (1992), 765–769.

[KM] Kollár, J. and Mori, S.: *Birational geometry of algebraic varieties*, Cambridge University Press 1998.

[LB] Le Brun, C.: Fano manifolds, contact structures and quaternionic geometry, *Int. Journ. of Math.* **6**, no. 3, 419-437, 1995.

[Mo] Mori, S.: Projective manifolds with ample tangent bundles, *Annals of Maths.* **110**, 593-606, 1979.

[Mk] Mok, N.: The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bisectional curvature, *J. Diff. Geom.* **27**, 179-214, 1989.

[Pe] Peternell, T.: Manifolds of semi-positive curvature, Transcendental methods in algebraic geometry (Cetraro, 1994), 98–142, *Lecture Notes in Math.*, 1646, Springer, Berlin, 1996.

[Ran] Ran, Z.: Hodge theory and deformation of maps, *Comp. Math.* **97**, 309-328, 1995.

[Sa] Sato, E.: Varieties which have two projective space bundle structures, *J. Math. Kyoto Univ.* **25**, no. 3, 445-457, 1985.

[SY] Siu, Y.T. and Yau, S.T.: Compact Kähler manifolds with positive bisectional curvature, *Inv. Math.* **59**, 189-204, 1980.

[SW] Szurek, M., Wiśniewski, J.A., Fano bundles on $\mathbf{P}^3$ and $\mathbf{Q}_3$, *Pacific J. Math.* **141** (1990), 197–208.

[Wa] Wahl, J. A cohomological characterization of $\mathbb{P}^n$, *Inv. Math.* **72** (1983), 315–322.

[Wie] Wierzba, J.: Contractions of symplectic varieties, *preprint* arXiv:math.AG/9910130v3, to appear in *J. Alg. Geom.*

[Wie2] Wierzba, J.: On 4-dimensional isolated symplectic singularities and a characterization of quadrics, *preprint* 2000.

[W1] Wilson, P.M.H., The Kähler cone on Calabi-Yau manifolds, *Inv. Math* **107** (1992), 561–583; Erratum *Inv. Math.* **114** (1993), 231–233.

[W2] Wilson, P.M.H., Symplectic deformations of Calabi-Yau threefolds, *J. Diff. Geom.* **45** (1997), 611–637.

[Wiś] Wiśniewski, J.A.: Uniform vector bundles on Fano manifolds and an algebraic proof of Hwang-Mok, *Complex Geometry (Göttingen, 2000)*, Springer, Berlin, 2002.

[Ye] Ye, Y., Extremal rays and null geodesics on a complex conformal manifold, *Inter. J. Math.* **5** (1994), 141–168.

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