Linearization Problems

Mariusz Koras and Peter Russell [1]

Abstract

In the paper we give a survey of known results on The Linearization Problem, both the local and global one. We pay special attention to reductive group actions on three dimensional affine space and, in particular, to the (in dimension 3) hardest case, torus actions.

1. Local linearization

Let k be an algebraically closed field. We recall that an algebraic group is called linearly reductive if and only if all its representations are direct sums of simple representations. Let a linearly reductive algebraic group G acts on a smooth algebraic variety X and let $p \in X^G$. We have the induced linear action on the tangent space T_pX. A. Bialynicki-Birula proved the following "local etale linearization theorem"

1.1 **Theorem**([BB1]):There exists an open G-invariant neighborhood U of p and an etale G-invariant morphism $\varphi : U \to T_pX, \varphi(p) = 0$.

Proof (different from the original one). By Sumihiro's Theorem [Sum] there exists an open quasi-affine G-invariant neighborhood W of p. Let $\psi_1, \cdots, \psi_n$ be a system of local parameters at p, with all functions defined on W. Let $\psi : W \to T_pX$ be given by $\psi(x) = (\psi_1(x), \cdots, \psi_n(x))$. We may assume that $d\psi(p) : T_pX \to T_pX$ is the identity. Let $x \in W$. Consider the function $\psi_x(g) = g(\psi(g^{-1}x))$, $\psi_x : G \to T_pX$. Let $\varphi(x) = R(\psi_x)$ where R denotes the Reynolds operator on G (see 1.2). Then $\varphi : W \to T_pX$ is G-invariant and $d\varphi(p) : T_pX \to T_pX$ is the identity map. Hence φ is etale at p.We put $U = \{x \in W : \varphi \text{ is etale at } x\}$.

1.2 **Reynolds operator**

Consider the action of G on $k[G]$, the algebra of regular functions, by translations: $g(f)(h) = f(g^{-1}h)$. It is well known that $k[G]$ is a union of finite dimensional G-modules. Let A be the subspace of G-invariant functions, i.e. A consists of constant functions. Since G is linearly reductive there exists a G-invariant subspace B such that $k[G] = A \oplus B$. The Reynolds operator on G is the projection of $k[G]$ onto A. This is an analogue of the Haar measure on a compact group.

1.3 **Corollary**: The fixed point set X^G is smooth.

Indeed , X^G is locally etale isomorphic to the fixed point space in the tangent space .

1.4.There exists a holomorphic version of the local linearization.

Theorem([K1]): Let G be a connected reductive complex Lie group that acts analytically on a compact Kähler manifold Y. Let $p \in Y^G$. Then there exists an open G-invariant neighborhood U of p and a G-equivariant analytic isomorphism $\varphi : U \to \varphi(U) \subset T_pX$, $\varphi(p) = 0$.

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The idea of the proof : Let K be the maximal connected compact subgroup of G. We may find a K-invariant open coordinate neighborhood $V$of p. We consider the map $V \to T_pX$ given by the coordinates and apply the averaging trick just as in the proof of Theorem 1.1, using the Haar measure on K in place of the Reynolds operator. Taking a smaller V if necessary we may assume that there exist local coordinates on V such that K acts linearly with respect to them. Hence we obtain a K-equivariant isomorphism $\alpha : V \to \varphi(V) \subset T_pX$. Now we want to extend that map to to a G-equivariant map $GV \to T_pX$. We put $\varphi(gx) = g(\alpha(x))$ for $gx \in GV$. This definition is correct if the following holds: for every $y \in Y$ the set $Gy \cap V$ is connected. The main result of [K1] says that we may find arbitrarily small V which satisfy this property.

1.5. **Corollary**: Let G be a reductive, connected complex algebraic group. Suppose that G acts algebraically on a smooth algebraic variety X. Let $p \in X^G$ and let $\varphi : U \to T_pX$ be the map constructed as in the proof of Theorem 1.1. Then there exists a G-invariant subset W, $p \in W$, open in the analytic topology such that $\varphi : W \to \varphi(W)$ is an analytic isomorphism.

2. Global linearization

Let k be field. We denote by $k^{[n]}$ the polynomial ring in n variables over k and by $\mathbf{A}_k^n$ affine n-space over k. Despite tremendous efforts to elucidate its structure, the group $A_n = Aut_k\, k^{[n]}$ (or $Aut_k \mathbf{A}_k^n$) remains mysterious.

2.1 **Key problem**: Find "normal forms" for actions of algebraic groups G on $\mathbf{A}_k^n$.

2.2 **Examples** of groups G we have in mind are
- (i) finite groups,
- (ii) multiplicative groups $k^*, (k^*)^r$,
- (iii) linearly reductive groups,
- (iv) additive groups k^+, $(k^+)^r$,
- (v) unipotent groups (e.g., upper triangular, all eigenvalues 1).

2.3 We note that (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iii) if $|G|$ is prime to *char* k. Also (iv) $\Rightarrow$ (v). Moreover

2.4 If we choose coordinates, i.e. write

$$k^{[n]} = k[x_1, \cdots, x_n],$$

then

$$SL_n \subset GL_n \subset Af_n \subset A_n$$

are defined. (Af_n is the group of *affine transformations.*) If *char* $k = 0$, SL_n and GL_n satisfy (iii).

We also define the *triangular group*

2.5

$$\Delta_n \subset A_n$$

as the group of transformations

$$x_i \mapsto \alpha_i x_i + \varphi_i(x_1, \cdots, x_{i-1}),$$

$\alpha_i \in k^*,\ \varphi_i$ a polynomial.

2.6.**Linearization Problem**: Let the algebraic group G act on $\mathbf{A}_k^n$. Does there exist a system of algebraic coordinates on $\mathbf{A}_k^n$ such that G acts by means of linear transformations in these coordinates?

A different choice of coordinates leads to conjugation in A_n. Hence the problem of linearization is the problem of whether a given algebraic subgroup of A_n can be conjugated into GL_n.

3. The situation is well understood in dimension 2. (Dimension 1 is left as an exercise.)

3.1 **Automorphism Theorem** (Jung [J], Van der Kulk [VdK], Nagata [N]):
A_2 is an amalgamated product

$$A_2 = Af_2 *_{L\Delta_2} \Delta_2, \qquad L\Delta_2 = Af_2 \cap \Delta_2.$$

We have the following general theorem about amalgamated products.

3.2 **Theorem** (Serre [Se]): Any subgroup $H \subset A *_U \Delta,\ U = A \cap \Delta,\ H$ of *bounded length*, is conjugate to a subgroup of A or Δ.

As an immediate consequence of 3.1 and 3.2 one finds (see [Wr])

3.3 **Theorem**: Every algebraic subgroup of A_2 is conjugate to a subgroup of Af_2 or Δ_2.

3.3.1 **Corollary**: Actions of linearly reductive groups on $\mathbf{A}_k^2$, in particular of k^* and of finite groups G, $|G|$ prime to *char* k, are linearizable.

3.3.2 In positive characteristic we have the following instructive example ([KP]):
char $k = 2,\ G = SL_2$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we put $g^{(2)} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$ and define an action of G on $\mathbf{A}_k^2$ by $g \cdot x = g^{(2)}x + \begin{pmatrix} ab \\ cd \end{pmatrix}$. This action has no fixpoint and cannot be linearized.

3.3.3 **Corollary**: $(k^+)^r$ actions on $\mathbf{A}_k^2$ are conjugate to Δ_2-actions.

4. We now assume $n \geq 3$ and (unless otherwise specified) k algebraically closed of characteristic 0. We begin with the following instructive

4.1 **Example** (Bass [Ba]): On $k[x, y, z]$ consider the k^+-action

$$\eta_t : \begin{array}{l} x \to x \\ y \to y + tz^2x \\ z \to z \end{array} \qquad\qquad t \in k^+$$

Consider also the automorphism

$$\varepsilon: \begin{array}{l} x \to x + z^{-1}y^2 \\ y \to y \\ z \to z \end{array}$$

of $k(z)[x,y]$. It is easy to verify that $\varphi_t = \varepsilon\eta_t\varepsilon^{-1} \in A_3$ (denominators cancel). Hence φ_t defines a $G = k^+$-action on $\mathbf{A}_k^3 = X$. We calculate that the fixpoint set X^G is the zero locus of $zx + y^2$, and it follows that φ_t is not conjugate to a triangular action since for triangular actions the fixed point set is of the form $Z \times \mathbf{A}_k^1$, where Z is union of lines or points; in particular it is smooth. The classification of k^+-actions is far from complete even for $n = 3$, and a very active area of research. (See [DF], [Fr] for a small sample.)

Let us now consider an action of a reductive group G on $X = \mathbf{A}_k^n$. We then have

4.2

$$\begin{array}{ccc} X^G & \subset & X \\ & \searrow & \downarrow \\ & & X//G \end{array}$$

Here X^G is the set of fixpoints and $X//G$ is the "categorical quotient", an affine variety that parametrizes the *closed* orbits of the action. Its ring of regular functions is the ring of regular G-*invariant* functions on X.

4.3 **Definition**: We call the action *fixpointed* if and only if

$$X^G = X//G,$$

i.e. the only closed orbits are the fixpoints, and hence every orbit has a fixpoint in its closure.

The following was proved by Kambayashi-Russell for $G = (k^*)^r$ [KaR] and in general by Bass-Haboush [BH1].

4.4 **Theorem**: For a fixpointed action of G on X, $X \to X//G = X^G$ is a vectorbundle with *linear* action of G on the fibres.

Remark . In case $k = \mathbf{C}$ it follows from 1.5 that there exists a covering of X by G-invariant coordinate subsets open in the analytic topology on which G acts linearly. It follows that the map $X \to X^G$, $x \to \overline{Gx} \cap X^G$, is a locally (in the analytic topology) trivial vector bundle with a representation of G as the fibre. One can show that it is locally trivial in the Zariski topology.

Note that $X//G = X^G$ is a retract of $X = \mathbf{A}_k^n$. We have the famous result of Quillen [Q] and Suslin [S] on the Serre-conjecture

4.5 **Theorem**: Vectorbundles over a retract of $\mathbf{A}_k^n$ are trivial,

Hence

4.6 **Theorem**: In the situation of 4.4

(i) $X \simeq_e X^G \times V$, where V is a (linear) representation of G,
(ii) the action is linearizable $\Leftrightarrow X^G \simeq \mathbf{A}_k^m$ for some m.
Here "$\simeq_e$" stands for *equivariantly isomorphic.*

The problem appearing in 4.6(ii), known as the *cancellation problem*, is quite unsolved for $m \geq 3$. The solution is relatively straightforward for $m = 1$. For $m = 2$ it is difficult and was given by Miyanishi-Sugie [MS] and Fujita [F1].

4.7 **Cancellation Theorem**: If
$$Y \times \mathbf{A}_k^d \simeq \mathbf{A}_k^{d+2},$$
then $Y \simeq \mathbf{A}_k^2$.

Sketch of the proof. We have the projection $\mathbf{A}_k^{d+2} \to Y$. Hence we have a dominant map $H \to Y$ for a generic plane $H \subset \mathbf{A}_k^{d+2}$. It follows that the logarithmic Kodaira dimension $\overline{k}(Y)$ equals $-\infty$. The famous result of Miyanishi-Sugie [MS] and Fujita[F1] now implies that Y contains a cylinderlike open subset, i.e. a set isomorphic to $C \times \mathbf{A}_k^1$, C a curve. Thus we obtain a rational map $f : Y \to C$ with $\mathbf{A}_k^1$ as the generic fibre. Suppose that the map f is not defined at a point q. Then q belongs to the closure of the generic fibre $f^{-1}(c)$ and that closure is a complete curve contained in Y. But Y is affine. Therefore f is regular on Y. Since Y is rational the curve C is also rational. It is clear that $Pic(Y) = 0$ and that invertible functions on Y are constant. From that we infer easily that $C \simeq \mathbf{A}_k^1$ and that every fibre of f is irreducible and has multiplicity 1. It is well known fact that the total space of an $\mathbf{A}_k^1$ fibration over $\mathbf{A}_k^1$ is isomorphic to $\mathbf{A}_k^2$.
So we find

4.8 **Theorem**: Fixpointed actions on $X = \mathbf{A}_k^n$ of a reductive group G are linearizable if $\dim X^G \leq 2$.

5. Research into actions on $X = \mathbf{A}_k^n$ of a (connected) reductive group G had long centered on a conjecture of Kambayashi [Ka], suggesting that such actions are linearizable. There were some further positive results.

5.1 **Theorem** (Kraft-Popov [KP], Panjushev [P]): Actions of a semi-simple group G on $\mathbf{A}_k^3$ and $\mathbf{A}_k^4$ are linearizable.

5.2 **Theorem** (Bialynicki-Birula [BB2]): Actions of $(k^*)^r$ on $\mathbf{A}_k^n$ are linearizable for $r \geq n - 1$.

So a quite new flavour was added to this field when Schwarz [Sc] discovered counterexamples to the conjecture by constructing non trivial G-vectorbundles over X.

5.3 **Definition**: By saying
$$p : E \to X$$
is a *G-vectorbundle* over X we mean:
(i) E is a vectorbundle;
(ii) E, X are G-spaces and p is G-equivariant;

(iii) for $x \in X,\ g \in G$ and $y = gx$, the map on fibres

$$g_x : E_x \to E_y$$

is linear.

Just as for vectorbundles, the usual constructions of linear algebra work for G-vectorbundles. We call (equivariantly) *trivial* the G-vectorbundles of the form $X \times W \to X$, where W is a representation of G and we act diagonally on $X \times W$. The starting point now is

5.4 **Theorem**: (Bass-Haboush [BH2]): Let G be reductive.

(i) G-vectorbundles over a (linear) representation $V(= X)$ of G are (equivariantly) stably trivial.

(ii) A non-trivial G-vectorbundle E over V gives a non-linearizable $k^* \times G$-action on $E \simeq \mathbf{A}_k^m$.

Proof of (ii). As a variety E is isomorphic to an affine space. We have an action of $\hat{G} = k^* \times G$ on E, where k^* acts by scalar multiplication on the fibres of E. Suppose that the $\hat{G}$-action on E is linearizable. Thus there exists a $\hat{G}$-equivariant isomorphism $F : E \to \mathbf{A}_k^m$, where $\hat{G}$ acts linearly on $\mathbf{A}_k^m$. We identify V with the zero section of E. Clearly V is the fixed point set of the k^*-action on E and also V is $\hat{G}$-invariant. The set $F(V)$ is also $\hat{G}$-invariant. Therefore, since $\hat{G}$ is reductive, there exists a $\hat{G}$-invariant linear subspace W of $\mathbf{A}_k^m$ such that $\mathbf{A}_k^m = F(V) \oplus W$. Take $x \in V$. It is clear that the fibre E_x is the set all points $z \in E$ such that $x \in \overline{k^* z}$. Since the tangent representations of $\hat{G}$ at y and at $F(y)$ are isomorphic for every $y \in V$, k^* acts on W by scalar multiplication. Therefore $\{u \in \mathbf{A}_k^m : F(x) \in \overline{\hat{G}u}\} = F(x) \times W$. It follows that F maps E_x onto $F(x) \times W$. Since $F|_{E_x}$ is equivariant with respect to the scalar multiplication by k^*, it is a linear map. The statement follows.

Let V_i denotes the irreducible representation of $SL_2(\mathbf{C})$ of dimension $i+1$ (V_i is the space of homogeneous forms of degree i in two variables). In [Sc] G.Schwarz proved the following

5.5.**Proposition**: For every $i \geq 3$ there exist infinitely many non-isomorphic SL_2-vector bundles E on V_2 whose zero fibre E_0 is SL_2-isomorphic to V_i.

Hence, by Theorem 5.4, there are nonlinerizable $\mathbf{C}^* \times SL_2$-actions on $\mathbf{C}^{\mathbf{n}}$ for every $n \geq 7$. Schwarz also showed that the induced SL_2-actions are also nonlinearizable ($n \neq 8$).

5.5 **Examples** of non-trivial G-vectorbundles, and hence of non-linearizable G-actions on $\mathbf{A}_k^n$, were constructed (among others) by

(1) Schwarz [Sc] for the orthogonal group $O_2,\ n = 4$;

(2) Masuda, Moser-Jauslin, Petrie [MM-JP1] for certain finite groups (e.g. G dihedral of order $\geq 10,\ n = 4$).

(3) Moser-Jauslin,Freudenburg [MF] for S_3 ,$n = 4$.

(4) Knop [Kn], for any non-commutative connected G, for suitable n.

All examples of non-linearizable actions of reductive groups known so far are of this type. The next result shows that the restriction to non-commutative G in 5.5(4) is essential.

5.6 **Theorem** ([MM-JP2]): For commutative reductive G, all G-vectorbundles over a represenation of G are trivial.

For completeness let us also mention

5.7 **Theorem** (Derksen, Kutschebauch [DK]): For reductive G, all holomorphic G-vectorbundles over a representation of G are holomorphically trivial

6. Torus actions

. Torus actions ($(k^*)^r$-actions) on $\mathbf{A}_k^n$, though group-theoretically very simple, pose rather surprising difficulties of a geometric nature, the difficulties seeming to increase rapidly with the codimension $s = n - r$. In view of 5.2, the case $s = 2$ is to be considered first. Since we use some topological arguments , we assume that $k = \mathbf{C}$.

6.1. Suppose that a r- dimensional torus $T = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$ acts algebraically on a smooth algebraic variety X. Let $p \in X^T$ be a fixed point. The tangent action of T on T_pX can be diagnolized, i.e. in some coordinate system T acts as follows:

$(t_1, \cdots, t_r)(x_1, \cdots, x_n) = (t_1^{a_{11}} \cdots t_r^{a_{r1}} x_1, \cdots, t_1^{a_{1n}} \cdots t_r^{a_{rn}} x_n)$ where the a_{ij} are integers. The vectors $(a_{1i}, \cdots, a_{ri})$, $i = 1, \cdots, n$ are called *the weights* of the action at p. The weights do not depend on a choice of coordinate system on T_pX and they do not change along a connected components of the set X^T. The weights depend on a choice of coordinates on T.

6.2. **Lemma**: Let a torus T acts on an affine variety X . There exist infinitely many prime numbers p such that $X^T = (\mathbf{C}^n)^{\omega_p}$, where ω_p is a subgroup of T of order p.

Proof. It is well known that one can embed X equivariantly into an affine space with a linear T-action. Hence we may assume that $X = \mathbf{C}^n$ and that T acts linearly. Let $a_i = (a_{1i}, \cdots, a_{ri})$, $i = 1, \cdots, n$ be the weights (at 0) of the action. We may assume that $a_i \neq 0 \Leftrightarrow i > s$. There exists an integral vector $(1, N_2, \cdots, N_r)$ such that $b_{1i} = a_{1i} + a_{2i}N_2 + \cdots + a_{ri}N_r \neq 0$ for $i > s$. Now consider the new system of coordinates $(u_1, \cdots, u_r)$ on T given by $t_1 = u_1, t_2 = u_1^{N_2} u_2, \cdots, t_r = u_1^{N_r} u_r$. The weights of the action with respect to the new coordinates are $b_i = (b_{1i}, \cdots, b_{ri})$, where b_{1i} is as above. Now let p be a prime such that p does not divide any b_{1i}, $i = 1, \cdots, s$. Let ω_p be the subgroup of T generated by $(\varepsilon, 1, \cdots, 1)$ where ε is a $p-th$ root of 1. Then $(\mathbf{C}^n)^T = (\mathbf{C}^n)^{\omega_p}$.

6.3.**Proposition**: Let a torus $T = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$ acts algebraically on an $\mathbf{Q}$-acyclic smooth irreducible affine variety X. Then the fixed point set X^T is nonempty and connected. Moreover X^T is $\mathbf{Z/pZ}$-acyclic for infinitely many primes p.

Proof. Suppose that $X^T = \emptyset$. Then every orbit of T is closed in X and is isomorphic to a torus T' . By the Luna Slice Theorem [Lu], X is an union of disjoint, locally closed subsets and each of them has a structure of a locally (in the analytic topology) trivial torus fibration. Hence the Euler characteristic χ of each of them equals 0 which gives $\chi(X) = 0$; contradiction. Let ω_p be as in the lemma 6.2. We may assume that X is $\mathbf{Z/pZ}$-acyclic since we have infinitely many choices for p and the $\mathbf{Z}$-homology groups of X are finitely generated. By Floyd's theorem ([Fl],thm.4.4), $\sum_{i\geq 0} h_i(X^{\omega_p}) \leq \sum_{i\geq 0} h_i(X) = 1$, where $h_i(W)$ denotes the dimension over $\mathbf{Z/pZ}$ of the $i-th$ homology group with coefficients in $\mathbf{Z/pZ}$ of a space W. From that $h_0(X^T) = 1$ and $h_i(X^T) = 0$ for $i \geq 1$. The proposition is proved.

The following (modest looking) result was established recently, with contributions by many researchers. It draws on a considerable part of the repertoire of methods and results in "affine algebraic geometry" established over the last three decades. See [KKM-LR], [KR1], [KR2], [KR3], [Kr], [KaR].

6.4 **Theorem**: $T = \mathbf{C}^*$ actions on $X = \mathbf{C}^3$ are linearizable.

We describe the major steps in the proof. We consider three cases according to the dimension of the fixpoint set. Let q be a fixed point and let a_1, a_2, a_3 be the weights of the action at q.

I. $a_i \geq 0,\ i = 1, 2, 3$. Then the action is fixpointed and dim $X^T \leq 2$ (dim X^T = number of i such that $a_i = 0$). We are done by 4.8.

II. $a_1 = 0,\ a_2 < 0,\ a_3 > 0$.

We can embed X equivariantly as a closed subvariety of $\mathbf{C}^n$ for some n with T acting linearly on $\mathbf{C}^n$. In $\mathbf{C}^n$ we have linear subspaces $(\mathbf{C}^n)^+$ and $(\mathbf{C}^n)^-$ on which T acts diagonally with non-negative and non-positive weights respectively, and

$(\mathbf{C}^n)^+ \cap (\mathbf{C}^n)^- = (\mathbf{C}^n)^T$

Let $Y = X^T$. We define $Y^+ = (\mathbf{C}^n)^+ \cap X$ and $Y^- = (\mathbf{C}^n)^-$. More directly $Y^+ = \{x \in X : \lim_{t\to 0} tx \in Y\}$. Similarly one defines Y^-. Thus Y^+ and Y^- are closed subvarieties in X. It follows from 1.1 that they are locally trivial (even in the Zariski topology) fibrations over Y with $\mathbf{C}^1$ as a fibre. By 6.3, Y is $\mathbf{Q}$-acyclic ; it follows that $Y \simeq \mathbf{C}^1$. Hence Y^+ and Y^- are isomorphic to the plane $\mathbf{C}^2$. Let $F = 0$ be the equation of Y^+. From 1.5 it follows that F is semi-invariant of weight a_2, i.e. $F(tx) = t^{a_2}F(x)$ for every $x \in X$. Let $G = 0$ be the equation of Y^-. G is semi-invariant of weight a_3. Let H_s denotes the s-level of a function H, i.e. $H_s = \{x \in X : H(x) = s\}$. We have $F_0 \cap G_0 = Y$. By the Abhyankar-Moh-Suzuki (AMS) Theorem [AM] (see 7.5), $L = F_1 \cap G_0 \simeq \mathbf{C}^1$. It is clear that F_1 is ω_{a_2}-invariant, where ω_{a_2} is the cyclic subgroup of T of order a_2. It is also clear that $X//T \simeq F_1//\omega_{a_2}$. The quotient $X//T$ is smooth, it has logarithmic Kodaira dimension $-\infty$, invertible functions on it are constant and it is not difficult to show that $Pic(X//T) = 0$. All that implies that $X//T \simeq \mathbf{C}^2$, see 4.7. The image of L under the quotient map $\alpha : F_1 \to F_1//\omega_{a_2} = \mathbf{C}^2$ is a coordinate line by the AMS Theorem. Hence the complement of $\alpha(L)$ is isomorphic to $\mathbf{C}^1 \times \mathbf{C}^*$. Since α is unramified outside L, the complement of L in F_1 is also isomorphic to $\mathbf{C}^1 \times \mathbf{C}^*$. It follows that F_1 is isomorphic to $\mathbf{C}^2$.

Now consider the linear action on $\mathbf{C}^3$ defined by $t(x, y, z) = (x, t^{a_2}, t^{a_3})$. Choose an ω_{a_2}-equivariant isomorphism $\varphi : F_1 \to \mathbf{C} \times \{1\} \times \mathbf{C}$. We extend φ to $\mathbf{C}^*$-equivariant isomorphism $\phi : X - F_0 \to \mathbf{C}^3 - \{x = 0\}$, putting $\phi(z) = t^{-1}\varphi(tz)$, where t is such that $tz \in F_1$. Since φ is ω_{a_2} equivariant, this definition does not depend on the choice of t. Using local analytic coordinates (see 1.5) one shows easily that ϕ extends to an equivariant isomorphism $X \to \mathbf{C}^3$.

III. $a_1 < 0,\ a_2, a_3 > 0$ (the "hyperbolic" case).

We have the following fundamental result

6.4.1 **Theorem**([KR3]): $S = X//T \simeq T_qX//T \simeq \mathbf{C}^2/\omega_{a_1}$.

The proof is long and technical. The following characterization of a quotient of the plane by a cyclic group action is used

6.4.2 **Theorem**([K2]): Let Z be an affine, normal surface with exactly one fixed point x_0 whose analytic type is that of the origin in $\mathbf{C}^2/\omega_a$. Assume that $\overline{k}(Z)$, the logarithmic Kodaira dimension of Z, equals $-\infty$, that $Z - x_0$ is not of logarithmic general type, i.e. $\overline{k}(Z - x_0) < 2$, and that Z is topologically contractible. Then Z is isomorphic to $\mathbf{C}^2/\omega_a$.

It is not hard to prove that S is contractible. From the Kawamata Addition Theorem it follows that $\overline{k}(S) = -\infty$ (one can use also the fact that for generic plane H in $\mathbf{C}^3$ we have a dominant map $H \to S$). The problem is to show that the smooth locus of S is not of general type. That fact is proved in [KR3]. In case $\overline{k}(Z - x_0) = -\infty$, the proof of 6.4.2 uses a fundamental results of Miyanishi-Tsunoda [MT1], [MT2].

Theorem 6.4.1 is the crucial step in the proof of 6.4. However linearization does not follow from it as easily as it was mistakenly stated in [KR4] . In fact the mistake led us to investigation a more general class of $\mathbf{C}^*$-varieties.

6.4.3 **Definition**: A smooth, contractible, affine variety X of dimension 3 with a hyperbolic T-action is called an LQ-variety if $X//T \simeq T_qX/T$, where q is the unique (by 6.3) fixpoint for T.

A complete description of all LQ-varieties is given in [KR2]. They all are hypersurfaces in $\mathbf{C}^4$ invariant with respect to a linear action of T. Also the defining equations are of some special type.

6.4.4 **Remark**: LQ-varieties are $\mathbf{Z}$-acyclic. Hence, in view of [F2], they are factorial and invertible functions on them are constant.

6.4.5 Consider an LQ-variety X. Let, as above, the weights of the T-action at the fixed point q be a_1, a_2, a_3, where $a_1 < 0$, $a_2, a_3 > 0$. Let $X^+ = \{x \in X : \lim_{t\to 0} tx = q\}$. Similarly we define X^-. Then X^+ is isomorphic to $\mathbf{C}^2$ and X^- is isomorphic to $\mathbf{C}^1$ (see 6.4 II). Let $(F, G, H) : X \to T_qX \simeq \mathbf{C}^3$ be a T-equivariant map etale at q, where T acts on T_qX diagonally. By 1.5 that map is an analytic isomorphism on a T-invariant neighborhood U of q. We may assume that F, G, H are irreducible (see 6.4.4). Then $F = 0$ is the defining equation of X^+ and F, G, H are semi-invariants of weights a_1, a_2, a_3 respectively.

6.4.6 **Lemma**: Suppose $F_1 = F^{-1}(1) \simeq \mathbf{C}^2$. Then $X \simeq \mathbf{C}^3$ and the action is linearizable.

Proof. The action of ω_{a_1} is linearizable, that is, we can choose an ω_{a_1}-equivariant isomorphism $\varphi : F_1 \to \mathbf{C}^2$, where ω_{a_1} acts on $\mathbf{C}^2$ by $t(u, w) = (t^{a_2}u, t^{a_3}v)$. Consider $Y = \mathbf{C}^1 \times \mathbf{C}^2$with T-action given by $t(w, u, v) = (t^{a_1}w, t^{a_2}u, t^{a_3}v)$. Let $W_0 = \{y \in Y : w(y) = 0\}$. We define $\phi : X - F_0 \to Y - W_0$ as follows : for $x \in X - F_0$ we choose $t \in T$ such that $tx \in F_1$ and put

$$\phi(x) = t^{-1}(1, \varphi(tx))$$

This is well defined since w has weight a_1 and φ is ω_{a_1}-invariant. We will show that it is possible to modify φ such that ϕ extends to (an isomorphism) $X \to Y = \mathbf{C}^3$. By 1.5 the action on X is locally linearizable, i.e. there exists an open T-invariant neighborhood U of q and analytic coordinates ξ, η, ζ on U such that for $t \in T$

$$t(\xi, \eta, \zeta) = (t^{a_1}\xi, t^{a_2}\eta, t^{a_3}\zeta)$$

. We have $F_0 \subset U$ and F_0 is defined by $\xi = 0$. We may assume , therefore, that $\xi = F$. Now η, ζ

are analytic coordinates on the neighborhood $U_1 = F_1 \cap U$ of $q_1 = X^- \cap F_1$ in F_1, and φ is given on U_1 by $\varphi(\eta, \zeta) = (\varphi_1(\eta, \zeta), \varphi_2(\eta, \zeta))$ where φ_1 and φ_2 are power series in η, ζ with, since φ is ω_{a_1}-invariant, only terms with weights of the form $b + ka \geq 0$ and $c + la \geq 0$ respectively.Hence for $(\xi, \eta, \zeta) \in U - F_0$

$$\phi(\xi, \eta, \zeta) = (\xi, \sum_{i,j \geq 0} a_{ij}\xi^k \eta^i \zeta^j, \sum_{i,j \geq 0} b_{ij}\xi^l \eta^i \zeta^j).$$

with $a_{ij} = 0$ unless $ib + jc = b + ka$ and $b_{ij} = 0$ unless $ib + jc = c + la$. We may assume that the linear forms of φ_1 and φ_2 are η and ζ respectively. ϕ extends to a morphism on X if $k \geq 0$ and $l \geq 0$ in each term present. Assume that a $k < 0$ appears and let k_0 be the smallest such. Then $i_0 = 0$ since $(i_0 - 1)b + j_0 c < 0$, and $j_0 c = b + k_0 a$. Hence $\varphi_1(\eta, \zeta) = a_{0j_0}\zeta^{j_0} + \widetilde{\varphi_1}(\eta, \zeta)$ where the lowest pure power of ζ appearing in $\widetilde{\varphi_1}$ has exponent larger than j_0. The same is then true for $\varphi_1' = \varphi_1 - a_{0j_0}{\varphi_2}^{j_0}$. Since $\varphi' = (\varphi_1', \varphi_2')$, $\varphi_2' = \varphi_2$, is again an ω_{a_1}-equivariant isomorphism, we can assure, repeating this process, that $a_{0j_0} \neq 0$ implies $jc - b = ka \geq 0$. If $a_{ij} \neq 0$ for $(i-1)b + c = 0$, i.e. $k = 0$, then again $i = 0$ and we can eliminate the corresponding term as before, or $i = 1$ and $j = 0$. We may argue similarly with reversed roles for i and j. So we may assume that ϕ extends to a morhism $X \to Y$ and that $\phi(0, \eta, \zeta) = (0, \eta, \zeta)$. Then ϕ is an isomorphism.

6.4.7 **Lemma**: Let the torus $T \simeq \mathbf{C}^*$ acts on a smooth, irreducible affine variety W . Let $W^T = \{q\}$ and suppose the weights of the action at q are $a < 0, b > 0$. Let W^+ and W^- be defined as in 6.4.5. Let $f = 0$ be a defining equation of W^+. Suppose that W is $\mathbf{R}$-acyclic and that for every $z \in W - (W^+ \cup W^-)$ the isotropy group T_z is trivial. Then all the curves $f_s = f^{-1}(s)$ are isomorphic to the line $\mathbf{C}^1$.

Proof. Let $W' = W/\omega_a$. W' is smooth since the fixed point set of ω_a has codimension 1 in W. ω_a acts trivially on the cohomology of W since it is contained in a continuous group of transformations. Hence W' is $\mathbf{R}$-acyclic. Let q' be the fixed point of the induced action on W'. T acts on W' with the weights $-1, b$. Let f' be the induced function on W'. W'^+ is isomorphic to $\mathbf{C}^1$. Using the duality $H_i(W', W' - W'^+; \mathbf{R}) \simeq H_c^{4-i}(W'^+) \simeq H_{i-2}(W'^+) \simeq H_{i-2}(\mathbf{C}^1)$ we compute $H_i(W' - W'^+) = 0$ for $i > 0$ and $H_1(W' - W'^+) = \mathbf{R}$. Since f' is semi-invariant of weight -1, we have $W' - W'^+ \simeq f_1' \times \mathbf{C}^*$. From the Künneth formula we obtain that f_1' is $\mathbf{R}$-acyclic. It follows that $f_1' \simeq \mathbf{C}^1$. Now the quotient map $f_1 \to f_1'$ is totally ramified at the point $f_1 \cap W^-$ and unramified elsewhere. Therefore f_1 is isomorphic to a line as well.

6.4.8. **Corollary**: Under the assumptions of Lemma 6.4.7, $W \simeq \mathbf{C}^2$.

6.4.9 **Lemma**: Suppose that $GCD(a_1, a_2) = GCD(a_1, a_2) = 1$. Then $X^{\omega_{a_1}} = X^-$. Also T acts with trivial isotropy on $X - (X^+ \cup X^-)$.

Proof. Take a prime p dividing a_1. Then $X^{\omega_{a_1}} \subset X^{\omega_p}$. By Floyd's theorem [Fl],X^{ω_p} is $\mathbf{Z/pZ}$-acyclic and $b_0(X^{\omega_p}; \mathbf{Z/pZ}) \leq b_0(X; \mathbf{Z/pZ}) = 1$. Hence X^{ω_p} is connected. Locally (in U as in 6.4.5) $X^{\omega_p}$is equal to X^-, as one easily computes using the local coordinates F, G, H. It follows that $X^{\omega_p} = X^-$ and that $X^{\omega_{a_1}} = X^{\omega_p} = X^-$. Now suppose that $t_0 x = x_0$ for some $x_0 \in X - (X^- \cup X^+)$ and $t_0 \neq 1$. We get $F(t_0 x_0) = t_0^{a_1} F(x_0) = F(x_0)$. Hence $t_0^{a_1} = 1$. Let a prime l divides ordt_0. Then $x_0 \in X^{\omega_l}$. Since l divides a_1, we have $X^{\omega_l} = X^{a_1} = X^-$ and $x_0 \in X^-$; contradiction.

6.4.10 **Lemma**: Let $d = GCD(a_1, a_2)$, $c = GCD(a_1, a_3)$. Suppose that $d > 1$ and

(i) $c = 1$

or

(ii) $c > 1$ and $X^{\omega_d} \cap X^{\omega_c} = X^-$

Then $H_0 = X^{\omega_d}$ and ω_d acts with trivial isotropy on $X - H_0$. Also $H_0 \simeq \mathbf{C}^2$.

Proof. Let p be a prime number dividing d. Then $X^{\omega_d} \subset X^{\omega_p}$.Locally, in U, $X^{\omega_d} \cap U = X^{\omega_p} \cap U = H_0 \cap U$. By Floyd[Fl] X^{ω_p} is connected. It follows that $X^{\omega_p} = H_0 = X^{\omega_d}$. It also follows that for any divisor k of d, $X^{\omega_k} = X^{\omega_d}$. This implies that ω_d acts with trivial isotropy outside H_0. Now let $z \in H_0 - (H_0^+ \cup H_0^-)$ and suppose $t_0 z = z$. Let r=ordt_0. Then r divides a_1 since $F(z) \neq 0$. Let p be a prime dividing r and let K be the maximal p-subgroup of ω_r. K acts trivially on $X^- = H_0^-$ and on z. Since X^K is connected by Floyd's theorem, $\dim(X^K) = 2$. Suppose that $X^K = G_0$. From the local linearizability it follows that p divides a_3. By Floyd's theorem $X^{\omega_c} = G_0$. We have a contradiction since then z belongs to $X^{\omega_d} \cap X^{\omega_c}$. Therefore $X^K \not\subset G_0$. Thus there exists a point w such that $w \notin G_0$ and $\omega(w) = w$ where ω is a generator of K. It follows that ord$K$ divides a_2. We obtain that r divides a_2. Hence r divides d. On H_0 we have an induced action of the torus $T' = T/\omega_d$ with weights $a_1/d, a_2/d$. We just proved that T' acts with trivial isotropy on $H_0 - (H_0^+ \cup H_0^-)$. By Lemma 6.4.7 and Corollary 6.4.8, H_0 is isomorphic to $\mathbf{C}^2$.

6.4.11 Let r be the number of connected componets of the set $X^{\omega_{a_2}} \cap X^{\omega_{a_3}}$. We introduce the following invariant

$$\varepsilon = (GCD(a_1, a_2) - 1)(GCD(a_1, a_3) - 1)(r - 1)$$

6.4.12 **Proposition**: If $\varepsilon = 0$ then X is equivariantly isomorphic to $\mathbf{C}^3$ with linear T-action.

Proof. One can see easily that the inclusion $F_1 \hookrightarrow X$ induces an isomorphism $F_1/\omega_{a_1} \simeq X//T$. Suppose first that $GCD(a_1, a_2) = GCD(a_1, a_3) = 1$. By 6.4.9, T acts with trivial isotropy on $X - (X^+ \cup X^-)$. In particular ω_{a_1} acts with trivial isotropy on $F_1 - q_1$, where $q_1 = X^- \cap F_1$. By 6.4.1, $F_1 - q_1/\omega_{a_1}$ is isomorphic to $\mathbf{C}^2 - 0/\omega_{a_1}$. It follows easily that F_1 is isomorphic to $\mathbf{C}^2$ and we are done by 6.4.6.

Now suppose that $d = GCD(a_1, a_2) > 1$. Let $X' = X/\omega_d$. One can show that X' is contractible: it is not difficult to show that X' is simply connected and that it has trivial

$\mathbf{Z}$-homology. One can also use the general result from [KPR] which says that a quotient of contractible algebraic variety by a reductive group action is contractible. On X' we have an induced T'-action with weights $a_1/d, a_2/d, a_3$ where $T' = T/\omega_d$. Hence X' is an LQ-variety. H_0 is isomorphic to a plane by 6.4.8 and the quotient map $X - H_0 \to X' - H_0'$ where H_0' is the image of H_0 is unramified. The map $H_0 \to H_0'$ is an isomorphism. Suppose that X' is equivariantly isomorphic to $\mathbf{C}^3$ with linear action of T'. By a result of Russell[R], H_0' is a coordinate plane in $\mathbf{C}^3$. Hence $X' - H_0' \simeq \mathbf{C}^2 \times \mathbf{C}^*$ which implies that $X - H_0 \simeq \mathbf{C}^2 \times \mathbf{C}^*$. Now it follows easily that X is T-equivariantly isomorphic to $\mathbf{C}^3$ with linear T-action. Repeating this process if necessary we come to an LQ-variety X'' where a torus T'' acts with weights a_1'', a_2'', a_3'' and $GCD(a_1'', a_2'') = GCD(a_1'', a_3'') = 1$. Then X'' is equivariantly isomorphic to $\mathbf{C}^3$ with linear action of T''.

6.4.13 **Theorem**(S. Kaliman, L. Makar-Limanov[KM-L]): If $\varepsilon > 0$ then X is not isomorphic, as an algebraic variety, to $\mathbf{C}^3$.

In the proof the authors made use of the Makar-Limanov invariant $AK(X)$, the ring of regular functions on X that are invariant under *all* $\mathbf{C}^+$-actions on X. It is interesting that their argument as well as the construction in [KR2] makes use of the Epimorphism Theorem 7.5.

That is the final step in the proof of the Theorem 6.4.

7. **Remarks, open problems**

7.1 We note that all $X \in LQ$ are "exotic affine spaces", algebraic varieties diffeomorphic, but not necessarily isomorphic to an affine space. A specific, simplest example is the hypersurface

$$X = \{(x, y, z, t) : x + x^2 y + z^2 + t^3 = 0\}$$

in $\mathbf{C}^4$. Here $\mathbf{C}^*$ acts with weights 6,-6,3,2 respectively on x, y, z, t. The Makar-Limanow invariant is the ring generated over $\mathbf{C}$ by the function x. In this case $\varepsilon = 2$, $r = 2$ and the subgroup ω_6 acts with disconnected fixed point set (it has 2 components).

In contrast to the situation in 5.4 and 5.6 the $X \in LQ$ can admit non stably trivial $\mathbf{C}^*$-vector bundles. The equivariant Grothendieck ring of X has been computed by Bell[Be]. Recently P.Murthy[Mu] proved a result which implies that all vector bundles on the above X are trivial. It is not known at the moment whether all vector bundles on an arbitrary LQ-variety are trivial.

Let us note that 5.1 and 6.4 together imply

7.2 **Theorem**: Actions of a connected reductive group on $\mathbf{C}^3$ are linearizable.

The problem for finite groups, in particular, seems to be wide open.

7.3 Theorem 6.4.1 now has a generalization

Theorem([KR6]): Let S be an affine normal topologically contractible surface. Suppose that the logarithmic Kodaira dimension $\overline{k}(S)$ equals $-\infty$ and that all singular points of S are quotient singularities. Then

(i) $\overline{k}(S - SingS) = -\infty$

(ii) If $SingS$ consists of only one point then S is isomorphic to a quotient $\mathbf{C}^2/G$ where G is a *small* subgroup of $GL(2;\mathbf{C})$. If $SingS$ has more than one point then S is constructed from a Hirzebruch surface F_n by a fully described procedure (we blow up some points in F_n and remove some divisors).

7.4 It is still an open problem whether a codimesion 2 torus action, i.e. an action of $(\mathbf{C}^*)^{n-2}$ on $\mathbf{C}^n$, is linearizable. Koras and Russell[KR5] proved that it is linearizable provided the fixed point set has dimension greater than 0. For an action with isolated fixed point the induction argument leads to the following question :

Let Y be an LQ-variety. Can $Y \times \mathbf{C}$ be isomorphic to $\mathbf{C}^4$ unless Y is isomorphic to $\mathbf{C}^3$?

If the answer is yes than we have an example of nonlinearizable action of $\mathbf{C}^* \times \mathbf{C}^*$ on $\mathbf{C}^4$.

A closed embedding $\varphi : \mathbf{A}_k^m \to \mathbf{A}_k^n$ is called *rectifiable* if φ is conjugate to a linear embedding. Very little is known about rectifiability except for

7.5. AMS-**Theorem**(Epimorphism Theorem[AM][Su]): If char $k = 0$, then all embeddings

$$\mathbf{A}_k^1 \subset \mathbf{A}_k^2$$

are rectifiable.

There are also very general positive results for "small" m ($n \geq 2m + 2$) [Ka], [Je], [Sr]. However, the cases $n = 3$, $m = 2$ and $m = 1$ are open. (the case $n = 3$, $m = 2$ would settle linearization for $\mathbf{C}^*$-actions on $\mathbf{C}^3$ without much fuss.) Let us record however

7.6 **Theorem**([R]): Closed embeddings of $\mathbf{A}_k^{n-1}$into $\mathbf{A}_k^n$ that are homogeneous for a linear $(k^*)^r$-action on $\mathbf{A}_k^n$, $r \geq n - 2$, are rectifiable.

7.7 Asanuma [A1] has described a general method that produces a non-linearizable k^*-action on $\mathbf{A}_k^{n+2}$ from a non-rectifiable closed embedding $\mathbf{A}^1 \subset \mathbf{A}_k^n$. This leads to the following examples.

7.7.1 As Shastri[Sh] has observed , one can realize any knot in $\mathbf{R}^3$ by a polynomial map $\varphi : \mathbf{R} \to \mathbf{R}^3$ in such a way that the extension $\varphi_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}^3$ is an embedding. Asanuma[A2] has used such φ to construct $\mathbf{R}^*$-actions on $\mathbf{R}^5$ that are not linearizable over $\mathbf{R}$. However they are linearizable over $\mathbf{C}$.

7.7.2 If char $k = p > 0$, there are counterexamples to the AMS-Theorem 7.5. Hence there exist non-linearizable k^*-actions on $\mathbf{A}_k^4$[A1].

7.7.3 Derksen and Kutschebauch[DK] have used holomorphic embeddings $\mathbf{C} \to \mathbf{C}^2$ that are not holomorphically rectifiable to construct holomorphic $\mathbf{C}^*$-actions on $\mathbf{C}^4$ that are not linearizable.

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Mariusz Koras
Dept. of Mathematics, Warsaw University, Warszawa ul.Banacha 2, Poland

Peter Russell,Dept. of Mathematics and Statistics, McGill University,Montreal, QC, Canada and
Centre Interuniversitaire en Calcul Mathématique Algébrique