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LUNA'S SLICE THEOREM AND APPLICATIONS

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Contents

1. Introduction

This course is devoted to the proof of Luna's étale slice theorem and to the study of some of its applications. A brief description of this theorem follows in 1.1 and 1.2. Our main source is the original paper [14]. The whole proof of this result is complicated, because the hypotheses are very general. We shall sometimes make some simplifications, for example by assuming that we consider only integral varieties or connected groups.

Luna's étale slice theorem is useful for the local study of good quotients by reductive groups. We will give here three applications. We will first give some general results on quotients by reductive groups. The second application is the local study of the moduli spaces of semi-stable vector bundles on curves, following a paper of Y. Laszlo [10]. There are also applications of Luna's theorem to the study of moduli spaces of semi-stable bundles on surfaces (cf. for example [20]). In the third application we will study the factoriality of the local rings of the points of some quotients. Luna's theorem will allow us to study the factoriality of the completions of these rings. This will give a counterexample, showing that some local properties of quotients cannot be derived from Luna's theorem.

In this paper we will suppose that the ground field is the field of complex numbers. Luna's theorem deals with affine varieties. This is why many definitions and results in this paper are given for affine varieties, although they can be sometimes generalized to other varieties.

1.1. Étales slices

Let G be a reductive algebraic group acting on an affine variety X. Let

$$\pi_X : X \longrightarrow X//G$$

be the quotient morphism. In general, $X//G$ is not an orbit space, but only the minimal identifications are made : if $x, x' \in X$, we have $\pi_X(x) = \pi_X(x')$ if and only if the closures of the orbits Gx, Gx' meet. The étale slice theorem is a tool for the local study of $X//G$.

Let $x \in X$ be a closed point such that Gx is closed, G_x its stabilizer in G, which is also reductive. Let $V \subset X$ be a G_x-invariant locally closed affine subvariety containing x. The multiplication morphism

$$\mu : G \times V \longrightarrow X$$

$$(g, v) \longmapsto gv$$

is G_x-invariant if the action of G_x on $G \times V$ is defined by : $h.(g, v) = (gh^{-1}, hv)$. Let

$$G \times_{G_x} V \ = \ (G \times V)//G_x.$$

Then μ induces a morphism

$$\psi : G \times_{G_x} V \longrightarrow X.$$

We say that V is an étale slice if ψ is *strongly étale*. This means that

(i) ψ is étale, and its image is a saturated open subset U of X.
(ii) $\psi_{/G} : (G \times_{G_x} V)//G \simeq V//G_x \longrightarrow U//G$ is étale.
(iii) ψ and the quotient morphism $G \times_{G_x} V \to V//G_x$ induce an isomorphism

$$G \times_{G_x} V \ \simeq \ U \times_{U//G} (V//G_x).$$

Luna's slice étale theorem asserts that an étale slice exists. The existence of a V satisfying only (i) and (ii) is known as 'weak Luna's étale slice theorem'.

Let $\pi_V : V \to V//G_x$ be the quotient morphism. The first consequence of (ii) is that $\widehat{\mathcal{O}}_{\pi_X(x)}$ and $\widehat{\mathcal{O}}_{\pi_V(x)}$ are isomorphic. So if we are interested only in the study of completions of the local rings of quotients, we can replace X and G by V and G_x respectively.

Property (iii) gives supplementary relations between the G-orbits in X near x and the G_x-orbits in V. More precisely, (iii) implies that for every closed point $v \in V//G_x$, if $u = \psi_{/G}(v)$ then we have a canonical isomorphism

$$G \times_{G_x} \pi_V^{-1}(v) \ \simeq \ \pi_X^{-1}(u).$$

1.2. Étale slice theorem for smooth varieties

Suppose that X is smooth at x. Then we can furthermore suppose that

(iv) V is smooth, and the inclusion $V \subset X$ induces an isomorphism $\sigma : T_x V \simeq N_x$, where N_x is the normal space to the orbit G_x in X.
(v) There exists a G_x-invariant morphism $\phi : V \to N_x$ such that $T\phi_x = \sigma$, which is strongly étale.

This implies that we can replace the study of X around Gx with the study of N_x around 0, which is generally much simpler.

1.3. General plan of the paper

Chapter 2 – *Reductive groups and affine quotients.*

We give here some basic definitions and results. We define algebraic reductive groups, categorical and good quotients, and list almost without proofs some of their properties.

Chapter 3 – *Zariski's main theorem for G-morphisms.*

We prove here theorem 3.3 which is used in the proof of Luna's theorem.

Chapter 4 – *Étale morphisms.*

in 4.1 we give a review of definitions and results concerning étale morphisms.

In 4.2 we define and study fiber bundles, principal bundles and extensions of group actions. These notions are used in the statement and the proof of Luna's theorem.

In 4.3 we study some properties of quotients by finite groups. This will throw a light on a very abstract algebraic lemma in [14].

In 4.4 we give some results on étale G-morphisms, which are important steps in the proof of Luna's theorem.

Chapter 5 – *Étale slice theorem.*

We give here the proof of Luna's étale slice theorem in the general case and prove also the supplementary results when X is smooth.

In 5.2 we give the first applications of Luna's theorem.

Chapter 6 – *G-bundles.*

In 6.1 we define and study G-bundles. In particular we prove the descent lemma, which is used in chapter 8.

In 6.2 we give a second application of Luna's theorem : the study of *models*. Let G be a reductive group acting on a smooth affine variety X. Let $x \in X$ be a closed point such that the orbit Gx is closed. Let N be the normal bundle of Gx. It is a G-bundle and its isomorphism class is called a *model*. We deduce in particular from Luna's theorem that there are only finitely many models associated to X. The proofs of this section are not difficult and left as exercises.

Chapter 7 – *Description of moduli spaces of semi-stable vector bundles on curves at singular points.*

In 7.1 we define *(semi-)stable vector bundles* on a smooth irreducible projective curve, and in 7.2 we recall the construction of their moduli spaces as good quotients of open subsets of Quot-schemes by reductive groups. Almost all the results of these sections are given without proofs.

In 7.4 we give some applications of Luna's theorem to the study of singular points of the moduli spaces. We use the description of moduli spaces as quotient varieties given in the preceeding sections.

Chapter 8 – *The étale slice theorem and the local factoriality of quotients.*

In 8.1 we recall some results on locally factorial varieties. Let G be a reductive group acting on a smooth affine variety X. Then under some hypotheses, we prove that $X//G$ is locally factorial if and only if for every character λ of G, and every closed point x of X such that Gx is closed, λ is trivial on the stabilizer of x.

Let $\pi_X : X \to X//G$ be the quotient morphism. Let $u \in X//G$ and $x \in \pi^{-1}(u)$ be such that Gx is closed. Using Luna's theorem, we can also prove that if $\widehat{\mathcal{O}}_u$ is factorial then the only character of G_x is the trivial one.

This proves that in general, if G_x is not trivial, $\widehat{\mathcal{O}}_u$ will not be factorial, even if $\mathcal{O}_u$ is. Hence in such cases we have no hope to prove the factoriality of $\mathcal{O}_u$ by using the good description of its completion given by Luna's theorem.

This phenomenon is illustrated by an example in 8.3. Let V be a non zero finite dimensional vector space, $q \geq 2$ an integer. Let $\mathbf{S}(V,q)$ be the variety of sequences $(M_1, \ldots, M_q)$, $M_i \in \mathrm{End}(V)$. On this variety there is an obvious action of $\mathrm{GL}(V)$

$$(g, (M_1, \ldots, M_q)) \longmapsto (gM_1g^{-1}, \ldots, gM_qg^{-1})$$

which is in fact an action of $\mathrm{PGL}(V)$. Let

$$\mathcal{M}(V,q) \ = \ \mathbf{S}(V,q)// \,\mathrm{PGL}(V).$$

We prove that $\mathcal{M}(V,q)$ is locally factorial whereas the completions of the local rings of its singular points may be non factorial.

1.4. Prerequisites and references

For basic results of algebraic geometry, see [7].

For basic results on reductive groups and algebraic quotients, see [19], [18].

Some results of algebra that will be used here can be found in [22], [15], [21], [4].

Some results on étale morphisms are taken from [6], [7].

The study of semi-stable vector bundles on curves and of their moduli spaces is done in [12], [23].

1.5. Notations and recall of some basic results

1.5.1. *Notations*

If X is an affine variety, $A(X)$ will denote the ring of regular functions on X, and $\mathbb{C}(X)$ its field of rational functions.

If $x \in X$, m_x will denote the maximal ideal of $\mathcal{O}_x$.

1.5.2. *Finite and quasi-finite morphisms*

Let X, Y be algebraic varieties.

Definition 1.1. *1 - A morphism $f : X \to Y$ is called* finite *if the $A(Y)$-module $A(X)$ is finitely generated.*

2 - A morphism $f : X \to Y$ is called quasi-finite *if its fibers are finite.*

Proposition 1.2. *1 - A finite morphism is quasi-finite and closed.*

2 - A morphism $X \to Y$ is quasi-finite if and only if for every closed point $y \in Y$, $A(X)/(A(X).m_y)$ is finite dimensional over $\mathbb{C}$.

1.5.3. *Finiteness of integral closure*

Theorem 1.3. *Let A be an integral domain which is a finitely generated $\mathbb{C}$-algebra. Let K be the quotient field of A, and L a finite algebraic extension of K. Then the integral closure of A in L is a finitely generated A-module, and is also a finitely generated $\mathbb{C}$-algebra.*

(see [22], Ch. V, 4, theorem 9, [4], theorem 4.14).

Corollary 1.4. *Let B be an integral domain which is a finitely generated $\mathbb{C}$-algebra. Let K be the quotient field of B, $A \subset B$ a finitely generated $\mathbb{C}$-subalgebra and $K' \subset K$ its quotient field. Suppose that K is a finite extension of K'. Then the integral closure of A in B is a finitely generated A-module, and is also a finitely generated $\mathbb{C}$-algebra.*

2. Reductive groups and affine quotients

2.1. Algebraic actions of algebraic groups

2.1.1. *Actions on algebraic varieties*

Definition 2.1. *An* algebraic group *is a closed subgroup of* $GL_n(k)$*.*

Definition 2.2. *A* left action *(ar more concisely an* action*) of an algebraic group on an algebraic variety* X *is a morphism*

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

such that for all $g, g' \in G$*,* $x \in X$ *we have* $g(g'x) = (gg')x$*, and* $ex = x$ *(where* e *denotes the identity element of* G*).*

Right actions are defined similarly in an obvious way. In any case we say that X is a G-variety.

If $x \in X$, the *stabilizer* G_x of x is the closed subgroup of G consisting of g such that $gx = x$.. The *orbit* of x is the subset $\{gx \mid g \in G\}$. Let

$$\begin{aligned} \psi : G \times X &\longrightarrow X \times X \\ (gx, x) &\longmapsto gx \end{aligned}$$

A subset $Y \subset X$ is called G$-$*invariant* if $gY \subset Y$ for every $g \in G$.

Definition 2.3. *The action of* G *on* X *is called*

> closed *if all the orbits of the action of* G *are closed.*
> proper *if* ψ *is proper.*
> free *if* ψ *is a closed immersion.*

Definition 2.4. *Let* X*,* X' *be algebraic varieties with an action of* G*. A* G-morphism *from* X *to* X' *is a morphism* $f : X \to X'$ *such that* $f(gx) = gf(x)$ *for all* $g \in G$*,* $x \in X$*. A* G*-isomorphism* $X \to X'$ *is a* G*-morphism which is an isomorphism of algebraic varieties.*

Let Z be an algebraic variety, X a G-variety and $f : X \to Z$ a morphism, We say that f is G-*invariant* if it is a G-morphism when we consider the trivial action of G on Z, i.e if $f(gx) = f(x)$ for every $x \in X$.

2.1.2. *Actions of algebraic groups on $\mathbb{C}$-algebras*

Definition 2.5. *Let G be an algebraic group and R a $\mathbb{C}$-algebra.*

1 - An action *of G on R is a map*

$$\begin{aligned} G \times R &\longrightarrow R \\ (g, r) &\longmapsto r^g \end{aligned}$$

such that

(i) $r^{gg'} = (r^g)^{g'}$, $r^e = r$ *for all $g, g' \in G$, $r \in R$.*
(ii) *For all $g \in G$ the map $r \mapsto r^g$ is a $\mathbb{C}$-algebra automorphism of R.*

2 - This action is called rational *if every element of R is contained in a finite dimensional G-invariant linear subspace of R.*

If G acts on R, R^G will denote the subalgebra of G-invariant elements of R.

Example : If G acts on an algebraic variety X then there is a natural action of G on $A(X)$, given by

$$(g, \phi) \longmapsto (x \longmapsto \phi^g(x) = \phi(gx))$$

The natural action of G on $A(X)$ is a rational action (cf. [19], lemma 3.1).

Definition 2.6. *Let G be an algebraic group. A* rational representation *of G is a morphism of groups $G \to GL(V)$, where V is a finite dimensional vector space. It induces an action of G on V.*

Lemma 2.7. *Let G be an algebraic group acting on an algebraic variety X, and W a finite dimensional $\mathbb{C}$-subspace of $A(X)$. Then*

(i) W is contained in a finite dimensional G-invariant $\mathbb{C}$-subspace of $A(X)$.

(ii) If W is G-invariant then the action of G on W is given by a rational representation.

(cf. [19], lemma 3.1).

2.2. Reductive groups

(see [12], chap. 6, [18] and [19]).

Definition 2.8. *An algebraic group G is called*

(i) reductive *if the radical of G (i.e. its maximal connected solvable normal subgroup) is a torus.*
(ii) linearly reductive *if every rational representation of G is completely reducible.*
(iii) geometrically reductive *if whenever $G \to GL(V)$ is a rational representation and $v \in V^G$, $v \neq 0$, then for some $r > 0$ there is a $f \in (S^r V^*)^G$ such that $f(v) \neq 0$.*

These 3 properties are equivalent (this is true for fields of characteristic 0). In non zero characteristic, (i) and (iii) are equivalent and are implied by (ii) (cf. [18]). Property (i) is mainly used to verify that some groups are reductive, and the other properties are useful to build algebraic quotients by G. Property (ii) is equivalent to the following : if V is a rational representation of G, and V' is a G-invariant linear subspace of V, then there exists a complementary G-invariant linear subspace of V.

For example, if G contains a subgroup K that is compact (for the usual topology), and Zariski dense, then it can be proved that G is reductive.

Definition 2.9. *Let V be a rational representation of G. Then there is a G-invariant linear subspace $W \subset V$ such that $V \simeq V^G \oplus W$ (as representations). The projection*

$$R_V : V \longrightarrow V^G$$

is called a Reynolds operator.

This map $R = R_V$ has the following properties :

(i) R is G-invariant.
(ii) $R \circ R = R$.
(iii) $\mathrm{Im}(R) = V^G$.

Conversely, if $R : V \to V^G$ is an operator such that (i), (ii) and (iii) are satisfied, then it can be proved that $R = R_V$.

More generally the Reynolds operator can be defined for every linear action of a reductive group G on a vector space V if for every $v \in V$ there exists a finite dimensional G-invariant linear subspace of V containing v. In particular this is the case if $V = A(X)$, where X is a G-variety (cf. lemma 2.7). So we have a well defined and unique Reynolds operator $A(X) \to A(X)^G$.

The Reynolds operator is *functorial* : this means that if we have linear actions of G on vector spaces V, W (such that each of them is the union of finite dimensional G-invariant linear subspaces), and if $f : V \to W$ is a linear G-invariant map, then f commutes with the Reynolds operators, i.e the following diagram is commutative :

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "R_V"] & W \arrow[d, "R_W"] \\ V^G \arrow[r, "f_{|V^G}"] & W^G \end{tikzcd}$$

Theorem 2.10. *(Nagata's theorem). Let G be a reductive algebraic group acting on a finitely generated $\mathbb{C}$-algebra. Then the $\mathbb{C}$-algebra R^G is finitely generated.*

(see [19], thm. 3.4).

2.3. Algebraic quotients

Definition 2.11. *Let G be an algebraic group acting on an algebraic variety X. a* categorical quotient *of X by G is a pair (Y, π), where Y is an algebraic variety and $\pi : X \to Y$ a G-invariant morphism such that for every algebraic variety Z, if $f : X \to Z$ is a G-invariant morphism then there is a unique morphism $\overline{f} : Y \to Z$ such that the following diagram is commutative :*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Y \arrow[dl, "\overline{f}"] \\ & Z & \end{tikzcd}$$

In this case we say also that Y is a categorical quotient of X by G.

We say that (Y, π) is a universal categorical quotient *if the following holds : Let Y' be an algebraic variety and $Y' \to Y$ a morphism. Consider the obvious action of G on $X \times_Y Y'$. If p_2 denotes the projection $X \times_Y Y' \to Y'$, then (Y', p_2) is a categorical quotient of $X \times_Y Y'$ by G. In this case we say also that Y is a universal categorical quotient of X by G.*

It is easy to see that categorical quotients are unique in an obvious way.

Let G be an algebraic group acting on algebraic varieties X and X', and $f : X \to X'$ a G-morphism. Suppose that there are categorical quotients (Y, π), (Y', π') of X by G, X' by G respectively. Then there is a unique morphism $f_{/G} : Y \to Y'$ such that the following diagram is commutative :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ Y \arrow[r, "f_{/G}"] & Y' \end{tikzcd}$$

Definition 2.12. *Let G be an algebraic group acting on an algebraic variety X. A* good quotient *of X by G is a pair (Y, π), where Y is an algebraic variety and $\pi : X \to Y$ a morphism such that*

(i) *π is G-invariant.*
(ii) *π is affine and surjective.*
(iii) *If $U \subset Y$ is open then the natural map*

$$A(U) \longrightarrow A(\pi^{-1}(U))^G$$

is an isomorphism.
(iv) *If W_1, W_2 are disjoint closed G-invariant subsets of X, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint closed subsets of X.*

A good quotient is a *categorical quotient.*

We will often say that Y is a good quotient of X by G and use the following notation : $Y = X//G$.

Lemma 2.13. *Let G be an algebraic group acting on an algebraic variety X, and suppose that a good quotient $(X//G, \pi)$ of X by G exists. Then for every closed point $y \in X//G$, $\pi^{-1}(y)$ contains a unique closed orbit.*

Proof. Since $\pi^{-1}(y)$ is G-invariant, it is the union of all the orbits of its points, and contains an orbit Γ whose dimension is minimal. We will see that Γ is closed. Suppose it is not. The closure $\overline{\Gamma}$ of Γ is also G-invariant, and so is $\overline{\Gamma}\backslash\Gamma$. It follows that this non empty set contains orbits whose dimension is smaller than that of Γ. This contradicts the definition of Γ and proves that it is closed.

If two orbits Γ_1, Γ_2 are distinct and closed, then they are disjoint, and it follows from the condition (iv) of definition 2.12 that $\pi(\Gamma_1) \neq \pi(\Gamma_2)$. Hence $\pi^{-1}(y)$ cannot contain more than one closed orbit. □

The preceeding lemma implies the following description of $X//G$: its closed points are the closed orbits of the closed points of X.

If $y \in X//G$ is a closed point, we will denote by $T(y)$ the unique closed orbit in $\pi^{-1}(y)$.

Lemma 2.14. *Let G be an algebraic group acting on algebraic varieties X, Y and $\phi : X \to Y$ a finite G-morphism. Then for every $x \in X$, Gx is closed in X if and only if $G\phi(x)$ is closed in Y.*

Proof. If Gx is closed, then $G\phi(x)$ is closed too, because ϕ is closed (cf. 1.5). Conversely, suppose that $G\phi(x)$ is closed. Suppose that Gx is not closed. Then there exists in $\overline{Gx}$ a closed orbit Gx_0 such that $\dim(Gx_0) < \dim(Gx)$. Now we have

$$G\phi(x_0) = \phi(Gx_0) \subset \phi(\overline{Gx}) \subset \overline{\phi(Gx)} = \overline{G\phi(x)} = G\phi(x).$$

It follows that $G\phi(x_0) = G\phi(x)$. But this is impossible, because $\dim(G\phi(x_0)) = \dim(Gx_0)$, $\dim(G\phi(x)) = \dim(Gx)$ (because ϕ is finite) and $\dim(Gx_0) < \dim(Gx)$. Hence Gx is closed. □

Let U be a G-invariant subset of X. We say that U is *saturated* if whenever $x \in U$ and $y \in X$ are such that $\overline{Gx} \cap \overline{Gy} \neq \emptyset$, then $y \in U$. A saturated subset is G-invariant. For example, if S is a set of G-invariant regular functions of X, then the set of points when all the elements of S vanish (resp. are non zero) is saturated. In particular, if $f \in A(X)^G$ then X_f is saturated. If a good quotient $\pi : X \to X//G$ exists, then U is saturated if and only if $U = \pi^{-1}(\pi(U))$.

Proposition 2.15. *Let G be an algebraic group acting on an algebraic variety X, and suppose that a good quotient $(X//G, \pi)$ of X by G exists. Then*

$$\begin{array}{rcl} X \text{ reduced} & \Longrightarrow & X//G \text{ reduced} \\ X \text{ connected} & \Longrightarrow & X//G \text{ connected} \\ X \text{ irreducible} & \Longrightarrow & X//G \text{ irreducible} \\ X \text{ normal} & \Longrightarrow & X//G \text{ normal}\ . \end{array}$$

Theorem 2.16. *Let G be a reductive group acting on an affine variety X. Then a good quotient of X by G exists, and $X//G$ is affine. Moreover it is a universal categorical quotient.*

(cf. [19], theorem 3.5, [18]). It follows from the definition of a good quotient that we have

$$X//G \;\simeq\; \mathrm{Spec}(A(X)^G).$$

Of course the preceeding theorem is a consequence of theorem 2.10. It follows that if Y is an affine variety and $Y \to X//G$ a morphism, then we have a canonical isomorphism (given by the projection on Y)

$$(X \times_{X//G} Y)//G \;\simeq\; Y.$$

We will note $\pi_X : X \to X//G$ the quotient morphism.

Definition 2.17. *Let G be an algebraic group acting on an algebraic variety X. a* geometric quotient *of X by G is a good quotient (Y,π) such that for $x_1, x_2 \in X$, $\pi(x_1) = \pi(x_2)$ if and only if $Gx_1 = Gx_2$.*

So a good quotient of X by G is geometric if and only all the orbits of the closed points of X are closed. For a geometric quotient (Y,π) of X by G we will use the notation $Y = X/G$, since in this case Y is an orbit space.

Proposition 2.18. *Let G be a reductive algebraic group acting on an affine variety X, and let Y be a G-invariant closed subvariety of X. Then $\pi_X(Y)$ is closed and the restriction of π_X*

$$Y \longrightarrow \pi_X(Y)$$

is a good quotient. In other words, we have $\pi_X(Y) = Y//G$.

Proof. Let $I \subset A(X)$ be the ideal of Y. We have only to prove that

$$A(X)^G/(I \cap A(X)^G) \;\simeq\; (A(X)/I)^G.$$

Since I is G-invariant, the quotient map $A(X) \to A(X)/I$ sends $A(X)^G$ to $(A(X)/I)^G$. Let $p : A(X)^G \longrightarrow (A(X)/I)^G$ be the induced morphism. Its kernel is $I \cap A(X)^G$. We need only to prove that p is surjective. This follows from the commutative diagram

$$\begin{tikzcd} A(X) \arrow[r, "R_{A(X)}"] \arrow[d] & A(X)^G \arrow[d, "p"] \\ A(X)/I \arrow[r, "R_{A(X)/I}"] & (A(X)/I)^G \end{tikzcd}$$

and the fact that the Reynolds operator $R_{A(X)/I}$ is surjective. □

2.4. Stabilizers

Proposition 2.19. *Let G be a reductive algebraic group acting on an affine variety X, and x a closed point of X such that the orbit Gx is closed. Then the isotropy group G_x is reductive.*

(see [16], and [14] for two proofs)

3. Zariski's main theorem for G-morphisms

3.1. Zariski's main theorem

Theorem 3.1. *Let $f : X \to Y$ be a birational projective morphism of integral algebraic varieties, with Y normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.*

(cf. [7], III.12) This theorem is called *Zariski's main theorem.* We deduce immediately the following consequence :

Corollary 3.2. *Let X, Y be affine irreducible normal varieties, $\phi : X \to Y$ a birational quasi-finite morphism. Then ϕ is an open immersion.*

We will prove in this chapter a similar result for G-morphisms.

3.2. Zariski's main theorem for G-morphisms

Theorem 3.3. *Let X, Y be affine varieties, G a reductive group acting on X and Y, and $\phi : X \to Y$ a quasi-finite G-morphism. Then*

1 - There exists an affine G-variety Z, an open immersion $i : X \to Z$ which is a G-morphism, and a finite G-morphism $\psi : Z \to Y$, such that $\psi \circ i = \phi$.

2 - Suppose that the image by ϕ of a closed orbit of X is closed, and that $\phi_{/G} : X//G \to Y//G$ is finite. Then ϕ is finite.

Proof. Let $\phi^* : A(Y) \to A(X)$ be the morphism induced by ϕ, $I = \ker(\phi^*)$, and Y_0 the closed subvariety of Y corresponding to the ideal I. Then $f(X) \subset Y_0$. By replacing Y with Y_0 we can suppose that ϕ^* is injective, and $A(Y)$ can be viewed as a subalgebra of $A(X)$.

Lemma 3.4. *There exists integral affine G-varieties X', Y' containing X, Y respectively as closed subvarieties, such that :*

(i) *The action of G on X, Y is induced by its action on X', Y' respectively.*
(ii) *There exists a dominant quasi-finite morphism $\phi : X' \to Y'$, inducing $\phi : X \to Y$.*
(iii) *X' is normal.*

We now prove theorem 3.3, 1-, assuming that lemma 3.4 is true. Let B' be the integral closure of $A(Y')$ in $A(X')$. Then B' is a finitely generated $\mathbb{C}$-algebra (cf. 1.5.3) (here we use the fact that X' is integral). Let B be the image of B' in $A(X)$. Let $Z' = \text{Spec}(B')$, $Z = \text{Spec}(B)$. Then Z' is normal.

Let $i' : X' \to Z'$ be the morphism induced by $B' \subset A(X')$, and $\psi' : Z' \to Y'$ the one induced by $A(Y') \subset B'$.

Let's prove first that i' is an open immersion. By corollary 3.2, we need only to prove that i' is quasi-finite and birational. Since $\phi' = \psi' \circ i'$ and ϕ' is quasi-finite, i' is also quasi-finite. Now we will prove that i' is birational. This is equivalent to the fact that $\mathbb{C}(X') = \mathbb{C}(Z')$. It is enough to prove that $A(X') \subset \mathbb{C}(Z')$. Let $\alpha \in A(X')$. Then since $\mathbb{C}(X')$ and $\mathbb{C}(Y')$ have the same transcendance degree (the dimension of X' and Y'), α is integral over $\mathbb{C}(Y')$, i.e. we have a relation

$$\alpha^n + \lambda_{n-1}\alpha^{n-1} + \cdots + \lambda_1\alpha + \lambda_0 \;=\; 0$$

with $\lambda_i \in \mathbb{C}(Y')$, $0 \leq i < n$. Let

$$\lambda_i \;=\; \frac{a_i}{b_i}$$

for $0 \leq i < n$, with $a_i, b_i \in A(Y')$, and $b = \prod_{0\leq i<n} b_i$. Then the preceeding relation implies that $b\alpha$ is integral over $A(Y')$. From the definition of B' it follows that $b\alpha \in B'$, hence $\alpha \in \mathbb{C}(Z')$.

Next we prove that ψ' is finite. This is because B' is finitely generated and all of its elements are integral over $A(Y')$, therefore B' is a finitely generated $A(Y')$-module (cf. 1.5.3).

Let $i : X \to Z$ be the morphism induced by $B \subset A(X)$, and $\psi : Z \to Y$ the one induced by $A(Y) \subset B$. Then i is an open immersion, and ψ is finite (this follows from the facts that i' is an open immersion and ψ' is finite). Hence the first part of theorem 3.3 is proved.

The proof of 2- is similar.

Proof of lemma 3.4. By lemma 2.7, since $A(X)$ is a finitely generated $\mathbb{C}$-algebra, there exists a finite dimensional (over $\mathbb{C}$) G-invariant subspace W of $A(X)$ that generates the $\mathbb{C}$-algebra $A(X)$. Let $S(W)$ be the symmetric algebra of W, with the action of G induced by its action on G. We have thus a $\mathbb{C}$-algebra G-morphism

$$\rho : S(W) \longrightarrow A(X)$$

which is surjective. Let $X'' = \mathrm{Spec}(S(W))$ and $I = \ker(\rho)$. Then X can be viewed as a closed sub-G-variety of X''. Let $V \subset S(W)$ be a finite dimensional $\mathbb{C}$-subspace containing a system of generators of the ideal I and whose image in $A(X)$ is contained in $A(Y)$ and contains a set of generators of this $\mathbb{C}$-algebra. By lemma 2.7 there exists a finite dimensional G-invariant $\mathbb{C}$-subspace V_0 of $S(W)$ containing V. Let A_0 be the G-invariant $\mathbb{C}$-subalgebra of $S(W)$ generated by V_0, and $Y' = \mathrm{Spec}(A_0)$. Then ρ maps A_0 on $A(Y)$, so we can view Y as a closed sub-G-variety of Y'. The inclusion $A_0 \subset S(W)$ defines a dominant G-morphism $\phi'' : X'' \to Y'$ inducing ϕ.

Let $y \in Y$, m_y the ideal of y in Y, and m'_y its ideal in Y'. We will prove that

$$\rho^{-1}(A(X)m_y) \;=\; A(X'')m'_y.$$

It is clear that $\rho(m'_y) \subset m_y$, hence $A(X'')m'_y \subset \rho^{-1}(A(X)m_y)$. Conversely, let $f \in \rho^{-1}(A(X)m_y)$. Then since $\rho(A(X'')) = A(X)$ and $\rho(m'_y) = m_y$, there exists $g \in A(X'')m'_y$ such that $\rho(f) = \rho(g)$, i.e. $f - g \in I$. Now, m'_y being a maximal ideal, we have $I \subset m'_y \subset A(X'')m'_y$, hence $f \in A(X'')m'_y$, and $\rho^{-1}(A(X)m_y) = A(X'')m'_y$. It follows that ρ induces an isomorphism

$$A(X'')/A(X'')m'_y \;\simeq\; A(X)/A(X)m_y.$$

Since ϕ is quasi-finite, $A(X)/A(X)m_y$ is a finite dimensional $\mathbb{C}$-vector space, and so is $A(X'')/A(X'')m'_y$, and this implies that ${\phi''}^{-1}(y)$ is finite. Let

$$X' \;=\; \{x'' \in X'' \mid {\phi''}^{-1}(\phi''(x'')) \text{ is finite}\}.$$

Then X' is open in X'' and G-invariant, and we have just seen that it contains X.

Let $\pi : X'' \to X''//G$ be the quotient morphism. It follows from the definition of a good quotient that $\pi''(X)$ and $\pi''(X''\backslash X')$ are disjoint closed subsets of $X''//G$. Hence there exists a $h \in A(X''//G)$ that vanishes on $X''\backslash X$, and does not vanish at any point of X. Let $X' \subset X''$ be the complement of the zero locus of $h \circ \pi$. It is an affine open G-invariant subset of X'' that contains X, and $\phi' = \phi''_{|X'} : X' \to Y'$ is quasi-finite. This proves lemma 3.4 and theorem 3.3. □

3.3. Remark : If X is integral it is easy from the preceeding proof that we can take $Z = \mathrm{Spec}(C)$, where C is the integral closure of the image of $A(Y)$ in $A(X)$, the morphisms ψ and i being induced by the inclusions of rings.

4. Étale morphisms

4.1. Étale morphisms

Definition 4.1. *Let X, Y be algebraic varieties, $f : X \to Y$ a morphism and $x \in X$. The morphism f is called* unramified *at x if we have $m_{f(x)}.\mathcal{O}_x = m_x$, and* étale *at x if it is flat at x and unramified at x.*

We say that f is unramified *(resp.* étale*) if it is* unramified *(resp.* étale*) at every point.*

if $x \in X$, an étale neighbourhood *of x is an étale morphism of algebraic varieties $f : Z \to X$ such that there exists $z \in Z$ such that $f(z) = x$.*

Proposition 4.2. *Let $f : X \to Y$ be a morphism of algebraic varieties.*

1 - Let $x \in X$. Then the following properties are equivalent :

(i) *f is étale at x.*
(ii) *f is flat at x and $\Omega_{X/Y} = 0$ at x.*
(iii) *The natural map $\widehat{\mathcal{O}}_{f(x)} \to \widehat{\mathcal{O}}_x$ is an isomorphism.*

2 - The set of points of X where f is unramified (resp. étale) is open.

(see [7], [6]).

If Let $f : X \to Y$ be a morphism of algebraic varieties, with X, Y irreducible. If f is unramified at some point then $\dim(X) \leq \dim(Y)$. If f is étale at some point of X then we have $\dim(X) = \dim(Y)$. Conversely, if $\dim(X) = \dim(Y)$ and f is unramified, it is not always true that f is étale. For example, let Y be a reduced irreducible curve, y an ordinary double point of Y, and $f : X = \widetilde{Y} \to Y$ the normalization of Y. If $x \in \widetilde{Y}$ is above y, then $\widehat{\mathcal{O}}_y$ is isomorphic to $k[[X,Y]]/(XY)$, $\widehat{\mathcal{O}}_x$ is isomorphic to $k[[X]]$, and $f^* : \widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$ is the obvious morphism. Hence f^* cannot be an isomorphism. However f is unramified and $\dim(X) = \dim(Y)$.

Proposition 4.3. *(i) An open immersion is étale.*

(ii) The composition of two étale morphisms is étale.

(iii) If $X \to Y$ is étale, and $T \to Y$ is any morphism of algebraic varieties, then the projection $X \times_Y T \to T$ is étale.

(iv) If $X \to Y$, $X' \to Y'$ are étale, and $Y \to T$, $Y' \to T$ morphisms, the the cartesian product $X \times_T X' \to Y \times_T Y'$ is étale.

(v) If $f : X \to Y$, $g : Y \to Z$ are morphisms of algebraic varieties such that g and $g \circ f$ are étale, then f is also étale.

(vi) If $f : X \to Y$ is a morphism of algebraic varieties, $x \in X$ is such that X is smooth at x and Y smooth at $f(x)$, then f is étale at x if and only if the tangent map $T_x f : T_x X \to T_{f(x)} Y$ is an isomorphism.

(vii) If $X \to Y$ is étale and $x \in X$, then X is smooth at x if and only if Y is smooth at $f(x)$.

(see [6]).

Theorem 4.4. *Let $f : X \to Y$ be a morphism of algebraic varieties and $x \in X$. Then*

(i) If f is étale at x, $\mathcal{O}_x$ is normal if and only if $\mathcal{O}_{f(x)}$ is.

(ii) If $\mathcal{O}_{f(x)}$ is normal, f is étale at x if and only it is unramified at x and $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is injective.

([6], Exposé 1, théorème 9.5, p. 17).

Definition 4.5. *Let X, Y be algebraic varieties, $f : X \to Y$ a morphism. Then f is called an* étale covering *if it is étale and finite.*

Proposition 4.6. *Let $f : X \to Y$ be an étale morphism, Y being connected. Then for every closed point $y \in Y$, $f^{-1}(y)$ contains only a finite number of points, and f is an étale covering if and only if this number does not depend on y. It is an isomorphism if and only if this number is 1.*

Let $f : X \to Y$ be an étale covering, with Y connected. The number of points of the inverse image of a point of Y is called the *degree* of f.

4.2. Fiber bundles

4.2.1. *Fibrations*

Definition 4.7. *Let E, B, F be algebraic varieties. A morphism $\pi : E \to B$ is called a* fibration *with total space E, basis B and fiber F if there exists a surjective étale morphism $\phi : B' \to B$ and an isomorphism $\sigma : F \times B' \to E \times_B B'$ such that the following diagram is commutative*

$$\begin{tikzcd} F \times B' \arrow[rr, "\sigma"] \arrow[dr] & & E \times_B B' \arrow[dl] \\ & B' & \end{tikzcd}$$

(the maps to B' being the projections).

We have a commutative diagram

$$\begin{tikzcd} E \times_B B' \arrow[r, "p"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "\phi"] & B \end{tikzcd}$$

where p is the first projection, which is étale (cf. prop. 4.3, (iii)) like ϕ. So the new morphism $E \times_B B' \to B'$ is a kind of *étale modification* of $E \to B$.

Remark : the preceeding definition is equivalent to the following : π is a fibration with total space E, basis B and fiber F if for every $b \in B$, there exists an étale neighbourhood $\psi : U \to B$ of b and an isomorphism $\sigma : F \times U \to E \times_B U$ such that the diagram analogous to the preceeding one is commutative. This is because if $f : U \to B$, $f' : U' \to B$ are étale morphisms, then the sum $f \coprod f' : U \coprod U' \to B$ is also étale.

Definition 4.8. *Let G be a reductive algebraic group acting on affine varieties X and F. The quotient morphism $X \to X//G$ is called a* fibration with fiber *F if there exists an affine variety Z, an étale surjective morphism $Z \to X//G$ and a G-isomorphism $F \times Z \simeq X \times_{X//G} Z$ such that the following diagram is commutative*

$$\begin{tikzcd} F \times Z \arrow[rr, "\simeq"] \arrow[dr] & & X \times_{X//G} Z \arrow[dl] \\ & Z & \end{tikzcd}$$

If $F = G$, we say that X (or $X \to X//G$) is a principal G-bundle.

Recall that the second projection induces a canonical isomorphism

$$(X \times_{X//G} Z)//G \ \simeq \ Z.$$

The quotient morphism
$$X \times_{X//G} Z \longrightarrow Z$$
is then an étale modification of $X \to X//G$. It is easy to see that if X is a principal G-bundle, then G acts freely on X.

Let G be a reductive group acting on an affine variety Y, and $X \to X//G$ a principal G-bundle. Consider the product action of G on the affine variety $X \times Y$, and the projection $X \times Y \to X$ is a G-morphism. It is easy to see that the quotient morphism $(X \times Y)//G \to X//G$ is a fibration with fiber Y.

4.2.2. *Extension of group actions*

Let G be a reductive group, $H \subset G$ a closed reductive subgroup. Then H acts on G in the following way
$$\begin{aligned} H \times G &\longrightarrow G \\ (h,g) &\longmapsto gh^{-1} \end{aligned}$$
and G is a principal H-bundle.

Suppose that H acts on an affine variety Y. Then the preceeding action of H on G and its action on Y induce an action of H on the product $G \times Y$ in such a way that $G \times Y$ is a principal H-bundle. Let
$$G \times_H Y \ = \ (G \times Y)//H.$$
There is an obvious action of G on $G \times_H Y$. We will say that $G \times_H Y$ is obtained from Y by *extending the action of H to G.*

If $(g,y) \in G \times Y$, we will denote by $\overline{(g,y)}$ the image of (g,y) in $G \times_H Y$.

Proposition 4.9. *Let $X = G \times_H Y$. Then*

1 - The projection $G \times Y \to Y$ induces an isomorphism $X//G \simeq Y//H$.

2 - For every $u \in X//G = Y//H$, we have a canonical isomorphism
$$\pi_X^{-1}(u) \ \simeq \ G \times_H \pi_Y^{-1}(u).$$
3 - Let $y \in Y$, $g \in G$, $u = \overline{(g,y)} \in X$. Then $G_u = gG_y g^{-1}$.

4 - Let $X' \subset X$ be a closed G-invariant subvariety. Then there exists a closed H-invariant subvariety such that $X' = G \times_H Y'$.

5 - The closed orbits of X are the subvarieties $G \times_H Hy$, where Hy is a closed orbit of Y.

6 - If $y \in Y$, and $\overline{y}$ is the image of (e,y) in X, then we have a canonical isomorphism
$$TX_{\overline{y}} \ \simeq \ (T_e G \oplus T_y Y)/T_e H,$$

where the inclusion $T_eH \subset T_eG \oplus T_yY$ *comes from the tangent map at* e *of the morphism*

$$\begin{aligned} H &\longrightarrow G \times Y \\ h &\longmapsto (h^{-1}, hx) \end{aligned}$$

Moreover if Y *is smooth at* y*, then* X *is smooth at* $\overline{y}$*.*

Proof. To prove 1- we consider the morphism

$$\begin{aligned} \phi : Y &\longrightarrow G \times Y \\ y &\longmapsto (e, y) \end{aligned}$$

and $\overline{\phi} = \pi_X \circ \pi_{G\times Y} \circ \phi : Y \to X//G$. It is clear that $\overline{\phi}$ is H-invariant, and that the induced morphism $Y//H \to X//G$ is the inverse of the morphism $X//G \to Y//H$ induced by the projection $G \times Y \to Y$.

Assertion 2- and 3- are straightforward.

Now we prove 4-. Let $Z = \pi_{G\times Y}^{-1}(X') \subset G \times Y$. It is a G-invariant closed subvariety, hence of the form $Z = G \times Y'$, where Y' is a H-invariant closed subvariety of Y. Now X' is isomorphic to $G \times_H Y'$, by proposition 2.18.

Assertion 5- is an easy consequence of the preceeding ones, and 6- comes from the fact that $G \times Y$ is a principal H-bundle. □

4.3. Quotients by finite groups

Let G be a finite group acting on a normal connected variety X. A finite group is reductive, so there exists a good quotient of X by G which in this case is a geometric quotient, since the orbits are closed. Let

$$\pi : X \longrightarrow X/G$$

be the quotient morphism. We will suppose that G acts *faithfully*, i.e that the only element of G that acts trivially is e (the identity element of G).

Proposition 4.10. *The extension* $\mathbb{C}(X)$ *of* $\mathbb{C}(X/G)$ *is finite and Galois, with Galois group isomorphic to* G *.*

Proof. Since the orbits of π are finite and π is surjective, we have $\dim(X) = \dim(X/G)$ and $\mathbb{C}(X)$ is finite over $\mathbb{C}(X/G)$.

Let $f \in \mathbb{C}(X)$. Then the polynomial $P_f = \prod_{g\in G}(T - f^g)$ has coefficients in $\mathbb{C}(X)^G \subset \mathbb{C}(X/G)$, vanishes at f and has all its roots in $\mathbb{C}(X)$. It follows that $\mathbb{C}(X)$ is a Galois extension of $\mathbb{C}(X/G)$.

Let $\mathcal{G}$ be the Galois group of $\mathbb{C}(X)$ over $\mathbb{C}(X/G)$. Let α be a primitive element of $\mathbb{C}(X)$ over $\mathbb{C}(X/G)$ (i.e. $\mathbb{C}(X) = \mathbb{C}(X/G)(\alpha)$, cf. [9], VII, 6). The action of G on X induces a canonical morphism $i : G \to \mathcal{G}$ which is injective, because G acts faithfully. On the other hand we have $\mathbb{C}(X) \simeq \mathbb{C}(X/G)[T]/(P_\alpha)$. Let $\sigma \in \mathcal{G}$. Then $\sigma\alpha$ is a root of P_α, therefore there exists $g \in G$ such that $\sigma\alpha = \alpha^g$. It follows that $\sigma = i(g)$ and i is surjective. □

Proposition 4.11. *Let $x \in X$. Then π is étale at x if and only if G_x is trivial.*

Proof. Suppose that G_x is trivial. We will prove that π is étale at x. From the definition of a good quotient we have $\mathcal{O}_{\pi(x)} \subset \mathcal{O}_x$. Since X is normal, the quotient $X//G$ is also normal (cf. prop. 2.15). By theorem 4.4 it follows that we need only to prove that π is unramified at x. Let $u \in m_x/m_x^2$ be such that $u \neq 0$. Then there exists $f \in A(X)$ such that its image in m_x/m_x^2 is u, and that does not vanish at any point of Gx distinct of x. Let

$$\phi \ = \ \prod_{g \in G} f^g.$$

Then ϕ in G-invariant, and can be considered as an element of $A(X/G)$. The image of ϕ in m_x/m_x^2 is a non-zero multiple of u. It follows from Nakayama's lemma that the morphism $\mathcal{O}_x.m_y \to m_x$ is surjective. Hence f is unramified at x.

Suppose now that G_x is not trivial. Let $g \in G_x$, $g \neq e$. Let

$$\nu : \mathcal{O}_x \longrightarrow \mathcal{O}_x, \qquad \nu_2 : m_x/m_x^2 \longrightarrow m_x/m_x^2$$

be the morphisms induced by the multiplication by g. We will prove that ν_2 is not the identity. For this we will show that if $\nu_2 = Id$, then for every $n \geq 2$, the morphism induced by ν

$$\nu_n : m_x/m_x^n \longrightarrow m_x/m_x^n$$

is also the identity. This clearly implies that ν is the identity and that g acts trivially on X. But this is impossible because G acts faithfully.

Suppose that ν_2 is the identity. We prove that $\nu_n = Id$ by induction on n. This is true for $n = 2$. Suppose that it is true for n. Let $x_1, \ldots, x_p \in m_x$ be such that their images in m_x/m_x^2 form a basis of this vector space. We will note also x_i the image of x_i in m_x/m_x^q, for every $q \geq 2$. To prove that $\nu_{n+1} = Id$, it suffices to show that $\nu_{n+1}(x_i) = x_i$ for $1 \leq i \leq p$. Since ν_n is the identity, we can write in m_x/m_x^{n+1}

$$\nu_{n+1}(x_i) \ = \ x_i + P_i(x_1, \ldots, x_p)$$

for $1 \leq i \leq p$, P_i being a homogeneous polynomial of degree n. Since G is finite, there exists an integer $m \geq 2$ such that $g^m = e$. It follows that $\nu \circ \cdots \circ \nu$ (m times) is the identity, and so is $\nu_{n+1} \circ \cdots \circ \nu_{n+1}$. We have

$$\nu_{n+1} \circ \cdots \circ \nu_{n+1}(x_i) \ = \ x_i + mP_i(x_1, \ldots, x_p) \ = \ x_i,$$

hence $P_i(x_1, \ldots, x_p) = 0$, and $\nu_{n+1} = Id$.

Let $\mu_g : X \longrightarrow X$ be the multiplication by g. Then

$$T(\mu_g)_x = \ {}^t\nu_2 : TX_x \longrightarrow TX_x$$

is not the identity. We have

$$T(\pi_X)_x \circ T(\mu_g)_x \ = \ T(\pi_X)_x,$$

hence $T(\pi_X)_x$ is not injective and π_X is not étale at x. □

Let $H \subset G$ be a subgroup. Then π is H-invariant, so it induces a morphism

$$\pi_{G,H} : X//H \longrightarrow X//G.$$

Let π_H denote the quotient morphism $X \to X//H$.

Proposition 4.12. *Let $x \in X$. Then $\pi_{G,H}$ is étale at $\pi_H(x)$ if and only if $G_x \subset H$.*

Proof. If H is a normal subgroup, it is a consequence of proposition 4.11, since in this case $\pi_{G,H} : X//H \longrightarrow X//G$ is the good quotient of $X//H$ by G/H.

In the general case, the proof is similar to that of proposition 4.11. $\square$

Now we give the algebraic version of proposition 4.12, as given in [14].

Proposition 4.13. *Let B be a $\mathbb{C}$-algebra of finite type, integral and integrally closed, L its field of fractions, K' a finite Galois extension of L with Galois group $\mathcal{G}$, A' the integral closure of B in K', m' a maximal ideal in A', $n = B \cap m'$. Let $\mathcal{H} \subset \mathcal{G}$ be a subgroup, K the fixed field of $\mathcal{H}$, A the integral closure of B in K, and $m = A \cap m'$. Then A_m is étale on B_n if and only if the decomposition group of m' is contained in $\mathcal{H}$.*

Proof. Let

$$X = \mathrm{Spec}(A), \quad Y = \mathrm{Spec}(B), \quad Z = \mathrm{Spec}(A').$$

Then it is easy to see that

$$X = Z/\mathcal{H}, \quad Y = Z/\mathcal{G},$$

and that the inclusion $B \subset A$ induces the canonical morphism $Z/\mathcal{H} \to Z/\mathcal{G}$. Let z be the closed point of Z corresponding to m'. Then the image x (resp. y) of z in $Z/\mathcal{H}$ (resp. $Z/\mathcal{G}$) corresponds to m (resp. n). Proposition 4.13 is then only a translation of proposition 4.12. $\square$

4.4. Étale G-morphisms

4.4.1. *Strongly étale morphisms*

Let G be an algebraic group acting on affine varieties X and Y. Let $\phi : X \to Y$ be a G-morphism.

Definition 4.14. *We say that ϕ is* strongly étale *if :*

(i) $\phi_{/G} : X//G \to Y//G$ *is étale.*
(ii) ϕ *and the quotient morphism $\pi_X : X \to X//G$ induce a G-isomorphism*

$$X \ \simeq \ Y \times_{Y//G} \ (X//G).$$

Proposition 4.15. *Let $\phi : X \to Y$ be a strongly étale G-morphism. Then*

1 - *ϕ is étale and surjective.*
2 - *For every $u \in X//G$, ϕ induces an isomorphism $\pi_X^{-1}(u) \simeq \pi_Y^{-1}(\phi_{/G}(u))$.*
3 - *For every $x \in X$, the restriction of ϕ to Gx is injective, and Gx is closed if and only $G\phi(x)$ is closed.*

Proof. Immediate from the definition of a strongly étale G-morphism. $\square$

4.4.2. *Properties of étale G-morphisms*

Let G be a reductive algebraic group acting on algebraic varieties X, Y, $\pi_X : X \to X//G$, $\pi_Y : Y \to Y//G$ the quotient morphisms. Let $\theta : X \to Y$ be a G-morphism, $u \in X//G$, $x \in T(u)$ (recall that $T(u)$ is the unique closed orbit in $\pi_X^{-1}(u)$, cf. lemma 2.13), and $y = \theta(x) \in Y$.

Proposition 4.16. *If X and Y are normal, θ is finite, θ étale at x, and $\theta_{|T(u)}$ injective, then $\theta_{/G}$ is étale at u.*

Proof. Here we give the proof for when G is connected and X, Y irreducible. For the complete proof, see [14].

Then since θ is étale at x, $\theta^* : \mathcal{O}_y \to \mathcal{O}_x$ is injective, and we can consider $A(Y)$ as a subring of $A(X)$ and $\mathbb{C}(Y)$ as a subfield of $\mathbb{C}(X)$. Let K be a finite Galois extension of $\mathbb{C}(Y)$ that contains $\mathbb{C}(X)$ (this is possible because $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(Y)$), with Galois group $\mathcal{G}$. Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ corresponding to $\mathbb{C}(X)$ (i.e. $K^{\mathcal{H}} = \mathbb{C}(X)$). Let C be the integral closure of $A(Y)$ in K, C' the integral closure of $A(Y)^G$ in K. Then C and C' are finitely generated $\mathbb{C}$-algebras (cf. 1.5.3). Let

$$Z = \mathrm{Spec}(C), \qquad Z' = \mathrm{Spec}(C').$$

Then C and C' are $\mathcal{G}$-invariant. We will prove that

(i) $Z/\mathcal{G} = Y$,
(ii) $Z'/\mathcal{G} = Y//G$,
(iii) $Z/\mathcal{H} = X$,
(iv) $Z'/\mathcal{H} = X//G$.

proof of (i). This is clear from the definition of Galois extensions.

proof of (ii). We have to show that $A(Y)^G = A(Z')^{\mathcal{G}}$. We have $A(Y)^G \subset C' = A(Z')$ (cf. definition of C'). On the other hand, the elements of $A(Y)^G$ are $\mathcal{G}$-invariant (because they are in $A(Y)$) , therefore $A(Y)^G \subset A(Z')^{\mathcal{G}}$. Conversely, $A(Z')^{\mathcal{G}}$ is integral over $A(Y)^G$ (because $A(Z')$ is), and we have $A(Z')^{\mathcal{G}} \subset K^{\mathcal{G}} = \mathbb{C}(Y)$. Hence, $A(Y)$ being integrally closed, we have $A(Z')^{\mathcal{G}} \subset A(Y)$. It is now enough to prove that $A(Y)^G$ is integrally closed in $A(Y)$. Let $f \in A(Y)$ and $a_1, \ldots, a_n \in A(Y)^G$ be such that

$$f^n + a_1 f^{n-1} + \cdots + a_1 f + a_0 \ = \ 0.$$

Then, for every $g \in G$ we have

$$(f^g)^n + a_1 (f^g)^{n-1} + \cdots + a_1 f^g + a_0 \ = \ 0.$$

Since the polynomial $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$ has a finite number of roots in K it follows that the orbit Gf is finite, and since G is connected, we have $f \in A(Y)^G$. Hence $A(Y)^G$ is integrally closed in $A(Y)$.

proof of (iii). This comes from the well known correspondence between subgroups of $\mathcal{G}$ and subfields of K containing $\mathbb{C}(Y)$.

proof of (iv). Similar to the proof of (ii).

The situation can be described with the following diagram :

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "\pi_{\mathcal{H}}"] & Z' \arrow[d] \\
Z/\mathcal{H} = X \arrow[r, "\pi_X"] \arrow[d, "\theta"] & Z'/\mathcal{H} = X//G \arrow[d, "\theta_{/G}"] \\
Z/\mathcal{G} = Y \arrow[r, "\pi_Y"] & Z'/\mathcal{G} = Y//G
\end{tikzcd}$$

where the morphism $Z \to Z'$ comes from the inclusion $C' \subset C$. Let $z \in Z$ over $x \in X$, $z' \in Z'$ the image of z in Z'. Then the image of z' in $X//G = Z'/\mathcal{H}$ is u. Since θ is étale at x, proposition 4.12 implies that $\mathcal{G}_z \subset \mathcal{H}$. We must prove that $\theta_{/G}$ is étale at $u = \pi_X(x)$. By proposition 4.12 this is equivalent to $\mathcal{G}_{z'} \subset \mathcal{H}$.

Let $\sigma \in \mathcal{G}_{z'}$. Then it is clear that $\theta(\pi_{\mathcal{H}}(\sigma z)) = y$ (because $Y = Z/\mathcal{G}$), and that $\pi_X(\pi_{\mathcal{H}}(\sigma z)) = u$. From lemma 2.14 and the facts that θ is finite and Gx is closed, we deduce that Gy and $G\pi_{\mathcal{H}}(\sigma z)$ are closed too. Since Gx is the only closed orbit over u, we have $\pi_{\mathcal{H}}(\sigma z) \in Gx$. Since $\theta_{|Gx}$ is injective, we have $\pi_{\mathcal{H}}(\sigma z) = x = \pi_{\mathcal{H}}(z)$. Since $X = \mathcal{Z}/\mathcal{H}$, there exists $\tau \in \mathcal{H}$ such that $\tau z = \sigma z$. Hence $\tau^{-1}\sigma \in \mathcal{G}_z \subset \mathcal{H}$ and $\sigma \in \mathcal{H}$.

□

Proposition 4.17. *Suppose that θ is étale at x, X normal at x, $\theta(Gx)$ closed and $\theta_{|Gx}$ injective. Then there exists an affine open subset $U' \subset X//G$ containing u such that, if $U = \pi_X^{-1}(U')$, the following properties hold :*

(i) *$\theta_{|U}$ and $(\theta_{/G})_{|U'}$ are étale.*
(ii) *The open subsets $V = \theta(U)$ and $V' = (\theta_{/G})(U')$ are affine, and $V = \pi_Y^{-1}(V')$.*
(iii) *$\theta_{|U} : U \to V$ sends closed orbits in U to closed orbits in V.*

Proof. We suppose that G is connected. The sets of normal points of X and Y are open and G-invariant, and contain respectively Gx and $\theta(Gx)$. Hence there exist $f \in k[X]^G$ and $g \in k[Y]^G$ such that X_f and Y_g are normal and contain respectively $T(u)$ and $T(\theta(u))$. By replacing X and Y by the saturated open subsets $X_f \cap \theta^{-1}(Y_g)$ and Y_g respectively, we can assume that X and Y are normal. We can also in the same way suppose that $X//G$ and $Y//G$ are irreducible and that the fibers of θ are finite.

Now we use theorem 3.3 to factorize θ. There exists an affine G-variety Z, an open immersion $i : X \to Z$ which is a G-morphism, and a finite G-morphism $\psi : Z \to Y$, such that $\psi \circ i = \theta$. Then Gx is closed in Z : since $\theta(Gx)$ is closed, $T = \psi^{-1}(\theta(Gx))$ is also closed. If O is an orbit in T, we have $\dim(O) \geq \dim(\theta(Gx)) = \dim(Gx)$. Since ψ is finite, we have $\dim(O) \leq \dim(Gx)$, hence $\dim(O) = \dim(Gx) = \dim(T)$. It follows that T contains a finite number of orbits that have the same dimension as Gx. Hence Gx, being a component of T, is closed.

Now ψ is étale at x, and finite. To apply lemma 4.16 to it we need to verify that Z is normal. This follows from 3.3, if we take for $A(Z)$ the integral closure of the image of $A(Y)$ in $A(X)$. Hence $\psi_{/G}$ is étale at $i_G(u)$. We can find a $f \in A(Z)^G \subset A(X)^G$ such that

- We have $Z_f = X_f$ (i.e. f vanishes on $Z_f \backslash X_f$).

- The restrictions of ψ and $\psi_{/G}$ to X_f and $(X//G)_f$ respectively are étale.

This follows from the fact that the set of points z of Z such that $z \notin X_f$ or ψ_z is not étale at z is closed, G-invariant, and does not meet the closed orbit $T(u)$. Hence there exists an element of $A(Z)^G$ that vanishes on the first closed subset and not on Gx. In the same way there exists $g \in A(Y)^G$ such that $\theta(Gx) \subset Y_g \subset \theta(X_f)$ (this follows from the fact that the restriction of θ to X_f is open). Let

$$U' \;=\; (X//G)_f \cap (\theta_{/G})^{-1}((Y//G)_g).$$

Now we will verify that U' has all the properties required by proposition 4.17.

We have

$$V' = (Y//G)_g, \quad U = X_f \cap \theta^{-1}(Y_g), \quad V = Y_g.$$

The first and third equalities come from the fact that $Y_g \subset \theta(X_f)$ and the second is straightforward.

Now we prove that the properties (i), (ii) and (iii) are satisfied. Property (i) is immediate from the definition of f, and (ii) also. To prove (iii), let $x \in U$ be such that Gx is closed in U. Then Gx is also closed in Z : this comes from the fact that U is saturated in Z. Since ψ is finite it is closed, and $\psi(Gx) = \theta(Gx)$ is closed in Y. □

Proposition 4.18. *Suppose that θ is étale at x, that X is normal at x, that the orbit $G\theta(x)$ is closed and that $\theta_{|Gx}$ is injective. Then there exists an affine open subset $U \in X$ containing x such that the following properties are satisfied :*

(i) *U is saturated (cf. 2.3).*
(ii) *$V = \theta(U)$ is an affine open saturated subset of Y.*
(iii) *The restriction of $\theta : U \to V$ is strongly étale.*

Proof. If we take the same U as in lemma 4.17 then properties (i), (ii), are satisfied, $\theta_{|U}$ is étale, and so is the quotient morphism

$$\theta_{/G} : \pi_X(U) \simeq U//G \longrightarrow \pi_Y(V) \simeq V//G.$$

We have only to prove that the G-morphism

$$\chi : U \longrightarrow V \times_{V//G} U//G$$

induced by $\theta : U \to V$ and the quotient morphism $\pi_U : U \to U//G$ is an isomorphism.

First we prove that

$$\big(V \times_{V//G} U//G\big)//G \;\simeq\; U//G.$$

Let $Z = V \times_{V//G} U//G$. The composition

$$U \xrightarrow{(\theta,\pi_U)} V \times_{V//G} U//G \xrightarrow{p_2} U//G$$

(where p_2 is the second projection) is equal to π_U. Hence taking quotients we obtain that the composition

$$U//G \longrightarrow Z//G \longrightarrow U//G$$

is the identity. We have only to prove that the composition λ

$$Z//G \longrightarrow U//G \longrightarrow Z//G$$

is the identity. We have, for $u \in U$, $v \in V$ such that $\theta_{/G}(\pi_U(u)) = \pi_V(v)$,

$$\lambda(\pi_Z(v, \pi_U(u))) \;=\; \pi_Z(\theta(u), \pi_U(u)).$$

But since $\pi_V(v) = \pi_V(\theta(u))$, the closures of the G-orbits of v and $\theta(u)$ meet. Hence the closures of the G-orbits of $(v, \pi_U(u))$ and $(\theta(u), \pi_U(u))$ in Z meet, which proves that λ is the identity.

It is clear that since θ sends closed orbits in U to closed orbits in V, χ sends closed orbits in U to closed orbits in Z. Hence, by theorem 3.3, 2, χ is finite.

Now we prove that χ is étale. If we identify V with $V \times_{V//G} V//G$, then we have

$$\theta \;=\; (I_V \times \theta_{/G}) \circ \chi.$$

It follows that χ is étale, from proposition 4.3, and the fact that θ and $I_V \times \theta_{/G}$ are étale.

It follows that χ is an étale covering. We have $\pi_Z \circ \chi = \pi_U$, hence

$$\pi_U(\chi^{-1}(\chi(Gx))) = \pi_Z \circ \chi(\chi^{-1}(\chi(Gx))) = \pi_Z(\chi(Gx)) = \pi_U(Gx),$$

and since Gx is closed, we have $\chi^{-1}(\chi(Gx)) = Gx$. Since $\theta_{|Gx}$ and $\theta = (I_V \times \theta_{/G}) \circ \chi$, $\chi_{|Gx}$ is also injective. It follows that the inverse images (by χ) of the elements of $\chi(Gx)$ can contain only one element. Therefore χ is of degree 1 and is an isomorphism from proposition 4.6. □

Proposition 4.19. *Let $Y' \subset Y$ be a G-invariant closed subvariety. Let $X' = Y' \times_Y X$, and*

$$\theta' = I_{Y'} \times \theta : X' = Y' \times_Y X \longrightarrow Y' \times_Y Y = Y'.$$

Suppose that θ and $\theta_{/G}$ are étale and that

$$\theta \times \pi_X : X \longrightarrow Y \times_{Y//G} \; (X//G)$$

is a G-isomorphism. Then θ' and $\theta'_{/G}$ are étale and

$$\theta' \times \pi_{X'} : X' \longrightarrow Y' \times_{Y'//G} \; (X'//G)$$

is a G-isomorphism.

Proof. By proposition 4.6, θ' is étale. We have

$$X' \simeq Y' \times_Y \big(Y \times_{Y//G} (X//G)\big) \simeq (Y' \times_Y Y) \times_{Y//G} (X//G) \simeq Y' \times_{Y//G} (X//G),$$

hence

$$X'//G \;\simeq\; (Y'//G) \times_{Y//G} (X//G),$$

and

$$\theta'_{/G} = I_{Y'//G} \times \theta_{/G} : X'//G = (Y'//G) \times_{Y//G} (X//G) \longrightarrow (Y'//G) \times_{Y//G} (Y//G) = Y'//G$$

is étale. We have

$$\begin{aligned} X' &\simeq Y' \times_{Y//G} (X//G) \simeq \big(Y' \times_{Y'//G} (Y'//G)\big) \times_{Y//G} (X//G) \\ &\simeq Y' \times_{Y'//G} \big((Y'//G) \times_{Y//G} (X//G)\big) \simeq Y' \times_{Y'//G} \; (X'//G). \end{aligned}$$

□

5. Étale slice theorem

5.1. The main results

Lemma 5.1. *Let G be a reductive algebraic group acting on an affine variety X. Let $x \in X$ and suppose that X is smooth at x, and that the isotropy group G_x of x is reductive. Then there exists a morphism $\phi : X \to T_xX$ such that*

1. *ϕ is G_x-invariant.*
2. *ϕ is étale at x.*
3. *$\phi(x) = 0$.*

Proof. Let $\mathbf{m} \subset A(X)$ be the maximal ideal corresponding to x. Then the quotient map $d : \mathbf{m} \to \mathbf{m}/\mathbf{m}^2 = (T_xX)^*$ is G_x-invariant. Let V be a finite dimensional linear subspace of $\mathbf{m}_x$ such that $d_{|V}$ is surjective, and V' a finite dimensional G_x-invariant linear subspace of $\mathbf{m}_x$ containing V. Since G_x is reductive, $\mathbf{m}/\mathbf{m}^2$ and V' can be decomposed in direct sums of irreducible representations of G_x. It follows that there exists a G_x-invariant linear subspace W of $A(X)$ such that $d_{|W} : W \to (T_xX)^*$ is a G_x-isomorphism. Let

$$\alpha = d_{|W}^{-1} : (T_xX)^* \longrightarrow W.$$

Then we can extend α to

$$S(\alpha) : S((T_xX)^*) \longrightarrow S(W)$$

(where for a vector space E, S(E) is the symetric algebra of E : $S(E) = \bigoplus_{n \geq 0} S^n E$). Using the canonical morphism $S(W) \to A(X)$, we obtain a morphism of $\mathbb{C}$-algebras

$$f : S((T_xX)^*) \longrightarrow A(X)$$

that corresponds to a morphism of algebraic varieties $\phi : X \to T_xX$. Now it is clear that conditions 1- and 3- are satisfied. It is easy to verify that the tangent map T_xf is an isomorphism, and 2- is a consequence of proposition 4.3, (vi). □

Lemma 5.2. *Let G be a reductive algebraic group acting on an affine variety X. Then there exists a smooth affine G-variety X_0 such that X is G-isomorphic to a closed G-invariant subvariety of X_0.*

Proof. Let $f_1, \dots, f_n$ be generators of the $\mathbb{C}$-algebra $A(X)$. From lemma 2.7 it follows that there exists a finite dimensional G-invariant linear subspace W of $A(X)$ containing $f_1, \dots, f_n$. Let $S(W)$ be the symetric algebra of W, and

$$S(W) \longrightarrow A(X)$$

the G-morphism of $\mathbb{C}$-algebras induced from the inclusion $W \subset A(X)$. This morphism is surjective, so we can take $X_0 = \mathrm{Spec}(S(W))$. □

Theorem 5.3. (**Luna's étale slice theorem**). *Let G be a reductive algebraic group acting on an affine variety X. Let $x \in X$ be such that the orbit Gx is closed. Then there exists a locally closed subvariety V of X such that*

(i) *V is affine and contains x.*
(ii) *V is G_x-invariant.*
(iii) *The image of the G-morphism $\psi : G \times_{G_x} V \to X$ induced by the action of G on X is a saturated open subset U of X.*
(iv) *The restriction of $\psi : G \times_{G_x} V \to U$ is strongly étale.*

Proof. Suppose first that X is smooth at x. Since Gx is closed, the group G_x is reductive (prop. 2.19). Hence we can apply lemma 5.1 : there exists a G_x-morphism $\phi : X \to T_x X$, étale at x and such that $\phi(x) = 0$. Since G_x is reductive, there exists a G_x-invariant linear subspace $N \subset T_x X$ such that

$$T_x X \ = \ T_x Gx \oplus N.$$

Let $Y = \phi^{-1}(N)$. It is a closed G_x-invariant subvariety of X containing x and it is smooth at x. Let $\overline{x} \in G \times_{G_x} Y$ be the image of (e, x). Then $G \times_{G_x} Y$ is smooth at $\overline{x}$ and the multiplication $G \times_{G_x} Y \to X$ is étale at $\overline{x}$. This comes from the description of the tangent space of $G \times_{G_x} Y$ at $\overline{x}$ (cf. prop. 4.9).

We can then apply lemma 4.18 to the morphism $G \times_{G_x} Y \to X$ and the theorem follows immediately.

In the general case, we embed X as a closed G-subvariety of a smooth affine G-variety X_0. The theorem is then true for X_0, and using proposition 4.19 one sees easily that this implies that it is true also for X. □

Note that (iii) implies that for every $x' \in V$, ψ induces an isomorphism

$$G \times_{G_x} \pi_V^{-1}(\pi_V(x')) \simeq \pi_X^{-1}(\pi_X(x')).$$

Therefore the restriction of ψ to every closed orbit is injective.

There is a more precise version of this theorem when X is smooth at x :

Theorem 5.4. *(***Luna's étale slice theorem at smooth points***). Let G be a reductive algebraic group acting on an affine variety X. Let $x \in X$ be such that X is smooth at x and the orbit Gx is closed. Then there exists a locally closed smooth subvariety V of X such that the properties (i) to (iv) of theorem 5.3, and an étale G_x-invariant morphism $\phi : V \to T_x V$ such that $\phi(x) = 0$, $T\phi_x = Id$, such that*

(v) *We have $T_x X = T_x(Gx) \oplus T_x V$.*
(vi) *The image of ϕ is a saturated open subset W of $T_x V$.*
(vii) *The restriction of $\phi : V \to W$ is a strongly étale G_x-morphism.*

Proof. This follows easily from the proof of theorem 5.3. □

In this case we have thus two strongly étale morphisms

$$\begin{tikzcd} & G \times_{G_x} V \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\ U & & W \subset G \times_{G_x} N_x \end{tikzcd}$$

where N_x denotes the normal space to Gx at x.

5.2. First applications to the study of quotients

Proposition 5.5. *Let G be a reductive group acting on an affine variety X. Let $x \in X$ be such that Gx is closed. Then there exists a saturated open subset $U \subset X$ containing x such that for every $x' \in U$ such that Gx' is closed, $G_{x'}$ is the conjugate of a subgroup of G_x.*

Proof. Take for U the U of theorem 5.3. Then $\psi^{-1}(Gx')$ is a finite set of closed G-orbits in $G \times_{G_x} V$. Let $y = \overline{(g_0, v)} \in \psi^{-1}(x')$ (with $g_0 \in G$, $v \in V$). Then by proposition 4.9, $G_x v$ is a closed orbit of V and $G_y = g_0 G_v g_0^{-1}$. By assertion (iv) of theorem 5.3, ψ induces an isomorphism

$$G.\overline{(g_0, v)} \;\simeq\; Gx'.$$

It follows that $G_{x'} = g_0 G_v g_0^{-1}$. □

Corollary 5.6. *Let G be a reductive group acting on an affine variety X. Let χ be a character of G. Let U_χ be the subset of $X//G$ of points u such that there exists $x \in \pi_X^{-1}(u)$ such that Gx is closed and χ is trivial on G_x. Then U_χ is open.*

The following propositions are easy consequences of theorem 5.4 :

Proposition 5.7. *Let G be a reductive group acting on an affine variety X. Let $X_0 \subset X$ be the set of points x such that Gx is closed and G_x is trivial. Then X_0 is a saturated open subset of X, and it is a principal bundle. Moreover, if $x \in X_0$ is a smooth point, then $\pi_X(x_0)$ is a smooth point of $X//G$.*

Proposition 5.8. *Let G be a reductive group acting on an affine smooth variety X. Then X^G is smooth.*

6. G-BUNDLES

6.1. G-bundles and the descent lemma

Let G be a reductive group acting on an algebraic variety X. Suppose that there exists a good quotient $\pi_X : X \to X//G$.

6.1.1. *G-bundles and G-line bundles*

Definition 6.1. *A* G-vector bundle *(or* G-bundle*) on X is an algebraic vector bundle E on X with a linear action of G over the action on X. A* G-line bundle *on X is a G-bundle of rank 1.*

More precisely this means that for every $x \in X$ and $g \in G$, we have $g.E_x = E_{gx}$, and the multiplication by g induces a linear isomorphism $E_x \simeq E_{gx}$. If F is an algebraic vector bundle on $X//G$ there is an obvious natural structure of G-bundle on $\pi_X^*(F)$. In particular, if V is a finite dimensional vector space, then $X \times V$ is a G-vector bundle, which is called *trivial.*

Two G-bundles E, E' are called *isomorphic* if there exists an isomorphism of vector bundles $E \simeq E'$ which is a G-morphism.

Let X' be another algebraic variety with an action of G, and $f : X' \to X$ a G-morphism. Let E be a G-bundle on X. Then we define in an obvious way the G-bundle $f^*(E)$ on X'.

If χ is a character of G, we define the G-line bundle $\mathcal{L}_\chi$ on X associated to χ as follows: the underlying line bundle is $\mathcal{O}_X = X \times \mathbb{C}$ and the action of G is

$$\begin{array}{ccc} G \times X \times \mathbb{C} & \longrightarrow & X \times \mathbb{C} \\ (g, x, t) & \longmapsto & (gx, \chi(g)t) \end{array}$$

In some particular cases all G-line bundles are of this type :

Proposition 6.2. *Suppose that X is connected, $Pic(X)$ is trivial and that one of the following conditions is satisfied :*

(1) Every invertible regular function on G is the product of a character and of a constant.

(2) Every regular function $X \to \mathbb{C}^*$ is constant.

Then every G-line bundle on X is isomorphic to some L_χ, where χ is a character of G.

Proof. Let L be a G-line bundle on X. Since $Pic(X)$ is trivial, the underlying line bundle of L is $X \times \mathbb{C}$, and the structure of G-bundle of L comes from a morphism

$$\theta : C \times X \longrightarrow \mathbb{C}^*$$

such that

$$(*) \qquad \theta(gg', x) \;=\; \theta(g, g'x)\theta(g', x)$$

for every $g, g' \in G$, $x \in X$. The action of G on L is

$$\begin{aligned} G \times X \times \mathbb{C} &\longrightarrow X \times \mathbb{C} \\ (g, x, t) &\longmapsto (gx, \theta(g,x)t) \end{aligned}$$

Suppose that we are in case (1) of the proposition. Then for every $x \in X$ there exists a character λ_x of G and $a_x \in \mathbb{C}^*$ such that $\theta(g,x) = a_x\lambda_x(g)$ for every $g \in G$. From $(*)$ we deduce that

$$a_x\lambda_x(gg') \;=\; a_{g'x}\lambda_{g'x}(g)a_x\lambda_x(g')$$

and it follows that

$$\lambda_x(g) \;=\; a_{g'x}\lambda_{g'x}(g).$$

Taking $g = e$ in the preceeding equality, we obtain $a_{g'x} = 1$. Hence all $a_x = 1$ for every $x \in X$. Now the group of characters of G is countable (cf. [1], prop. 10.7), hence for every $g \in G$, the regular function

$$\begin{aligned} X &\longrightarrow \mathbb{C}^* \\ x &\longmapsto \lambda_x(g) \end{aligned}$$

takes only countably many values. Hence it is constant, i.e. $\lambda_x(g)$ is independant of x : $\lambda_x = \lambda_0$ for every $x \in X$. It follows that $L = \mathcal{L}_{\lambda_0}$.

Suppose now that we are in case (2). It follows that $\theta(g,x)$ depends only on g : $\theta(g,x) = \lambda(g)$. From $(*)$ we deduce immediately that λ is a character and that $L = \mathcal{L}_\lambda$. □

For example, we are in case (2) of proposition 6.2 if X is a vector space.

Lemma 6.3. *Let $U \subset X$ be a G-invariant open subset such that* $\operatorname{codim}_X(X\backslash U) \geq 2$. *Suppose that X is normal. Then*

(i) *If E, E' are two G-bundles on X whose restrictions to U are isomorphic, then E and E' are isomorphic.*
(ii) *Every G-bundle on U can be extended to a G-bundle on X.*

Proof. Let

$$f : E_{|U} \longrightarrow E'_{|U}$$

be an isomorphism. Since X is normal, it can be extended to an isomorphism between the underlying algebraic vector bundles E, E'. By continuity, it is clear that this isomorphism is also an isomorphism of G-bundles. This proves (i).

Now we prove (ii). Let E be a G-bundle on U. Since X is normal, the underlying vector bundle can be extended to an algebraic vector bundle E' on X. Since $\operatorname{codim}_X(X\backslash U) \geq 2$ the multiplication $G \times E \to E$ can be extended to $G \times E' \to E'$, and by continuity it is easy to check that this defines a G-bundle structure on E' whose restriction to U is E. This proves (ii). □

Proposition 6.4. *Let E be a G-vector bundle on X. Suppose that X is affine. Then for every section s of E, there exists a G-invariant finite dimensional linear subspace W of $H^0(X,E)$ such that $s \in W$.*

Proof. Let
$$\begin{aligned}\sigma : G\times X &\longrightarrow E\\ (g,x) &\longmapsto gs(g^{-1}x)\end{aligned}$$
Let $p_X : G\times X \to X$ be the projection. Then since
$$H^0(G\times X, p_X^*(E)) \;=\; A(G)\otimes H^0(X,E)$$
there exists an integer $n>0$ and $\phi_1,\ldots,\phi_n \in A(G)$, $\psi_1,\ldots,\psi_n \in H^0(X,E)$ such that
$$s \;=\; \sum_{i=1}^n \phi_i \otimes \psi_i .$$
Let $V\subset H^0(X,E)$ the linear subspace generated by $\psi_1,\ldots,\psi_n$. Then $Gs\subset V$. We can take for W the linear subspace spanned by Gs. $\square$

It follows that there exists a Reynolds operator $H^0(X,E)\to H^0(X,E)^G$.

6.1.2. *G-bundles, extension of group actions and homogeneous spaces*

Let $H\subset G$ be a reductive subgroup. Let Y be an algebraic variety with an action of H, and F a H-bundle on Y. Consider the action of H on $G\times Y$ used to construct $G\times_H Y$ (cf. 4.2). Then the projection
$$p_Y : G\times Y \longrightarrow Y$$
is a H-morphism, so we obtain the H-bundle $p_y^*(F)$ on $G\times_H Y$. On this bundle there is also a trivial structure of G-bundle, compatible with its structure of H-bundle. Since $G\times Y$ is a principal H-bundle, $p_y^*(F)$ descends to $G\times_H Y$ and defines a G-bundle F_G on $G\times_H Y$. For every $g\in G$, $y\in Y$, we have a canonical isomorphism
$$F_{G,\overline{(g,y)}} \;\simeq\; F_y$$
such that if $h\in H$ we have a commutative diagram
$$\begin{tikzcd} F_{G,\overline{(g,y)}} \arrow[r, equal] \arrow[d, "\simeq"] & F_{G,\overline{(gh^{-1},hy)}} \arrow[d, "\simeq"] \\ F_y \arrow[r, "\times h"] & F_{hy} \end{tikzcd}$$
It is easy to see that every G-bundle on $G\times_H Y$ can be obtained in this way.

Here we will call *homogeneous spaces* the G-varieties which are isomorphic to some G/H where $H\subset G$ is a reductive subgroup.

Let N be a vector space with a linear action of H. Then $G\times_H N$ has a natural structure of G-bundle over G/H. This is a particular case of the preceeding construction : take the variety

$*$ (with one point) with the trivial action of H. Then N can be viewed as a H-bundle on $*$, and we have $G/H \simeq G \times_H *$, $G \times_H N \simeq N_G$. Every G-bundle on G/H is of this type.

Let $x \in X$ be a closed point such that Gx is closed. Then $gx \simeq G/G_x$ is an homogeneous space. Suppose that X is smooth at x (hence along Gx). Let $N(x)$ denote the normal bundle to Gx. This is an example of G-bundle on G/G_x.

Let $H \subset G$ be a reductive subgroup and $y \in Y$ such that Hy is closed. Suppose that $X = G \times_H Y$, and let $x = \overline{(e,y)} \in X$. Then Gx is closed and $G_x = H_y$. We have canonical isomorphisms

$$G \times_H Hy \simeq Gx, \quad N(x) \simeq N(y)_H.$$

6.1.3. *Descent lemma*

Definition 6.5. *We say that a G-bundle E on X* descends *to $X//G$ if there exists an algebraic vector bundle F on $X//G$ such that the G-bundles E and $\pi_X^*(F)$ are isomorphic.*

Lemma 6.6. *Let E be a G-bundle on X. Then E descends to $X//G$ if and only if for every $z \in X//G$ there exists an affine open neighbourhood U of z in $X//G$ such that $E_{|\pi^{-1}(U)}$ is trivial.*

Proof. If E descends to F, we can take for U an open affine neighbourhood of z such that $F_{|U}$ is trivial.

Conversely, suppose that for every $z \in X//G$ there exists an affine open neighbourhood U of z in $X//G$ such that $E_{|\pi^{-1}(U)}$ is trivial. We can suppose that X is connected. Let r be the rank of E. Let $(U_i)_{i\in I}$ be an affine open cover of $X//G$ such that $E_{|\pi^{-1}(U_i)}$ is trivial for every $i \in I$. Then for every $i \in I$ there exists a G-isomorphism

$$f_i : E_{|\pi^{-1}(U_i)} \ \simeq \ \mathcal{O}_{\pi^{-1}(U_i)} \otimes \mathbb{C}^r.$$

Let

$$g_{ij} = f_j \circ f_i^{-1} : \mathcal{O}_{\pi^{-1}(U_i \cap U_j)} \otimes \mathbb{C}^r \ \simeq \ \mathcal{O}_{\pi^{-1}(U_i \cap U_j)} \otimes \mathbb{C}^r.$$

The g_{ij} can be viewed as $r \times r$ matrices of G-invariant functions on $U_i \cap U_j$. These functions descend to $X//G$ and define isomorphisms

$$\overline{g_{ij}} : \mathcal{O}_{U_i \cap U_j} \otimes \mathbb{C}^r \ \simeq \ \mathcal{O}_{U_i \cap U_j} \otimes \mathbb{C}^r.$$

Let F be the vector bundle on $X//G$ defined by the cocycle (g_{ij}). Then it is easy to see that $E \simeq \pi^*(F)$. □

Proposition 6.7. *If π_X is a geometric quotient and G acts freely on X, then every G-bundle on X descends to $X//G$*

(cf. [11], prop. 4). This result can be generalized :

Theorem 6.8. *Let E be a G-bundle on X. Then E descends to $X//G$ if and only if for every closed point $x \in X$ such that the orbit Gx is closed, the stabilizer G_x of x acts trivially on E_x.*

(see [3], théorème 2.3. For the case where Z is not integral, cf. [13], lemme 1.4).

Proof. If E descends to $X//G$ it is clear that G_x acts trivially on E_x.

Conversely, suppose that for every closed point $x \in X$ such that the orbit Gx is closed, G_x acts trivially on E_x. From lemma 6.6, we have only to show that for every $z \in X//G$, there exists an affine open neighbourhood U of z such that $E_{|\pi^{-1}(U)}$ is trivial. Let $x \in X$ such that $x \in \pi^{-1}(z)$ and that Gx is closed.

We first prove that $E_{|Gx}$ is trivial. Let $u_1, \ldots, u_r$ be a basis of E_x,

$$\begin{aligned} \phi : G &\longrightarrow Gx \\ g &\longmapsto gx \end{aligned}$$

$$\begin{aligned} \gamma_i : G &\longrightarrow E_{|Gx} \\ g &\longmapsto gu_i \end{aligned}$$

for $1 \leq i \leq r$. Then we have a commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\gamma_i"] \arrow[dr, "\phi"'] & & E_{|Gx} \arrow[dl] \\ & Gx & \end{tikzcd}$$

Since G_x acts trivially on E_x this diagram induces the following commutative one

$$\begin{tikzcd} G/G_x \arrow[rr, "\overline{\gamma}_i"] \arrow[dr, "\psi"'] & & E_{|Gx} \arrow[dl] \\ & Gx & \end{tikzcd}$$

where ψ is an isomorphism. We obtain

$$\sigma_i = \overline{\gamma}_i \circ \psi^{-1} : Gx \longrightarrow E_{|Gx},$$

which is a G-invariant section of $E_{|Gx}$. These sections define a G-isomorphism

$$\mathcal{O}_{Gx} \otimes \mathbb{C}^r \ \simeq \ E_{|Gx},$$

showing that $E_{|Gx}$ is trivial.

Let V be an affine open neighbourhood of z, and

$$R : H^0(\pi^{-1}(V), E) \longrightarrow H^0(\pi^{-1}(V), E)^G, \qquad R' : H^0(Gx, E) \longrightarrow H^0(Gx, E)^G$$

the Reynolds operators (cf. prop. 6.4). Let

$$\sigma = (\sigma_1, \ldots, \sigma_r) : \mathcal{O}_{Gx} \otimes \mathbb{C}^r \longrightarrow E_{|Gx}$$

be a G-isomorphism. We can extend it to a morphism

$$(\overline{\sigma}_1, \ldots, \overline{\sigma}_r) : \mathcal{O}_{\pi^{-1}(V)} \otimes \mathbb{C}^r \longrightarrow E_{|\pi^{-1}(V)}.$$

Then

$$\alpha = (R(\overline{\sigma}_1), \ldots, R(\overline{\sigma}_r)) : \mathcal{O}_{\pi^{-1}(V)} \otimes \mathbb{C}^r \longrightarrow E_{|\pi^{-1}(V)}.$$

is a morphism of G-bundles. By functoriality of the Reynolds operators, we have

$$R(\overline{\sigma}_i)_{|Gx} \ = \ R'(\sigma_i) \ = \ \sigma_i$$

for $1 \leq i \leq r$.

Let $W \subset \pi^{-1}(V)$ be the open subset of points y such that α_y is an isomorphism. It suffices now to show that there exists an open affine neighbourhood U of z such that $\pi^{-1}(U) \subset W$. This is an easy consequence of the fact that $\pi^{-1}(V)\backslash W$ and Gx are disjoint G-invariant closed subvarieties of $\pi^{-1}(V)$. □

Let $H \subset G$ be a reductive subgroup. Let Y be an algebraic variety with an action of H, and F a H-bundle on Y. Let $u = \overline{(g, y)} \in G \times_H Y$. Then we have $G_u = gH_y g^{-1}$ and it is easy to check that G_u acts trivially on $E_{G,u}$ if and only if H_y acts trivially on E_y.

From this and from the descent lemma it follows immediately that E_G descends to $(G \times_H Y)//G = Y//H$ if and only E descends to $Y//H$ and the corresponding bundles on $Y//H$ are of course the same.

6.2. Models and principal models

The results of this section are left as exercises (for some proofs, see [14]).

Definition 6.9. *A* model *is a G-bundle over an homogeneous space. Two models E, F over G/H, G/K respectively are* isomorphic *if there exists $g_0 \in G$ such that $g_0^{-1}Hg_0 = K$, and that the G-bundles E and $\phi^*(F)$ on G/H are isomorphic, where $\phi : G/H \to G/K$ is the isomorphism induced by the automorphism of G*

$$g \longmapsto g_0^{-1} g g_0$$

Let X be a smooth connected G-variety. Let $x \in X$ be a closed point such that Gx is closed. Then $N(x)$ is a model (cf. 6.1.2). If $x' \in Gx$, then $N(x)$ and $N(x')$ are isomorphic.

Let $\mathcal{N}(G)$ denote the set of isomorphism classes of models, and let

$$\mu_X : X//G \longrightarrow \mathcal{N}$$

be the mapping that associates to u the isomorphism class of $N(x)$ where $x \in X$ is such that Gx is the unique closed orbit of $\pi_X^{-1}(u)$.

Proposition 6.10. *The image of μ_X is finite, and for every $\lambda \in \mathrm{Im}(\mu_X)$, $\mu^{-1}(\lambda)$ is a locally closed smooth subvariety of X.*

Hint : This can be proved by induction on $\dim(X)$.

Suppose that G acts linearly on a finite dimensional vector space E. Let E_G be the kernel of the Reynolds operator $E \to E^G$. It is the unique complementary G-invariant subspace of E^G. We have a canonical isomorphism

$$E//G \simeq E^G \times E_G//G.$$

Definition 6.11. *Let G/H, G/K be homogeneous spaces. We say that G/H is* bigger than *G/K if some conjugate of H is contained in K. We obtain in this way an order relation on the set of homogeneous spaces (i.e. the set of reductive subgroups of G).*

Proposition 6.12. *Let $u \in X//G$, $x \in \pi^{-1}(u)$ be such that Gx is closed, N_x the normal space to Gx at x. Then the following conditions are equivalent :*

(i) *$\mu_X^{-1}(N(x))$ is open.*
(ii) *Gx is maximal in the set of closed orbits of X.*
(iii) *$\pi_{N_x}^{-1}(\pi_{N_x}(0)) = (N_x)_{G_x}$.*
(iv) *π_X is smooth at x.*

If the conditions of the preceeding proposition are satisfied, we say that $N(x)$ is the *principal model* of X. Of course this notion is interesting only if there is no open G-invariant subset of X on which G acts freely.

7. Description of moduli spaces of semi-stable vector bundles on curves at singular points

Let X be a projective smooth irreducible algebraic curve of genus $g \geq 2$. Some results of 7.1, 7.2 and 7.3 are given without proof (see [12], [23], [3]).

7.1. Semi-stable vector bundles on curves

If E is an algebraic vector bundle of rank $r > 0$ and degree d on X, the rational number

$$\mu(E) \ = \ \frac{d}{r}$$

is called the *slope* of E.

A subsheaf (or a subbundle) F of E is called *proper* if $F \neq 0$ and $F \neq E$.

We say that E is *simple* if the only endomorphisms of E are the homotheties.

Recall that if E, F are algebraic vector bundles on X, then an *extension* of F by E is an exact sequence of vector bundles on X

$$0 \longrightarrow E \longrightarrow \mathcal{E} \longrightarrow F \longrightarrow 0.$$

The vector bundle $\mathcal{E}$ in the middle is also called an extension of F by E.

Definition 7.1. *Let E be an algebraic (non-zero) vector bundle on X. Then E is called* semi-stable *(resp. stable) if for every proper subbundle $F \subset E$ we have*

$$\mu(F) \ \leq \ \mu(E) \qquad (\text{resp.} \ \ \mu(F) \ < \ \mu(E) \ )$$

If E is semi-stable (resp. stable) and $F \subset E$ is a proper subsheaf, then F is locally free and $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$), so in the definition we can replace subbundle by subsheaf.

Proposition 7.2. *Let E, F be semi-stable vector bundles on X.*

1 - If $\mu(F) < \mu(E)$ then $\mathrm{Hom}(E, F) = \{0\}$.

2 - If E, F are stable, $\mu(E) = \mu(F)$, and $\mathrm{Hom}(E, F) \neq \{0\}$ *then E and F are isomorphic.*

3 - If E is stable then E is simple.

4 - If $\mu(F) = \mu(E)$, then any extension of F by E is a semi-stable vector bundle of the same slope as E and F.

5 - If $\mu(F) = \mu(E)$, and $f : E \to F$ is a morphism, then f is of constant rank, and $\ker(f)$, $\mathrm{coker}(f)$ *are semi-stable vector bundles which have the same slope as E and F if they are non zero.*

Definition 7.3. *An algebraic vector bundle E on X is called* polystable *if it is isomorphic to a direct sum of stable bundles of the same slope.*

Proposition 7.4. *Let E be a polystable vector bundle on X. Then*

1 - There exist positive integers $m_1, \ldots, m_p$ and stable vector bundles $E_1, \ldots, E_p$ on X such that

(i) $\mu(E_i) = \mu(E)$ *for* $1 \leq i \leq p$.
(ii) *if* $1 \leq i < j \leq p$ *then E_i and E_j are not isomorphic.*
(iii) $E \simeq \displaystyle\bigoplus_{1 \leq i \leq p} E_i \otimes \mathbb{C}^{m_i}$.

2 - If the integers $n_1, \ldots, n_q$ and the stable bundles $F_1, \ldots, F_q$ have the same properties, then $p = q$ and there exists a permutation σ of $\{1, \ldots, p\}$ such that

$$n_i = m_{\sigma(i)}, \qquad F_i \simeq E_{\sigma(i)}$$

for $1 \leq i \leq p$.

Using proposition 7.2, it is easy to see that is μ is a rational number, then the category $\mathcal{C}(\mu)$ of semi-stable vector bundles on X of slope μ, together with the morphisms between them, is an abelian, artinian category. The stable bundles of slope μ are the simple objects in this category, i.e. they don't have proper subbundles of slope μ. Therefore we have the Jordan-Hölder theorem for $\mathcal{C}(\mu)$:

Proposition 7.5. *Let E be a semi-stable vector bundle on X. Then there exists a filtration of E by subbundles*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{p-1} \subset E_p = E$$

such that for $1 \leq i \leq p$, E_i/E_{i-1} is stable, and $\mu(E_i/E_{i-1}) = \mu(E)$.

Moreover, the isomorphism class of the polystable vector bundle $\bigoplus_{1 \leq i \leq p} E_i/E_{i-1}$ depends only on E.

Let $Gr(E) = \bigoplus_{1 \leq i \leq p} E_i/E_{i-1}$. If E, F are semi-stable vector bundles on X, we say that they are S-equivalent if $Gr(E) \simeq Gr(F)$. Note that if E is stable, E and F are S-equivalent if and only if they are isomorphic.

7.2. Moduli spaces of semi-stable vector bundles

Let r, d be integers, with $r \geq 1$.

Definition 7.6. *Let S be an algebraic variety. A* family *of vector bundles of rank r and degree d parametrized by S is a vector bundle $\mathcal{F}$ on $X \times S$ such that for every closed point $s \in S$, the vector bundle $\mathcal{F}_s$ (restriction of $\mathcal{F}$ to $X = X \times \{s\}$) is of rank r and degree d. We say that $\mathcal{F}$ is a family of semi-stable (resp. stable) vector bundles on X if for every closed point $s \in S$, $\mathcal{F}_s$ is semi-stable (resp. stable).*

Two families $\mathcal{E}$, $\mathcal{F}$ of vector bundles parametrized by S are isomorphic *if there exists a line bundle L on S such that $\mathcal{F} \simeq \mathcal{E} \otimes p_S^*(L)$ (where p_S denotes the projection $X \times S \to S$).*

If $f : T \to S$ is a morphism of algebraic varieties and $\mathcal{F}$ a family of vector bundles of rank r and degree d on X parametrized by S, let

$$f^\sharp(\mathcal{F}) \ = \ (I_X \times f)^*(\mathcal{F})$$

which is a family of vector bundles of rank r and degree d on X parametrized by T.

Proposition 7.7. *Let S be an algebraic variety and $\mathcal{F}$ a family of vector bundles of rank r and degree d on X parametrized by S. Then the subset of $S(\mathbb{C})$ of points s such that $\mathcal{F}_s$ is semi-stable (resp. stable) is Zariski open.*

We now define the contravariant functor $\mathcal{M}(r, d)$ from the category of algebraic varieties to the category of sets. If S is an algebraic variety, $\mathcal{M}(r, d)(S)$ is the set of isomorphism classes of families of semi-stable vector bundles of rank r and degree d on X parametrized by S. If

$f : T \to S$ is a morphism of algebraic varieties, then

$$\mathcal{M}(r,d)(f) = f^\sharp : \mathcal{M}(r,d)(S) \longrightarrow \mathcal{M}(r,d)(T).$$

We define in the same way the contravariant functor $\mathcal{M}^s(r,d)$ by considering families of stable bundles of rank r and degree d on X.

Let $S(r,d)$ be the set of S-equivalence classes of semi-stable vector bundles of rank r and degree d on X.

Theorem 7.8. *There exists a coarse moduli space $M(r,d)$ for the functor $\mathcal{M}(r,d)$. It is an integral projective algebraic variety of dimension $r^2(g-1)+1$. There is a canonical bijection $M(r,d) \simeq S(r,d)$. The subset $M^s(r,d)$ of $M(r,d)$ corresponding to stable bundles is a dense open subset. If r and d are coprime, $M(r,d)$ is a fine moduli space.*

Let's recall what means 'a coarse moduli space' : there is a morphism of functors

$$\mathcal{M}(r,d) \longrightarrow \mathrm{Mor}(-, M(r,d))$$

such that for any algebraic variety N and every morphism of functors $\mathcal{M}(r,d) \to \mathrm{Mor}(-,N)$, there is a unique morphism $f : M(r,d) \to N$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{M}(r,d) \arrow[r] \arrow[dr] & \mathrm{Mor}(-, M(r,d)) \arrow[d, "f"] \\ & \mathrm{Mor}(-, N) \end{tikzcd}$$

So to each family $\mathcal{F}$ of semi-stable bundles of rank r and degree d on X parametrized by an algebraic variety S one associates a morphism $S \to M(r,d)$ such that for every closed point $s \in S$, $f(s)$ is the point of $M(r,d)$ corresponding to the S-equivalence class of $\mathcal{F}_s$. The coarse moduli space $M(r,d)$ is unique up to isomorphism.

Suppose now that r and d are coprime. Then semi-stable vector bundles of rank r and degree d are actually stable, and the closed points of $M(r,d)$ are the isomorphism classes of stable vector bundles of rank r and degree d on X. In this case $M(r,d)$ represents the functor $\mathcal{M}(r,d)$. So there exists a *universal vector bundle* $\mathcal{E}$ on $X \times M(r,d)$, i.e. for every closed point $s \in S$, the isomorphism class of $\mathcal{E}_s$ corresponds to s. For every family $\mathcal{F}$ of stable bundles of rank r and degree d on X parametrized by an algebraic variety S, if $f : S \to M(r,d)$ is the the associated morphism, then the two families $\mathcal{F}$ and $f^\sharp(\mathcal{E})$ are isomorphic.

We now give some properties of these moduli spaces :

Theorem 7.9. *1 - Except when $g=2$, $r=2$ and d is even, the singular locus of $M(r,d)$ is exactly the subvariety corresponding to semi-stable non stable vector bundles.*

2 - The variety $M(r,d)$ is locally factorial, hence normal.

7.3. Construction of moduli spaces of semi-stable vector bundles

Let r, d be integers with $r \geq 1$. We will build $M(r,d)$ as a good quotient of a smooth variety on which a reductive group acts. The construction uses *Quot-schemes* (cf. [5], [8], [12]).

7.3.1. *Quot-schemes*

Let L be a line bundle of degree 1 on X. Consider the polynomial

$$P(T) \ = \ rT + d + r(1-g).$$

If E is a vector bundle of rank r and degree d on X we have $\chi(E \otimes L^n) = P(n)$ (Riemann-Roch's theorem). Let n be an integer, and $p = P(n)$.

Let $\mathrm{Quot}_X(L^{-n} \otimes \mathbb{C}^p, P)$ be the *Quot-scheme* of quotients of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P. We will give a short description of this variety.

A *family of quotients* of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P parametrized by an algebraic variety S is a surjective morphism of coherent sheaves on $X \times S$

$$\Phi : p_S^\sharp(L^{-n} \otimes \mathbb{C}_p) \longrightarrow \mathcal{F}$$

where $\mathcal{F}$ is flat on S, such that for every closed point s of S, $\mathcal{F}_s$ is a sheaf of rank r and degree d on X. Two such families $p_S^\sharp(L^{-n} \otimes \mathbb{C}^p) \to \mathcal{F}$ and $p_S^\sharp(L^{-n} \otimes \mathbb{C}^p) \to \mathcal{F}'$ are called *isomorphic* if there exists an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ such that the following diagram is commutative

$$\begin{tikzcd}
p_S^\sharp(L^{-n} \otimes \mathbb{C}^p) \arrow[r, two heads] \arrow[d, equal] & \mathcal{F} \arrow[d, "\alpha"] \\
p_S^\sharp(L^{-n} \otimes \mathbb{C}^p) \arrow[r, two heads] & \mathcal{F}'
\end{tikzcd}$$

If $f : Z \to S$ is a morphism of algebraic varieties and $\Phi : p_S^\sharp(L^{-n} \otimes \mathbb{C}^p) \to \mathcal{F}$ is a family of quotients of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P parametrized by S, then
$f^*(\Phi) : p_Z^\sharp(L^{-n} \otimes \mathbb{C}^p) \to f^*(\mathcal{F})$ is a family of quotients of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P parametrized by Z. In this way we define an obvious functor F from the category of algebraic varieties to the category of sets that associates to S the set of isomorphism classes of families of quotients of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P parametrized by S.

This functor is representable, by the projective variety $\mathbf{Q} = \mathrm{Quot}_X(L^{-n} \otimes \mathbb{C}^p, P)$. Precisely this means the following : on $X \times \mathbf{Q}$ there is a *universal family of quotients*, i.e. a family of quotients $\Phi_0 : p_{\mathbf{Q}}^\sharp(L^{-n} \otimes \mathbb{C}^p) \to \mathcal{F}_0$ such that for every algebraic variety S and family Φ of quotients of $L^{-n} \otimes \mathbb{C}^p$ with Hilbert polynomial P parametrized by S, there exists a unique morphism $S \to \mathbf{Q}$ such that $\Phi \simeq f^*(\Phi_0)$.

The closed points of $\mathrm{Quot}_X(L^{-n} \otimes \mathbb{C}^p, P)$ are the isomorphism classes of quotients
$L^{-n} \otimes \mathbb{C}^p \to F$ with F of rank r and degree d.

There is a canonical action of $\mathrm{GL}(p)$ on $\mathbf{Q}$: if $g \in \mathrm{GL}(p)$ and $q \in \mathbf{Q}$ is represented by the quotient

$$\phi : L^{-n} \otimes \mathbb{C}^p \longrightarrow E$$

then gq is represented by the quotient

$$L^{-n} \otimes \mathbb{C}^p \xrightarrow{g^{-1}} L^{-n} \otimes \mathbb{C}^p \xrightarrow{\phi} E$$

Clearly if g is an homothety then the quotient that represents gq is isomorphic to the one that represents q, hence we have in this case $gq = q$. So we have in fact an action of $\mathrm{PGL}(p)$ on $\mathbf{Q}$.

7.3.2. *Construction of moduli spaces*

Proposition 7.10. *There exists an integer n_0 such that for $n \geq n_0$ and every semi-stable vector bundle E on X of rank r and degree d we have*

1 - E(n) is generated by its global sections.

2 - $h^1(E \otimes L^n) = 0$.

Suppose that $n \geq n_0$, and let $\mathbf{Q} = \mathrm{Quot}_X(L^{-n} \otimes \mathbb{C}^p, P)$, with $p = P(n)$, and universal morphism

$$\Phi_0 : p_{\mathbf{Q}}^{\sharp}(L^{-n} \otimes \mathbb{C}^p) \longrightarrow \mathcal{F}_0.$$

Let R^{ss} be the open subset of $\mathbf{Q}$ whose closed points are the q such that

(i) $\mathcal{F}_{0q}$ is locally free and semi-stable.
(ii) Φ_0 induces an isomorphism $\mathbb{C}^p \simeq H^0(\mathcal{F}_{0q} \otimes L^n)$.

Let $R^s \subset R^{ss}$ be the open subset consisting of points q such that $\mathcal{F}_{0q}$ is stable. We will see later that R^{ss} is smooth.

It follows from (i) and (ii) that if closed points q, q' of R^{ss} are in the same $PGL(p)$-orbit if and only if $\mathcal{F}_{0q} \simeq \mathcal{F}_{0q'}$.

Lemma 7.11. *Let $q \in R^{ss}$. Then there are canonical isomorphisms* $\mathrm{GL}(p)_q \simeq \mathrm{Aut}(\mathcal{F}_{0q})$, $\mathrm{PGL}(p)_q \simeq \mathrm{Aut}(\mathcal{F}_{0q})/\mathbb{C}^*$.

Proof. This follows immediately from the identification of $\mathbb{C}^p$ with $H^0(\mathcal{F}_{0q} \otimes L^n)$. □

Theorem 7.12. *There exists an integer $n_1 \geq n_0$ such that if $n \geq n_1$ then there exists a good quotient $R^{ss}//\,\mathrm{PGL}(p)$ which is a coarse moduli space for semi-stable vector bundles of rank r and degree d on X.*

So if $n \geq n_1$, we have $M(r,d) = R^{ss}//\,\mathrm{PGL}(p)$. Let $\pi : R^{ss} \to M(r,d)$ be the quotient morphism. The open subset R^s of R^{ss} is saturated, its image in $M(r,d)$ is $M^s(r,d)$ and the restriction of π $R^s \to M^s(r,d)$ is a geometric quotient.

If $q \in R^{ss}$, then $q' \in \pi^{-1}(\pi(q))$ if and only if $Gr(\mathcal{F}_{0q}) \simeq Gr(\mathcal{F}_{0q'})$.

7.3.3. *Differential study of the Quot-scheme*

We will study here the Quot-scheme $\mathbf{Q} = \mathrm{Quot}_X(L^{-n}\otimes\mathbb{C}^p, P)$. Recall that we have an universal morphism on $X\times\mathbf{Q}$

$$\Phi_0 : p_{\mathbf{Q}}^{\sharp}(L^{-n}\otimes\mathbb{C}^p) \longrightarrow \mathcal{F}_0.$$

Let $\mathcal{N} = \ker(\Phi_0)$, so that we have an exact sequence of coherent sheaves on $X\times\mathbf{Q}$

$$0 \longrightarrow \mathcal{N} \xrightarrow{i} p_{\mathbf{Q}}^{\sharp}(L^{-n}\otimes\mathbb{C}^p) \xrightarrow{\Phi_0} \mathcal{F}_0 \longrightarrow 0$$

Let $q\in\mathbf{Q}$. Then the tangent space $T_q\mathbf{Q}$ is canonically isomorphic to $\mathrm{Hom}(\mathcal{N}_q,\mathcal{F}_{0q})$, and if $\mathrm{Ext}^1(\mathcal{N}_q,\mathcal{F}_{0q}) = \{0\}$, then $\mathbf{Q}$ is smooth at q (cf. [5], [12]).

Now suppose that $n\geq n_1$ and let $q\in\mathbf{Q}$. Then we have the exact sequence

$$0\longrightarrow \mathcal{N}_q \longrightarrow L^{-n}\otimes\mathbb{C}^p \longrightarrow \mathcal{F}_{0q}\longrightarrow 0$$

Using the fact that $h^1(\mathcal{F}_{0q}\otimes L^n) = 0$, it follows that $\mathrm{Ext}^1(\mathcal{N}_q,\mathcal{F}_{0q}) = \{0\}$. Therefore R^{ss} is smooth. We have the exact sequence

$$0\longrightarrow \mathrm{End}(\mathcal{F}_{0q})\longrightarrow \mathrm{Hom}(L^{-n}\otimes\mathbb{C}^p,\mathcal{F}_{0q})\longrightarrow \mathrm{Hom}(\mathcal{N}_q,\mathcal{F}_{0q})\longrightarrow \mathrm{Ext}^1(\mathcal{F}_{0q},\mathcal{F}_{0q})\longrightarrow 0.$$

From the definition of R^{ss}, we have a canonical isomorphism $\mathrm{Hom}(L^{-n}\otimes\mathbb{C}^p,\mathcal{F}_{0q})\simeq M(p)$, where $M(p)$ denotes the vector space of $p\times p$ matrices. Let

$$\begin{aligned}\alpha : \mathrm{GL}(p) &\longrightarrow R^{ss}\\ g &\longmapsto gq\end{aligned}$$

Then the map $\mathrm{Hom}(L^{-n}\otimes\mathbb{C}^p,\mathcal{F}_{0q})\to\mathrm{Hom}(\mathcal{N}_q,\mathcal{F}_{0q})$ of the preceeding exact sequence is nothing but the tangent map $T_I\alpha : M(p) = T_I\,\mathrm{GL}(p)\to T_qR^{ss}$. On the other end, the map $T_qR^{ss} = \mathrm{Hom}(\mathcal{N}_q,\mathcal{F}_{0q})\to\mathrm{Ext}^1(\mathcal{F}_{0q},\mathcal{F}_{0q})$ is the *Kodaïra-Spencer map* ω_q of the family $\mathcal{F}_0$ at q. Hence the preceeding exact sequence can be written as follows :

$$0\longrightarrow \mathrm{End}(\mathcal{F}_{0q})\longrightarrow M(p)\xrightarrow{T_I\alpha} T_qR^{ss}\xrightarrow{\omega_q}\mathrm{Ext}^1(\mathcal{F}_{0q},\mathcal{F}_{0q})\longrightarrow 0$$

The following lemma is an easy consequence of this exact sequence and of lemma 7.11 :

Lemma 7.13. *Let $q\in R^{ss}$ be a closed point and N_q the normal space to $\mathrm{GL}(p)q$ at q. Then there is a canonical $\mathrm{Aut}(\mathcal{F}_{0q})$-isomorphism*

$$N_q \;\simeq\; \mathrm{Ext}^1(\mathcal{F}_{0q},\mathcal{F}_{0q}).$$

7.4. Study of singular points

(cf. [10])

Lemma 7.14. *Let $q \in R^{ss}$ be a closed point. Then the orbit* $\mathrm{PGL}(p)q$ *is closed if and only if $\mathcal{F}_{0q}$ is polystable.*

Proof. The orbit $\mathrm{PGL}(p)q$ is closed if and only if $\pi^{-1}(\pi(q)) = \mathrm{PGL}(p)q$. But $\pi^{-1}(\pi(q))$ consists of the points q' such that $Gr(\mathcal{F}_{0q'}) \simeq Gr(\mathcal{F}_{0q})$, and must contain exactly one closed orbit. It follows that $\pi^{-1}(\pi(q))$ contains exactly one closed orbit if and only if $\mathcal{F}_{0q} \simeq Gr(\mathcal{F}_{0q})$, if and only $\mathcal{F}_{0q}$ is polystable. □

Let E be a polystable vector bundle of rank r and degree d on X and $w \in M(r,d)$ the corresponding point. Suppose that E is not stable. Then w is a singular point (except if $g = 2$, $r = 2$ and d is even). We want to study this singularity. Let

$$E \;=\; \bigoplus_{1\leq i\leq n} (E_i \otimes \mathbb{C}^{m_i})$$

with E_i stable, of the same slope as E, E_i not isomorphic to E_j if $i \neq j$, $m_i > 0$ for $1 \leq i \leq n$. Then we have

$$\mathrm{Aut}(E) \;\simeq\; \prod_{1\leq i\leq n} \mathrm{GL}(m_i),$$

$$\mathrm{Ext}^1(E,E) \;\simeq\; \Big(\bigoplus_{1\leq i\leq n} \mathrm{Ext}^1(E_i,E_i)\Big) \oplus \Big(\bigoplus_{1\leq i,j\leq n, i\neq j} \big(\mathrm{Ext}^1(E_i,E_j) \otimes L(\mathbb{C}^{m_i},\mathbb{C}^{m_j})\big)\Big).$$

Let $q \in R^{ss}$ be such that $\mathcal{F}_{0q} \simeq E$. Then the orbit $\mathrm{PGL}(p)q$ is closed. Let N_q be the normal space to the orbit at q. By Lemma 7.13, we have $N_q \simeq \mathrm{Ext}^1(E,E)$. The action of $\mathrm{PGL}(p)_q = \mathrm{Aut}(E,E)$ on N_q is induced by the actions of $\mathrm{GL}(m_i)$ on $M(m_i)$ by conjugation, and by the natural actions of $GL(m_i)$, $GL(m_j)$ on $L(\mathbb{C}^{m_i},\mathbb{C}^{m_j})$ if $i \neq j$.

Let

$$\mathbf{N} \;=\; \mathrm{Ext}^1(E,E)//\,\mathrm{Aut}(E),$$

and let 0 denote the image of 0 in $\mathbf{N}$. Luna's slice theorem implies that there exists an affine variety Z, $z \in Z$, and étale morphisms

$$\begin{tikzcd} & Z \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ M(r,d) & & \mathbf{N} \end{tikzcd}$$

such that $\phi(z) = w$, $\psi(z) = 0$. It follows that we have ring isomorphisms

$$\widehat{\mathcal{O}}_{M(r,d),w} \;\simeq\; \widehat{\mathcal{O}}_{Z,z} \;\simeq\; \widehat{\mathcal{O}}_{\mathbf{N},0},$$

and we obtain the

Corollary 7.15. *the local ring $\widehat{\mathcal{O}}_{M(r,d),w}$ depends only on g, n, m_i, $rk(E_i)$ and $deg(E_i)$ for $1 \leq i \leq n$.*

7.4.1. *The case $n = 2$, $m_1 = m_2 = 1$*

We have then $E = E_1 \oplus E_2$, $\mathrm{Aut}(E) = \mathbb{C}^* \times \mathbb{C}^*$ and

$$\mathrm{Ext}^1(E,E) \;=\; \mathrm{Ext}^1(E_1,E_1) \oplus \mathrm{Ext}^1(E_2,E_2) \oplus \mathrm{Ext}^1(E_1,E_2) \oplus \mathrm{Ext}^1(E_2,E_1).$$

The action of $\mathrm{Aut}(E)$ on $\mathrm{Ext}^1(E,E)$ is given by

$$(s,t).(u_{11},u_{22},u_{12},u_{21}) \;=\; (u_{11},u_{22},\frac{t}{s}u_{12},\frac{s}{t}u_{21}).$$

Let $Y \subset \mathrm{Ext}^1(E_1,E_2) \otimes \mathrm{Ext}^1(E_2,E_1)$ be the closed subvariety of decomposable elements. It is also the affine cone over the Segre variety

$$\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)) \times \mathbb{P}(\mathrm{Ext}^1(E_2,E_1)) \;\subset\; \mathbb{P}(\mathrm{Ext}^1(E_1,E_2) \otimes \mathrm{Ext}^1(E_2,E_1)).$$

The following result is immediate :

Proposition 7.16. *We have*

$$\mathrm{Ext}^1(E,E)//\,\mathrm{Aut}(E) \;\simeq\; \big(\mathrm{Ext}^1(E_1,E_1) \oplus \mathrm{Ext}^1(E_2,E_2)\big) \times Y.$$

Corollary 7.17. *The Zariski tangent space of $M(r,d)$ at w is*

$$\mathrm{Ext}^1(E_1,E_1) \oplus \mathrm{Ext}^1(E_2,E_2) \oplus \big(\mathrm{Ext}^1(E_1,E_2) \otimes \mathrm{Ext}^1(E_2,E_1)\big).$$

Moreover the multiplicity of $M(r,d)$ at w is

$$\begin{pmatrix} 2r_1r_2(g-1)-2 \\ r_1r_2(g-1)-1 \end{pmatrix},$$

with $r_1 = rk(E_1)$, $r_2 = rk(E_2)$.

Proof. To prove this we can replace $M(r,d)$ and w with $\big(\mathrm{Ext}^1(E_1,E_1) \oplus \mathrm{Ext}^1(E_2,E_2)\big) \times Y$ and 0 respectively. The result follows from the description of Y as a cone : for the first assertion we use the fact that Y generates $\mathrm{Ext}^1(E_1,E_2) \otimes \mathrm{Ext}^1(E_2,E_1)$. For the second, we use the fact that the multiplicity of a cone at the origin is equal to its degree (cf. [7], [17]). $\square$

7.4.2. *The case $n = 1$*

In this case we have $E \simeq E_1 \otimes \mathbb{C}^{m_1}$ and $\mathrm{Aut}(E) = \mathrm{PGL}(r)$. Using Luna's slice theorem, the multiplicity of $\mathcal{O} \oplus \mathcal{O}$ in $M(2,0)$ is computed in [10].

8. The étale slice theorem and the local factoriality of quotients

8.1. Local factoriality of quotients

Let Z be a smooth irreducible variety and G a reductive group acting on Z. suppose that there exists a good quotient

$$\pi : Z \longrightarrow M.$$

The variety M is normal (by prop. 2.15) but in general it is not smooth.

8.1.1. *Locally factorial varieties*

We will be interested in the factoriality of local rings of closed points of M. Recall that a normal variety X is called *locally factorial* if all the local rings of its points are unique factorization domains (in fact it is sufficient to consider only the closed points). Recall that an integral domain A is called a *unique factorization domain* (or a *factorial ring*) if every non zero element of A has a unique factorization into irreducible elements.

Proposition 8.1. *Let X be a normal variety, and $x \in X$ a closed point. Then $\mathcal{O}_x$ is an UFD if and only if for every prime divisor $H \subset X$ containing x, the ideal of H in $\mathcal{O}_x$ is principal. Hence X is locally factorial if and only if the ideal sheaf of every prime divisor of X is locally free.*

Proof. See for example [15], p. 141. □

Let $Cl(X)$ denote the *divisor class group of X* (cf. [7], II, 6). Then we have a canonical morphism of groups

$$\phi : Pic(X) \longrightarrow Cl(X)$$

and proposition 8.1 implies that X is locally factorial if and only if this morphism is an isomorphism. More generally, if H is a prime divisor of X. Then the point of $Cl(X)$ corresponding to H is in $\mathrm{Im}(\phi)$ if and only if $\mathcal{I}_H$ is locally free, i.e. H is locally principal.

Proposition 8.2. *Let X be an integral algebraic variety, $x \in X$ a closed point. Then if $\widehat{\mathcal{O}}_x$ is an UFD, then so is $\mathcal{O}_x$.*

This result is known as *Mori's theorem* (cf. [7], V, 5, Ex. 5.8, [21]).

8.1.2. *Application of the descent lemma*

Theorem 8.3. *Suppose that there exists a saturated open subset $Z_0 \subset Z$ such that :*

(i) $\mathrm{codim}_Z(Z \backslash Z_0) \geq 2$.
(ii) $\pi_{|Z_0} : Z_0 \longrightarrow \pi(Z_0)$ *is a geometric quotient.*
(iii) *G acts freely on Z_0.*

Then M is locally factorial if and only if for every G-line bundle L on Z, and every closed point $z \in Z$ such that Gz is closed, G_z acts trivially on L_z.

Proof. We first prove the following assertion : M is locally factorial if and only if every line bundle on $\pi(Z_0)$ can be extended to a line bundle on M. We have a commutative diagram

$$\begin{tikzcd} Pic(M) \arrow[r, "\phi"] \arrow[d, "r_1"] & Cl(M) \arrow[d, "r_2"] \\ Pic(\pi(Z_0)) \arrow[r, "\phi_0"] & Cl(\pi(Z_0)) \end{tikzcd}$$

where ϕ, ϕ_0 are the canonical morphisms and r_1, r_2 the restrictions. The hypotheses imply that $\pi(Z_0)$ is smooth, hence ϕ_0 is an isomorphism. Since $\text{codim}_M(M\backslash\pi(Z_0)) \geq 2$, r_2 is an isomorphism. We have seen that M is locally factorial if and only if ϕ is an isomorphism. The preceeding diagram implies that ϕ is an isomorphism if and only if r_1 is an isomorphism. Since M is normal, and $\text{codim}_M(M\backslash\pi(Z_0)) \geq 2$, r_1 is injective. The assertion follows immediately.

Now we have to prove that every line bundle on $\pi(Z_0)$ can be extended to M if and only if for every G-line bundle L on Z, and closed point $z \in Z$ such that Gz is closed, G_z acts trivially on L_z. Suppose that every line bundle on $\pi(Z_0)$ can be extended to a line bundle on M, and let L be a G-line bundle on Z. Then $L_{|Z_0}$ descends to a line bundle $\mathcal{L}_0$ on $\pi(Z_0)$, by proposition 6.7. Let $\mathcal{L}$ be an extension of $\mathcal{L}_0$ to M. Then the G-bundles $\pi^*(\mathcal{L})$ and $L_{|\pi^{-1}(U)}$ are isomorphic on Z_0. Hence by lemma 6.3, (i), they are isomorphic. It follows that G_z acts trivially on L_z.

Conversely suppose that for every G-line bundle L on Z, and every closed point $z \in Z$ such that Gz is closed, G_z acts trivially on L_z. Let $\mathcal{L}_0$ be a line bundle on $\pi(Z_0)$. By lemma 6.3 there exists a G-line bundle L on Z extending $\pi^*(\mathcal{L}_0)$. By the hypothesis and the descent lemma, this bundle descends to a line bundle $\mathcal{L}$ on M. Then $\mathcal{L}$ is the desired extension of $\mathcal{L}_0$. □

Using proposition 6.2 we obtain the

Theorem 8.4. *Suppose that the conditions (i), (ii) and (iii) of theorem 8.3 are satisfied, that $Pic(Z)$ is trivial and that one of the following conditions is satisfied :*

(1) Every invertible regular function on G is the product of a character and of a constant.

(2) Every regular function $X \to \mathbb{C}^*$ is constant.

Then M is locally factorial if and only if for every character λ of G and every $z \in Z$ such that Gz is closed, λ is trivial on G_z.

In particular, if Z is a vector space and if the action of G is linear, then M is locally factorial if and only if the only character of G is the trivial one.

8.2. Étale slice theorem and local factoriality

As in 8.1 we consider a smooth irreducible variety and a reductive group G acting on Z. We suppose that there exists a good quotient

$$\pi : Z \longrightarrow M.$$

We suppose also that there is an open saturated subset $Z_0 \subset Z$ having properties (i), (ii) and (iii) of theorem 8.3.

We have seen that if M is locally factorial, then for every $x \in M$, if $z \in \pi^{-1}(x)$ is such that Gz is closed, then every character of G is trivial on G_z (the converse being true in some cases, for example when Z is a vector space).

Let $x \in M$, $z \in \pi^{-1}(x)$ be such that Gz is closed. We could also try to prove the factoriality of $\mathcal{O}_x$ by using proposition 8.2, i.e. we could try to prove that $\widehat{\mathcal{O}}_x$ is factorial.

Let $Y = T_zZ/T_z(Gz)$, with the obvious action of G_z. Let

$$\pi_Y : Y \longrightarrow Y//G_z = N$$

be the quotient morphism. Then by theorem 5.4, there is a canonical isomorphism

$$\widehat{\mathcal{O}}_x \ \simeq \ \widehat{\mathcal{O}}_{\pi_Y(0)}.$$

Hence if $\widehat{\mathcal{O}}_x$ is factorial, so is $\widehat{\mathcal{O}}_{\pi_Y(0)}$, and also $\mathcal{O}_{\pi_Y(0)}$, by proposition 8.2. Now this implies, using the fact that $G_0 = G_z$ in Y, that the only character of G_z is the trivial one. Finally we obtain the

Proposition 8.5. *Let $x \in M$, $z \in \pi^{-1}(x)$ be such that Gz is closed. Then if $\widehat{\mathcal{O}}_x$ is factorial then the only character of G_z is the trivial one.*

8.3. Moduli spaces of sequences of matrices

8.3.1. *Definition and properties of the moduli spaces*

Let V be a finite dimensional vector space, q a positive integer. Let $\mathbf{S}(V,q) = \mathrm{End}(V) \otimes \mathbb{C}^q$. We can view the elements of $\mathbf{S}(V,q)$ as sequences $(A_1, \ldots, A_q)$ of endomorphisms of V. On $\mathbf{S}(V,q)$ we have the following action of $\mathrm{GL}(V)$:

$$\begin{aligned} \mathrm{GL}(V) \times \mathbf{S}(V,q) &\longrightarrow \mathbf{S}(V,q) \\ (g,(A_1,\ldots,A_q)) &\longmapsto (gA_1g^{-1},\ldots,gA_qg^{-1}) \end{aligned}$$

The homotheties act trivially, so we have in fact an action of $\mathrm{PGL}(V)$ on $\mathbf{S}(V,q)$. We call the elements of $\mathbf{S}(V,q)$ *sequences of matrices.*

There is a similar quotient problem, involving projective varieties with an action of $G = \mathrm{SL}(V) \times \mathrm{SL}(V)$. We consider $\mathbf{S}(V,q+1)$ with the following action of $\mathrm{GL}(V) \times \mathrm{GL}(V)$:

$$\begin{aligned} (\mathrm{GL}(V) \times \mathrm{GL}(V)) \times \mathbf{S}(V,q+1) &\longrightarrow \mathbf{S}(V,q+1) \\ ((g,h),(A_1,\ldots,A_{q+1})) &\longmapsto (hA_1g^{-1},\ldots,hA_{q+1}g^{-1}) \end{aligned}$$

We have an induced action of G on $\mathbb{P} = \mathbb{P}(\mathbf{S}(V, q+1))$. We can embed $\mathbf{S}(V,q)$ as a locally closed subvariety $Y \subset \mathbb{P}$ in the following way : to $(A_1, \dots, A_q)$ one associates $\mathbb{C}.(I_V, A_1, \dots, A_q)$. Then $U = GY$ is an affine open subset of $\mathbb{P}$.

We need now some results of Geometric Invariant theory ([17], [19]). There is an obvious linearization of the action of G on $\mathbb{P}$. The semi-stable (resp. stable) points of $\mathbb{P}$ correspond to elements $(A_0, A_1, \dots, A_n) \in \mathbf{S}(V, q+1)$ such that if $V' \subset V$ is a proper linear subspace, then we have

$$\dim\big(\sum_{0\leq i\leq q} A_i(V')\big) \ \geq\ \dim(V') \qquad (\text{resp.} \quad > \ ).$$

Now it is clear that U is contained in the set of semi-stable points of $\mathbb{P}$. It is easy to check that

$$\mathbf{S}(V,q)//\operatorname{PGL}(V) \ \simeq\ U//G.$$

Definition 8.6. *Let V, W be finite dimensional vector spaces, $\alpha = (A_1, \dots, A_q) \in \mathbf{S}(V,q)$, $\beta = (B_1, \dots, B_q) \in \mathbf{S}(W,q)$. A morphism $\alpha \to \beta$ is a linear map $f : V \to W$ such that for $1 \leq i \leq q$ the following diagram is commutative :*

$$\begin{tikzcd} V \arrow[r, "A_i"] \arrow[d, "f"] & V \arrow[d, "f"] \\ W \arrow[r, "B_i"] & W \end{tikzcd}$$

It is easy to prove that the sequences of matrices with fixed length q and the morphisms between them form an abelian category $\mathcal{S}_q$. If $\phi : \alpha \to \beta$ is a morphism of sequences of matrices as in definition 8.6, defined by a linear map $f : V \to W$, we have for $1 \leq i \leq q$

$$A_i(\ker(f)) \subset \ker(f), \qquad B_i(\operatorname{Im}(f)) \subset \operatorname{Im}(f),$$

and we have

$$\begin{aligned} \ker(\phi) &= (A_{1|\ker(f)}, \dots, A_{q|\ker(f)}) \in \mathbf{S}(\ker(f), q), \\ \operatorname{Im}(\phi) &= (B_{1|\operatorname{Im}(f)}, \dots, B_{q|\operatorname{Im}(f)}) \in \mathbf{S}(\operatorname{Im}(f), q), \\ \operatorname{coker}(\phi) &= (\overline{B}_1, \dots, \overline{B}_q) \in \mathbf{S}(\operatorname{coker}(f), q), \end{aligned}$$

where $\overline{B}_i$ is induced by B_i. The morphism ϕ is *injective* (resp. *surjective*, an *isomorphism*) if and only if f is.

Let $\alpha = (A_1, \dots, A_q) \in \mathbf{S}(V,q)$. Then α is a *simple* object of the category $\mathcal{S}_q$ if and only the only linear subspaces V' of V such that $A_i(V') \subset V'$ for $1 \leq i \leq q$ are $\{0\}$ and V. Since $\mathcal{S}_q$ is obviously artinian we have the Jordan-Hölder decomposition of elements of $\mathbf{S}(V,q)$:

Proposition 8.7. *Let $\alpha = (A_1, \dots, A_q) \in \mathbf{S}(V,q)$. Then there exists a filtration of V by linear subspaces*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that we have :

(i) $A_i(V_j) \subset V_j$ *for* $1 \leq i \leq q$, $0 \leq j \leq n$.

(ii) *Let $\alpha_j \in S(V_j, q)$ be the sequence induced by α. Then α_j/α_{j-1} is simple for $1 \leq j \leq n$. Moreover the isomorphism class of $\bigoplus_{1\leq j\leq n} \alpha_j/\alpha_{j-1}$ (as an element of $\mathbf{S}(\bigoplus_{1\leq i\leq q} V_i/V_{i-1}, q)$). depends only of α.*

We will note $Gr(\alpha)$ the isomorphism class of $\bigoplus_{1\leq j\leq n} \alpha_j/\alpha_{j-1}$. The following is an easy exercise :

Proposition 8.8. *Let p be a positive integer, $V_1, \ldots, V_p$ finite dimensional vector spaces, $m_1, \ldots, m_p$ positive integers and σ_i a simple sequence in $\mathbf{S}(V_i, q)$ for $1 \leq i \leq p$, such that σ_i and σ_j are not isomorphic if $i \neq j$. Let*

$$\sigma \;=\; \bigoplus_{1\leq i\leq p} m_i\sigma_i \;\in\; \mathbf{S}(V, q)$$

with $V = \bigoplus_{1\leq i\leq p} (V_i \otimes \mathbb{C}^{m_i})$. Then we have a canonical isomorphism

$$\mathrm{GL}(V)_\sigma \;\simeq\; \prod_{1\leq i\leq p} \mathrm{GL}(m_i).$$

Proposition 8.9. *Let $\alpha, \beta \in \mathbf{S}(V, q)$. Then we have $\pi_{\mathbf{S}(V,q)}(\alpha) = \pi_{\mathbf{S}(V,q)}(\beta)$ if and only if $Gr(\alpha) = Gr(\beta)$. Moreover $\mathrm{GL}(V)\alpha$ is closed if and only if $\alpha \simeq Gr(\alpha)$.*

Proof. Let $\alpha = (A_1, \ldots, A_n) \in \mathbf{S}(V, q)$ and $V' \subset V$ a proper linear subspace such that $A_i(V') \subset V'$ for $1 \leq i \leq q$. Consider now a basis $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_n)$ such that $(e_1, \ldots, e_m)$ is a basis of V'. We can then represent the elements of $\mathrm{End}(V)$ as $n \times n$ matrices. We have

$$A_i \;=\; \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix},$$

where a_i (resp. b_i, c_i) is $m \times m$ (resp. $m \times (n-m)$, $(n-m) \times (n-m)$) matrix. Let $t \in \mathbb{C}^*$ and

$$g_t \;=\; \begin{pmatrix} tI_m & 0 \\ 0 & I_{n-m} \end{pmatrix} \;\in\; \mathrm{GL}(V).$$

We have

$$g_tA_ig_t^{-1} \;=\; \begin{pmatrix} a_i & tb_i \\ 0 & c_i \end{pmatrix}.$$

Hence the closure of $\mathrm{GL}(V)\alpha$ contains $(A_1', \ldots, A_q')$, with

$$A_i' \;=\; \begin{pmatrix} a_i & 0 \\ 0 & c_i \end{pmatrix}$$

for $1 \leq i \leq q$. It follows then easily by induction that $\overline{\mathrm{GL}(V)\alpha}$ contains sequences isomorphic to $Gr(\alpha)$. It is then clear that if $\alpha, \beta \in \mathbf{S}(V, q)$ are such that $Gr(\alpha) = Gr(\beta)$, then $\pi_{\mathbf{S}(V,q)}(\alpha) = \pi_{\mathbf{S}(V,q)}(\beta)$

To prove the converse we have only to show that if α is isomorphic to $Gr(\alpha)$, then the orbit $\mathrm{GL}(V)\alpha$ is closed. Suppose that $\alpha \simeq Gr(\alpha)$. Then there is a direct sum decomposition

$$V \ \simeq \ \bigoplus_{1\leq i\leq p} W_i$$

such that $\alpha = \bigoplus_{1\leq i\leq p} \alpha_i$, where $\alpha_i \in \mathbf{S}(W_i, q)$ is simple. Let

$$\{0\} = V_0^0 \subset V^0 \subset \cdots \subset V_n^0 = V$$

be the corresponding filtration of V (i.e. $V_i^0 = \bigoplus_{1\leq j\leq i} W_i$). Let $\beta \in \overline{\mathrm{GL}(V)\alpha}$. Let $\mathbf{D}$ be the variety of filtrations

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of V by linear subspaces such that $V = \bigoplus_{1\leq i\leq p} V_i$ and $\dim(V_i) = \dim(V_i^0)$. It is a projective variety with an obvious action of $\mathrm{GL}(V)$. Let

$$\Psi : G \longrightarrow \mathbf{D} \times \overline{\mathrm{GL}(V)\alpha}$$

$$g \longmapsto (g(V_i^0), g\alpha)$$

Then since $\mathbf{D}$ is projective, the projection $\overline{\mathrm{Im}(\Psi)} \to \overline{\mathrm{GL}(V)\alpha}$ is surjective. Let $((W_i), \beta) \in \overline{\mathrm{Im}(\Psi)}$. There exists an open neighbourhood U of (W_i) in $\mathbf{D}$, and a morphism $\lambda : U \to \mathrm{GL}(V)$ such that for every $u \in U$ we have $\lambda(u)u = (V_i^0)$. Using this morphism we can suppose that $(W_i) = (V_i^0)$ and that $\beta \in \Gamma\alpha$ where Γ denotes the stabilizer of (V_i^0). Let $\beta_i \in \mathbf{S}(V_i^0/V_{i-1}^0, q)$ be the sequence induced by β. Then we have $\beta_i \in \overline{\mathrm{GL}(V_i^0/V_{i-1}^0)\alpha_i}$. Suppose that our result is true for $p = 1$ (i.e. when α is simple). Then it follows that β_i is isomorphic to α_i, and by what we have seen in the beginning of the proof of this proposition this implies that $\alpha \in \overline{\mathrm{GL}(V)\beta}$, hence $\beta \in \mathrm{GL}(V)\alpha$, and $\mathrm{GL}(V)\alpha$ is closed.

Now we have only to prove that the orbit of a simple sequence is closed. This can be easily deduced from the fact that simple sequences correspond to stable points in $\mathbb{P}$. $\square$

8.3.2. *Local factoriality of the moduli spaces*

Let $\mathcal{M}(V, q) = \mathbf{S}(V, q)//\,\mathrm{PGL}(V)$.

Theorem 8.10. *The variety $\mathcal{M}(V, q)$ is locally factorial.*

Proof. Let $Z_0 \subset \mathbf{S}(V, q)$ be the open saturated subset consisting of simple sequences. Then one checks easily that

$$\mathrm{codim}_{\mathbf{S}(V,q)}(\mathbf{S}(V, q)\backslash Z_0) \ \geq \ 2$$

except when $q = 2$, $\dim(V) = 3$. When $q = 2$, $\dim(V) = 3$, one sees easily that $\mathcal{M}(V, q)$ is isomorphic to an open subset of $\mathbb{P}_5$. In the other cases we can apply theorem 8.4, using the fact that $\mathrm{PGL}(V)$ has no non trivial character. $\square$

Let p be a positive integer, $V_1, \ldots, V_p$ finite dimensional vector spaces, $m_1, \ldots, m_p$ positive integers and σ_i a simple sequence in $\mathbf{S}(V_i, q)$ for $1 \leq i \leq p$, such that σ_i and σ_j are not isomorphic if $i \neq j$. Let

$$\sigma \;=\; \bigoplus_{1 \leq i \leq p} m_i \sigma_i \;\in\; \mathbf{S}(V, q)$$

with $V = \bigoplus_{1 \leq i \leq p} (V_i \otimes \mathbb{C}^{m_i})$. Then we have a canonical isomorphism

$$\mathrm{PGL}(V)_\sigma \;\simeq\; \big(\prod_{1 \leq i \leq p} \mathrm{GL}(m_i) \big) / \mathbb{C}^*$$

(proposition 8.8).

We now apply proposition 8.5, and get the

Proposition 8.11. *Let $x \in \mathcal{M}(V, q)$ be the point corresponding to σ. If $p > 1$, then $\widehat{\mathcal{O}}_x$ is not factorial.*

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