Birational Geometry of Symplectic 4-folds

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Abstract: We prove the ¤op conjecture for smooth symplectic 4-folds. As an application we show that any birational map between smooth symplectic 4-folds is a composition of £nitely many Mukai ¤ops. This answers a problem posed by Mukai in 1988. We also give an interpretation of these results in terms of cones of movable and ample divisors.

1 Introduction

This paper is concerned with the birational geometry of complex projective smooth symplectic varieties of dimension 4, which we from now on shall simply call *symplectic* 4*-folds*.

An example of a birational transformation of a symplectic 4-folds is a *Mukai ¤op*. It is obtained by blowing up a $\mathbb{P}^2$ and blowing it down ”in the other direction”. This process actually works for any $\mathbb{P}^2$ inside a symplectic 4-fold, but the resulting space X' is not necessarily algebraic, let alone projective. To ensure the latter it is sometimes necessary to simultaneously ¤op several disjoint $\mathbb{P}^2$'s. This happens when several $\mathbb{P}^2$'s are covered by curves in the same ray. We shall call a birational transformation between two symplectic 4-folds that ¤ops precisely the locus of one ray a *projective Mukai ¤op*.

The results of this paper roughly state that the birational geometry of symplectic 4-folds is governed by Mukai ¤ops. The £rst result states that the ¤op conjecture [KMM, 5-1-10, 5-1-13] holds for symplectic 4-folds.

A divisor D on a symplectic 4-fold X is called *small extremal*, if D is effective and the locus of negative curves $\bigcup_{(C.D)<0} C \subset X$ is of codimension at least 2.

Theorem 1. *Let X_1 be a symplectic 4-fold and let D_1 be a small extremal divisor on X_1. There exists a £nite sequence of projective Mukai ¤ops*

$$X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n$$

such that the strict transform $D_n \subset X_n$ of D_1 is nef.

In order to prove the theorem we £rst construct a sequence of Mukai ¤ops by employing the classi£cation of small contractions of symplectic 4-folds [WW] and by showing that the resulting spaces are projective varieties again. Then easy geometric arguments show that this process must stop after £nitely many steps.

A direct consequence of Theorem 1 is the following result. It answers a problem posed by S. Mukai in 1988 [M, Problem 14].

Theorem 2. *Any birational transformation $X \dashrightarrow X'$ of two symplectic 4-folds can be factorised into a £nite sequence of projective Mukai ¤ops*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_n = X'.$$

Another interpretation of Theorem 1 is in terms of certain cones. The existence and termination of ¤ops implies that the movable cone has a locally £nite cover by ample cones of birational models. This has been worked out for Calabi-Yau 3-folds in [K]. For the Horrocks-Mumford quintic, which is are examples of a Calabi-Yau 3-fold, [F] has studied the movable cone in detail including an explicit veri£cation of the theorem for this example.

We brie¤y outline the results for symplectic 4-folds. For a symplectic 4-fold X we denote by $\mathrm{Amp}(X)$ and $\mathrm{Mov}(X)$ the (open) cones of ample and movable divisors respectively. Let $\phi : X \dashrightarrow X'$ be a birational transformation of symplectic 4-folds which is not an automorphism. By taking strict transformations we get a map $\phi^* : N^1(X) \to N^1(X')$. Since ϕ is necessarily an isomorphism in codimension 1, the map ϕ^* is an isomorphism. It is easy to see that

$$\begin{aligned} \phi^*(\mathrm{Mov}(X')) &= \mathrm{Mov}(X), \\ \phi^*(\mathrm{Amp}(X')) \cap \mathrm{Amp}(X) &= 0. \end{aligned}$$

Therefore $\mathrm{Mov}(X)$ has a partial decomposition into ample cones of birational symplectic 4-folds. Theorem 1 now implies that the ample cones £ll out the movable cone. Indeed, any movable divisor D on X is small extremal and by Theorem 1 there is a birational transformation $\phi : X \to X'$ such that the strict transform D' is nef and therefore in $\overline{\mathrm{Amp}}(X)$, the closure of the ample cone.

To end this introduction we shall also mention related work by D. Burns, Y. Hu and T. Luo [BHL], which addresses the same problems as this paper. However, we feel that in [BHL] the issues of projectivity and of termination of ¤ops are not adequately addressed in order to complete the proofs.

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2 Existence of projective Mukai Flops

Proposition 2.1. *Let X be a symplectic 4-fold and let $D \subset X$ be an effective divisor. Let $1 \gg \varepsilon > 0$ such that $(X, \varepsilon D)$ is KLT. Let $R \in \overline{\mathrm{NE}}(X)$ be an $(X, \varepsilon D)$-extremal ray such that its locus*

$$E = \bigcup_{[C] \in R} C$$

is at most 2-dimensional.
(i) E is a disjoint union of £nitely many copies of $\mathbb{P}^2$.
(ii) There exists a projective Mukai ¤op $X \dashrightarrow X'$ which ¤ops precisely the components of E.
(iii) Let $E' \subset X'$ be the union of the ¤opped $\mathbb{P}^2$'s. Then the strict transform $D' \subset X'$ of D is positive on all curves in E'.

Proof. Let $c_R : X \to Y$ be the contraction of R. The c_R-exceptional set is precisely E. Therefore c_R is a small contraction. Let E_i be a connected component of E. Let $\overline{c}_R : \overline{X} \to \overline{Y}$ obtained from $c_R : X \to Y$ by passing to the category of analytic spaces and by shrinking Y to a small neighbourhood around $c_R(E_i)$. Then $\overline{c}_R : \overline{X} \to \overline{Y}$ is a projective small contraction of an analytic symplectic 4-fold. The classi£cation theorem [WW, Thm 1] implies that $E_i \cong \mathbb{P}^2$. Therefore E is a disjoint union of £nitely many copies of $\mathbb{P}^2$.
(ii) Let $\pi : Z = \mathrm{Bl}_E X \to X$ be the blow up of E. Let $E = E_1 \cup \cdots \cup E_r$ be the decomposition into irreducible components. For all i, let $G_i = \pi^{-1}(E_i)$. Since $E_i \cong \mathbb{P}^2$ and $\mathrm{H}^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}) = 0$, it follows that E_i is lagrangian, i.e. $\mathcal{N}^\vee_{E_i/X} \cong \mathcal{T}_{E_i}$. Therefore $G_i \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2})$, i.e. G_i is the ¤ag variety $\mathbb{F}_{1,2}$ which classi£es complete ¤ags in $\mathbb{C}^3$. The restriction of π to $\alpha_i = \pi|_{G_i} : G_i \to E_i$ associates to a ¤ag $\{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \mathbb{C}^3\}$ the point in $\mathbb{P}^2$ representing the line V_1 in $\mathbb{C}^3$. There is another projection $\beta_i : G_i \to E_i^\vee \cong \mathbb{P}^2$ to the dual of E_i, which takes a ¤ag $\{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \mathbb{C}^3\}$ to the line in E_i which represents V_2 in $\mathbb{C}^3$. Let $A_i \cong \mathbb{P}^1$ be a £bre of α_i and let $B_i \cong \mathbb{P}^1$ be a £bre of β_i.

Claim 2.2. *$F = \sum_i \mathbb{R}_+[B_i]$ is an extremal face in $\overline{\mathrm{NE}}(Z/X)$.*

Proof. Let $[C] \in \overline{\mathrm{NE}}(Z/Y)$, where $C \subset Z$ is an irreducible curve. Then C lies in the exceptional locus of π, i.e. for some i we have $C \subset G_i$. Then it is well known that $C =_{num, G_i} \lambda A_i + \mu B_i$ for some positive integers λ and μ and therefore $C =_{num, X} \lambda A_i + \mu B_i$. This shows that $\overline{\mathrm{NE}}(Z/Y) = \sum_i \mathbb{R}_+[A_i] + \mathbb{R}_+[B_i]$. Since $K_Z = \sum_i G_i$ and $K_{G_i} = K_Z|_{G_i} + G_i|_{G_i}$ it follows that $K_Z|_{G_i} = \frac{1}{2} K_{G_i}$. Since A_i and B_i are £bres of a $\mathbb{P}^1$-£bration of G_i, it follows that $(K_{G_i}.A_i) = (K_{G_i}.B_i) = -2$. Therefore for any curve $C \subset G_i$ we have $(K_Z.C)_Z = \frac{1}{2}(K_{G_i}.C) < 0$. This shows that $\overline{\mathrm{NE}}(Z/Y)_{K \geq 0} = (0)$. Now the cone theorem implies that $\overline{\mathrm{NE}}(Z/Y)$ is rational polyhedral.

It remains to show that $F \subset \overline{\mathrm{NE}}(Z/Y)$ is a face. Suppose $M = \sum_i \lambda_i [A_i] + \mu_i [B_i] \in F$. By de£nition of F we have $M = \sum_i \kappa_i [B_i]$. Let $H \subset X$ be an ample divisor and $\overline{H} = \pi^* H$. Then for all i we have $(\overline{H}.A_i) = 0$ and $(\overline{H}.B_i) > 0$. Then

$$(\overline{H}.M) = \sum_i \mu_i(\overline{H}.B_i) = \sum_i \kappa_i(\overline{H}.B_i).$$

Therefore $\sum_i(\mu_i - \kappa_i)(\overline{H}.B_i) = 0$. Since $(G_j.A_i) = (G_j.B_i) = -\delta_{i,j}$ we get $(G_j.M) = -\lambda_i - \mu_i = -\kappa_i$. Therefore we obtain

$$\sum_i \lambda_i(B_i.\overline{H}) = 0.$$

Since for all i we have $(B_i.\overline{H}) > 0$ and $\lambda_i \geq 0$ it follows that all $\lambda_i = 0$. This shows that $F \subset \overline{NE}(Z/Y)$ is an extremal face. $\square$

Now continue the proof of the Proposition. Let $\pi' : Z \to X'$ be the contraction of F. Then by construction $\pi'|_{G_i} = \beta_i : G_i \to E_i^\vee$. Locally at all irreducible components E_i the birational transform $\pi' \circ \pi^{-1} : X \dashrightarrow X'$ is an analytic Mukai ¤op. Since $\pi' \circ \pi^{-1}$ ¤ops precisely the ray R and since X' is projective, it follows that $\pi' \circ \pi^{-1}$ is a projective Mukai ¤op. This shows (ii).

Let $E_i' = \pi'(G_i)$ and $E' = \pi'(G)$. Then $\pi^* D - \pi'^* D' = \sum a_i D_i$ for some integers a_i. Since $(G_i.B_j) = -\delta_{i,j}$ we have

$$-a_i = ((\pi^* D - \pi'^* D').B_i) = (D.\pi(B)) - (D'.\pi'(B_i)) = (D.\pi(B_i)) < 0.$$

Therefore

$$(D'.\pi'(A_j)) = (\pi'^* D'.A_j) = \sum_i a_i(G_i.A_j) + \pi(A).D = a_j > 0.$$

Since all E_i' are isomorphic to $\mathbb{P}^2$, all curves on E_i' are up to numerical equivalence just multiples of $\pi'(A_i)$. Therefore we have just seen that D' is positive on all curves in E'. $\square$

3 Proof of Theorem 1

We construct a sequence of projective Mukai ¤ops inductively as follows. Suppose we have already constructed

$$X_1 \overset{\phi_1}{\dashrightarrow} X_2 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} X_i.$$

Let $D_i \subset X_i$ be the strict transform of $D_1 \subset X_1$. Since D_1 is small extremal and since X_i is isomorphic to X_1 in codimension 1, it follows that D_i is small extremal. We are done if D_i is nef. Suppose D_i is not nef. For suf£ciently small $\varepsilon > 0$, the pair $(X_i, \varepsilon D_i)$ is KLT and since D_i is not nef, we can £nd an $(X_i, \varepsilon D_i)$-extremal ray R. Since D_i is small extremal, the locus of R in X_i does not contain a divisor. Therefore we can apply Proposition 2.1 to obtain a projective Mukai ¤op $\phi_i : X_i \dashrightarrow X_{i+1}$ that ¤ops precisely the locus of R.

It now remains to show that after a £nite number of steps this process must terminate.

Let us introduce some new notation. For any irreducible surface $S^i \subset X_i$ we can associate an irreducible surface $S^{i+1} \subset X_{i+1}$ as follows. If $\phi_i : X_i \dashrightarrow X_{i+1}$ is de£ned at the generic point of S^i, then we let S^{i+1} be the strict transform of S^i. Otherwise, S^i must be the center of a projective Mukai ¤op and we let S^{i+1} be the ¤opped center. Similarly we can associate a surface $S^{i-1} \subset X_{i-1}$ to each $S^i \subset X_i$. By induction we can associate to $S^i \subset X_i$ a surface $S^j \subset X_j$ for all $j \geq 1$. We say that $S^j \subset X_j$ is the *corresponding* surface of $S^i \subset X_i$.

Lemma 3.1. *A surface can be ¤opped only once. More formally: for any irreducible surface $S^1 \subset X_1$ we associate the series of corresponding surfaces $S^2 \subset X_2, S^3 \subset X_3, S^4 \subset X_4, \ldots$ as above. Then for at most one index i, the S^i is the center of a projective Mukai ¤op $\phi_i : X_i \dashrightarrow X_{i+1}$.*

Proof. Suppose not. Let N be the set of positive integers n, such that S^n is a center of a $\phi_n : X_n \dashrightarrow X_{n+1}$. Let $i, j \in N$ with $i \neq j$ and $|i - j|$ minimal. Then by construction, $S^{i+1} \cong \mathbb{P}^2$ with $\deg_{S^{i+1}}(D_{i+1}) > 0$ and $S^j \cong \mathbb{P}^2$ with $\deg_{S^j}(D_j) < 0$. Since $|i - j|$ is minimal, the map

$$\phi_{j-i} \circ \cdots \circ \phi_{i+1} : X_{i+1} \dashrightarrow X_j$$

is an isomorphism at the generic point of S^{i+1}. Therefore we get a birational map $\psi : S^{i+1} \dashrightarrow S^j$ and $D_j|_{S^j}$ is the strict transform of $D_{i+1}|_{S^{i+1}}$ with respect to ψ. Let $C \subset S^{i+1}$ be a complete curve such that ψ is de£ned on C. Then

$$0 < \deg_{S^{i+1}}(D_{i+1}) \deg_S^{i+1}(C) = (D_{i+1}.C)_{S^{i+1}} = (D_j.\psi(C))_{S^j}.$$

Since $S^{i+1} \cong \mathbb{P}^2$ the curve C is not contracted under ψ. Therefore

$$(D_j.\psi(C))_{S^j} = \deg_{S^j}(D_j) \deg_{S^j}(\psi(C)) < 0.$$

Contradiction. □

Lemma 3.2. *For some $n > 0$, the divisor $D_n \subset X_n$ is nef.*

Proof. Suppose not. Then for each positive integer i we £nd an surface $S_i \cong \mathbb{P}^2 \subset X_i$, such that $\deg_{S_i}(D_i) < 0$. Let $T_i \subset X_1$ be corresponding surface to S_i in X_1. By the last lemma, all T_i are different. Moreover, the map $\phi_{i-1} \circ \cdots \circ \phi_1 : X_1 \dashrightarrow X_i$ is an isomorphism at the generic point of T_i. Therefore we get a birational map $T_i \dashrightarrow S_i$. Since $\deg_{S_i}(D_i) < 0$, it follows that T_i is covered by D_1-negative curves. By assumption, D_1 is small extremal and therefore the locus of D_1-negative curves can only contain a £nite number of surfaces. □

This £nishes the proof of Theorem 1.

4 Proof of Theorem 2

Proposition 4.1. *Let $\phi : X \dashrightarrow X'$ be a birational transformation of symplectic 4-folds. Let $H' \subset X'$ be an ample divisor and let $H \subset X$ be its strict transform.*
(i) ϕ is an isomorphism in codimension 1.
(ii) H is small extremal.
(iii) If H is nef, then ϕ is an isomorphism.

Proof. (i) The map ϕ is de£ned on an open subset $U \subset X$ whose complement has at least codimension 2. Since K_X is trivial and X' is smooth, it follows that ϕ must be an isomorphism on U.
(ii) Let $C \subset X$ be an irreducible curve, such that ϕ is de£ned on its generic point η_C. Since mH' is very ample for $m \gg 0$, some element $G' \in |mH'|$ does not contain $\phi(\eta_C)$ and therefore its strict transform $G \subset X$ does not contain C. Therefore $0 \leq (C.G)_X = m(C.H)_X$. We conclude that the locus of H-negative curves is contained in $X \setminus U$, which by (i) is of codimension at least 2.
(iii) Since H' is ample it follows that H is big. Since H is nef, it follows from [KMM, 3-2-1] that $|mH|$ is basepoint free for $m \gg 0$. Therefore ϕ is a morphism. Since K_X is trivial and X' is smooth, it must be an isomorphism. □

Proof of Theorem 2. Let $\phi : X \dashrightarrow X'$ be a birational transformation of symplectic 4-folds. Let $H' \subset X'$ be an ample divisor and let $H \subset X$ be its strict transform. By Proposition 4.1.ii, H is small extremal. We now apply Theorem 1 to conclude that there is a £nite sequence of projective Mukai ¤ops

$$\psi : X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n,$$

such that the strict transform $H_n \subset X_n$ of H is nef. Now Proposition 4.1.iii implies that the birational transform $\phi \circ \psi^{-1} : X_n \dashrightarrow X'$ is an isomorphism. □

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