THE BOGOMOLOV–MIYAOKA–YAU INEQUALITY FOR LOG CANONICAL SURFACES

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0. Introduction

Let X be a smooth projective surface of non-negative Kodaira dimension. Bogomolov [1, Theorem 5] proved that $c_1^2 \leq 4c_2$. This was improved to $c_1^2 \leq 3c_2$ by Miyaoka [12, Theorem 4] and Yau [19, Theorem 4]. Equality $c_1^2 = 3c_2$ is attained, e.g., if the universal cover of X is a ball (if $\kappa(X) = 2$ then this is the only possibility). Further generalizations of inequalities for Chern classes for some singular surfaces with (fractional) boundary were obtained by Sakai [16, Theorem 7.6], Miyaoka [13, Theorem 1.1], Kobayashi [6, Theorem 2; 7, Theorem 12], Wahl [18, Main Theorem] and Megyesi [10, Theorem 10.14; 11, Theorem 0.1].

In [8] we introduced Chern classes of reflexive sheaves, using Wahl's local Chern classes of vector bundles on resolutions of surface singularities. Here we apply them to obtain the following generalization of the Bogomolov–Miyaoka–Yau inequality.

Theorem 0.1. *Let X be a normal projective surface defined over $\mathbb{C}$. Let (X, D) be a log canonical pair with reduced boundary (i.e., D is a reduced Weil divisor). Let $\mathcal{F}$ be a rank 2 reflexive subsheaf of $\Omega_X(\log D)$. Assume that $c_1\mathcal{F}$ is pseudoeffective and let $c_1\mathcal{F} = P + N$ be its Zariski decomposition. Then*

$$3c_2\mathcal{F} + \frac{1}{4}N^2 \geq (c_1\mathcal{F})^2.$$

In this form the theorem was known only for smooth surfaces (see [13, Remark 4.18]) and it was used to obtain a higher dimensional analogue of the Bogomolov–Miyaoka–Yau inequality for smooth varieties (see [14], Theorem 1.1). Similarly, our theorem yields a similar inequality for log canonical varieties with boundary (see Theorem 5.8). Our proof of Theorem 0.1 simplifies Miyaoka's original approach from [13] (see also [18]). In fact, it resembles Miyaoka's first proof of this inequality for nef $c_1\mathcal{F}$ (see [12]).

At first one might think that the theorem is not the most general one, since analogues of the Bogomolov–Miyaoka–Yau inequality are also known for surfaces with fractional boundary (see [7, Theorem 12] and [11, Theorem 0.1]). However, our result implies all the results known previously. We introduce pull backs of logarithmic 1-forms with poles along a $\mathbb{Q}$-divisor to obtain the following corollary:

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Corollary 0.2. *Let (X,D) be a log canonical surface with fractional boundary. Assume that K_X+D is pseudoeffective and let $K_X+D=P+N$ be its Zariski decomposition. Then*

$$3c_2(X,D)+\frac{1}{4}N^2\geq c_1^2(X,D)=(K_X+D)^2.$$

Corollary 0.2 was known (see [7, Theorem 12] for the case $\kappa(X,D)=2$ and [11, Theorem 0.1] for the general statement). Megyesi formulated his theorem in a different way using a certain inequality ([11, Theorem 4.2]) which he called a local version of the Bogomolov–Miyaoka–Yau inequality. Here we explain that this inequality is in fact an immediate consequence of the asymptotic Riemann–Roch formula and positivity of the modified Euler characteristic (see Proposition 2.6 and Remark 5.6), recovering all Megyesi's results.

In Section 1 we recall the definition and basic properties of Chern classes of reflexive sheaves. In Section 2 we introduce a pull back of logarithmic poles with poles along a $\mathbb{Q}$-divisor and we define the Chern numbers of a surface with boundary. Then we give a short proof of the generalization of [11, Theorem 4.2]. Section 3 is devoted to a covering trick which allows us to reduce the proof of Corollary 0.2 to Theorem 0.1. In Section 4 we prove the Logarithmic Ramification Formula for log canonical surfaces. Section 5 contains the proof of Theorem 0.1 and its corollaries. In Section 6 we use local theory to compute the Chern numbers of log canonical surfaces with fractional boundary. The appendix contains a short elementary proof of Bogomolov's instability theorem for rank 2 vector bundles.

Notation and Definitions.

A $\mathbb{Q}$-divisor $D=\sum d_iD_i$, where D_i are distinct integral Weil divisors and $0\leq d_i\leq 1$, is called a *boundary divisor*. The boundary divisor D is called *fractional* if for every i either $d_i=1$ or $1/(1-d_i)$ is an integer. D is called *reduced* if all $d_i=1$.

A pair (X,D) consisting of a normal variety X and an effective $\mathbb{Q}$-divisor D such that $\lceil D\rceil$ is a reduced divisor is called a *variety with boundary*. We will assume that either X is a surface or D is a $\mathbb{Q}$-Cartier divisor. This assumption allow us to define the pull back of D under a generically finite dominant map.

The *singular locus* $\operatorname{Sing}(X,D)$ of the pair (X,D) is the set of points of X where either X or $\operatorname{Supp}D=\sum D_i$ is singular.

For the definitions of the Zariski decomposition, log canonical and Kawamata log terminal singularities we refer the reader to [5, 0.2 and 7.3].

We write $\hat{S}^n\mathcal{F}(D)$ to denote $(S^n\mathcal{F}\otimes\mathcal{O}(D))^{**}$.

1. Chern classes and their properties

In this section we define Chern classes of reflexive sheaves on normal surfaces. The definition is slightly different from that given in [8] and follows [9].

Let (X,x) be a germ of a normal surface singularity and $f\colon(\tilde{X},E)\to(X,x)$ a good resolution of (X,x) (i.e., we assume that the exceptional curve E has only normal crossings).

Definition 1.1. *([17, (3.1)]) For a vector bundle $\mathcal{F}$ on $\tilde{X}$ we define the* modified Euler characteristic

$$\chi(x,\mathcal{F})=\dim\left((f_*\mathcal{F})^{**}/f_*\mathcal{F}\right)+\dim R^1f_*\mathcal{F}.$$

Definition 1.2. *([17, (2.1)]) Let $\mathcal{F}$ be a rank 2 vector bundle on $(\tilde{X}, E)$.*
(1) The first Chern class *$c_1(x, \mathcal{F})$ is an f-exceptional $\mathbb{Q}$-divisor whose intersection with any f-exceptional divisor F is equal to the degree of $\mathcal{F}|_F$.*
(2) The second Chern class *of $\mathcal{F}$ is defined by*

$$c_2(x, \mathcal{F}) = \inf \left\{ \frac{c_1(x, \mathcal{L}_1) \cdot c_1(x, \mathcal{L}_2)}{\deg g} \right\},$$

where the infimum is taken over all generically finite, proper maps $g\colon (Y, F) \to (\tilde{X}, E)$, $F = g^{-1}(E)$, such that there exists a filtration of $g^*\mathcal{F}$ with line bundles $\mathcal{L}_i$ as quotients.

Wahl's Conjecture 1.3. *(the local asymptotic Riemann–Roch formula for rank 2 vector bundles; [17, Main Conjecture]) If $\mathcal{F}$ is a rank 2 vector bundle on $(\tilde{X}, E)$ then*

$$\chi(x, S^n\mathcal{F}) = -\frac{n^3}{6}(c_1(x, \mathcal{F})^2 - c_2(x, \mathcal{F})) + O(n^2).$$

Wahl's conjecture is proved for a quotient singularity (see [8, Theorem 5.1]) and for a simply elliptic singularity or its quotient (see [8, Corollary 7.5]). To prove it for all log canonical surface singularities it would be sufficient to prove it for cusps (see [8, Theorem 4.20]). Note also that Wahl's conjecture holds for logarithmic 1-forms $\Omega_{\tilde{X}}(\log E)$ on good resolutions $(\tilde{X}, E) \to (X, x)$ of cusps (see Corollary 4.3 and [17, Proposition 2.13]) and hence of all log canonical surface singularities.

To avoid proving Wahl's conjecture we introduce the following definition:

Definition 1.4. *([9, Definition 2.2]) Let $\mathcal{F}$ be a rank 2 vector bundle on $(\tilde{X}, E)$. We define the* second RR Chern class *of $\mathcal{F}$ by*

$$c_2'(x, \mathcal{F}) = \frac{1}{4}c_1(x, \mathcal{F})^2 + \frac{3}{4}\liminf_{n\to\infty} \frac{\chi(x, S^{2n}\mathcal{F}(-n\det\mathcal{F}))}{n^3}.$$

If Wahl's conjecture holds for $\mathcal{F}$ then $c_2'(x, \mathcal{F}) = c_2(x, \mathcal{F})$ (see [8, Corollary 4.19]). Our second Chern class, although its definition is somewhat artificial, is well behaved and sufficient for most applications (see, e.g., [9]). For example, it is "natural" under pull backs:

Theorem 1.5. *(see [8, Theorems 4.14 and 4.20]) Consider the following commutative diagram*

$$\begin{tikzcd} (\tilde{Y}, F) \arrow[r, "g"] \arrow[d, "\tilde{\pi}"] & (Y, y) \arrow[d, "\pi"] \\ (\tilde{X}, E) \arrow[r, "f"] & (X, x) \end{tikzcd}$$

where π is a finite proper holomorphic map and f, g are good resolutions of singularities. Let $\mathcal{E}$ be a vector bundle on $(\tilde{X}, E)$ and set $\mathcal{F} = \tilde{\pi}^*\mathcal{E}$. Then

$$\chi(y, S^{2n}\mathcal{F}(-n\det\mathcal{F})) = \deg\pi \cdot \chi(x, S^{2n}\mathcal{E}(-n\det\mathcal{E})) + \mathrm{O}(n^2).$$

In particular, $c_2'(x, \mathcal{F}) = \deg\pi \cdot c_2'(x, \mathcal{E})$.

Now we can introduce Chern classes of reflexive sheaves. If the reader is interested only in Corollary 0.2 (or believes in Wahl's conjecture for cusps) then one can use Chern classes of reflexive sheaves defined by using Wahl's local Chern classes (see [8, Section 3]).

Definition–Proposition 1.6. *(see [9, Definition–Proposition 2.8]) Let $\mathcal{E}$ be a rank 2 reflexive sheaf on a normal surface X. The first Chern class $c_1\mathcal{E}$ of $\mathcal{E}$ is the class of* $\det \mathcal{E}$ *in* $\mathrm{Weil}\, X$. *Let $f\colon Y \to X$ be any good resolution of singularities and $\mathcal{F}$ any vector bundle on Y such that $(f_*\mathcal{F})^{**} = \mathcal{E}$. Then the number*

$$c_2\mathcal{E} = c_2\mathcal{F} - \sum_{x\in \mathrm{Sing}\, X} c_2'(x, \mathcal{F})$$

does not depend on the choice of f and $\mathcal{F}$ and we call it the second RR Chern class *of $\mathcal{E}$.*

By Theorem 1.5 if $\pi\colon Y \to X$ is a finite map then $c_2((\pi^*\mathcal{E})^{**}) = \deg \pi \cdot c_2\mathcal{E}$. By Definition 1.4 we have the asymptotic Riemann–Roch formula for $\chi(X, \hat{S}^{2n}\mathcal{E}(-nc_1\mathcal{E}))$. By an abuse of notation we write it in the following form

$$\chi(X, \hat{S}^{2n}\mathcal{E}(-nc_1\mathcal{E})) = \frac{1}{3}(c_1^2\mathcal{E} - 4c_2\mathcal{E})n^3 + \mathrm{O}(n^2),$$

which is true if Wahl's conjecture holds for singularities occurring on X. This does not affect any computations we do, since all we need are some asymptotic inequalities.

For further properties of the above Chern classes we refer the reader to [8], [9, Section 2] and [17].

2. Logarithmic 1-forms and their Chern classes

In this section we define the "virtual" reflexive sheaf of logarithmic 1-forms with poles in a $\mathbb{Q}$-divisor.

Definition 2.1. *Let X be a smooth variety (it is sufficient to assume that X is locally factorial). Let $D = \sum a_i D_i$, where D_i are irreducible reduced divisors and $0 \le a_i \le 1$. Any generically finite dominant morphism $f\colon Y \to X$ such that $f^*(a_iD_i)$ is a Cartier divisor for any i is called a* good morphism for the pair *(X, D).*

For example, any finite dominant morphism f between smooth varieties and such that f^*D has integral coefficients is good for (X, D). However, this is no longer true if f is only generically finite since the divisors $f^*(a_iD_i)$ do not need to have integral coefficients.

Take a point $y \in f^{-1}(\mathrm{Supp}\, D)$ and set $x = f(y)$. Choose a coordinate neighbourhood V_y around y and U_x around x such that

(1) $f(V_y) \subset U_x$,
(2) D_i is given by $z_i = 0$ in U_x,
(3) $f^*(a_iD_i)$ is given by $t_i = 0$ in V_y.

Since $a_i \le 1$ the function $f^*(z_i)/t_i$ is regular on V_y.

Definition 2.2. *The* pull back of the $\mathcal{O}_X$-module of logarithmic 1-forms *of X along D to Y is defined to be the $\mathcal{O}_Y$-module*

$$f^*\Omega_X(\log D) = \sum_i \frac{f^*dz_i}{t_i}\mathcal{O}_Y + f^*\Omega_X.$$

It is easy to see that $\Omega_X(\log D)$ is a well defined torsion free $\mathcal{O}_Y$-submodule of $f^*\Omega_X(f^*D)$. If D is a simple normal crossing divisor then $f^*\Omega_X(\log D)$ is a locally free sheaf. Moreover, if D is reduced then our definition agrees with the usual pull back of logarithmic 1-forms.

Set $\widehat{f^*\Omega}_X(\log D) = (f^*\Omega_X(\log D))^{**}$. In the literature there are two different definitions of $\Omega_X(\log D)$ for smooth X and arbitrary D. One of them agrees with our $\Omega_X(\log D) = (\mathrm{id})^*\Omega_X(\log D)$ and the other with $\widehat{\Omega}_X(\log D) = \widehat{(\mathrm{id})^*\Omega}_X(\log D)$. Note also that the pull back in Definition 2.2 can be defined in either the algebraic or the analytic case. In this way we usually get two different sheaves. For example, in the analytic case $\Omega_X(\log D)$ is locally free if D is a normal crossing divisor, which is not true in the algebraic case. We usually take reflexivizations, so we avoid the problem of distinguishing between these two cases.

Proposition 2.3. *Let (X, D) be a variety with boundary and let $g: Z \to Y$ and $f: Y \to X$ be generically finite morphisms. Assume that X is smooth and f is a good morphism for the pair (X, D). Moreover, assume that either Y is smooth or g is finite. Then*

$$(gf)^*\Omega_X(\log D) = g^*(f^*\Omega_X(\log D)).$$

Proof. If either Y is smooth (or at least locally factorial) or g is finite, then gf is a good morphism for the pair (X, D) and $\operatorname{div} g^*t_i = g^* \operatorname{div} t_i$ (the last equality does not hold if g is only generically finite and Y is not locally factorial). Now the lemma follows immediately from the definition. Q.E.D.

Definition 2.4. *Let (X, D) be a variety with boundary. Let $\pi: Y \to X$ be a finite dominant morphism of normal varieties such that π^*D is a Weil divisor. Let U be an open subset of X such that $\pi|_{\pi^{-1}U}$ is good for the pair $(U, D|_U)$ and* $\operatorname{codim}(X - U) \geq 2$. *We define the* pull back $\widehat{\pi^*\Omega}_X(\log D)$ of the module of logarithmic 1-forms *of X along D to Y as the reflexivization of $(\pi|_U)^*\Omega_U(\log D|_U)$.*

Lemma 2.5. *(Covering Lemma, cf. [5, Theorem 1-1-1]) Let (X, D) be a projective variety with boundary. Then there exists a finite morphism $\pi: Y \to X$ such that π^*D is a Weil divisor. Moreover, if X is smooth and* $\operatorname{Supp} D$ *is a simple normal crossing divisor, then there exists a finite good morphism for the pair (X, D) from a smooth variety Y.*

Proof. The second part of the lemma follows from [5, Theorem 1-1-1]. To prove the first one take a log resolution f of (X, D) and construct a finite good morphism $\tilde{\pi}$ for the resulting pair. Taking the Stein factorization of $f\tilde{\pi}$ we get the required morphism. Q.E.D.

Definition 2.6. *Let (X, D) be a surface with boundary and $\pi: Y \to X$ any finite morphism such that π^*D is a Weil divisor. We define the* Chern numbers *of the pair (X, D) by $c_1^2(X, D) = (K_X + D)^2$ and $c_2(X, D) = c_2(Y, \widehat{\pi^*\Omega}_X(\log D))/\deg \pi$.*

By Theorem 1.5, Proposition 2.3 and Lemma 2.5 the number $c_2(X, D)$ is well defined. Note that $c_1^2(X, D) = c_1^2(Y, \widehat{\pi^*\Omega}_X(\log D))/\deg \pi$.

The following theorem generalizes [11, Theorem 4.2], which was one of the main ingredients in Megyesi's proof of his generalized Bogomolov–Miyaoka–Yau inequality.

Proposition 2.7. *(cf. [11, Theorem 4.2]) Let $f\colon (X,D)\to (X',D')$ be a birational morphism between normal projective surfaces with boundary (i.e., $f_*(K_X+D)=K_{X'}+D'$). Then*

$$c_1^2(X,D)-4c_2(X,D)\le c_1^2(X',D')-4c_2(X',D').$$

Proof. There exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

such that π, π' are finite and π^*D, $(\pi')^*D'$ are Weil divisors. Set $\mathcal{E}=\widehat{\pi^*\Omega}_X(\log D)$ and $\mathcal{F}=\widehat{\pi'^*\Omega}_{X'}(\log D')$. From the definition of the pull back of logarithmic 1-forms we see that

$$(g_*(\hat S^{2n}\mathcal{E}(-nc_1\mathcal{E})))^{**}=\hat S^{2n}\mathcal{F}(-nc_1\mathcal{F}).$$

Therefore

$$\begin{aligned}\chi(Y,\hat S^{2n}\mathcal{E}(-nc_1\mathcal{E}))&=\chi(Y',\hat S^{2n}\mathcal{F}(-nc_1\mathcal{F}))-\sum_{x\in Y'}\chi(x,\hat S^{2n}\mathcal{E}(-nc_1\mathcal{E}))\\ &\le \chi(Y',\hat S^{2n}\mathcal{F}(-nc_1\mathcal{F})).\end{aligned}$$

Since

$$\chi(Y,\hat S^{2n}\mathcal{E}(-nc_1\mathcal{E}))=\frac{1}{3}(c_1^2\mathcal{E}-4c_2\mathcal{E})n^3+\mathrm{O}(n^2)$$

and

$$\chi(Y',\hat S^{2n}\mathcal{F}(-nc_1\mathcal{F}))=\frac{1}{3}(c_1^2\mathcal{F}-4c_2\mathcal{F})n^3+\mathrm{O}(n^2),$$

we get

$$c_1^2\mathcal{E}-4c_2\mathcal{E}\le c_1^2\mathcal{F}-4c_2\mathcal{F}.$$

The last inequality is equivalent to the asserted inequality. Q.E.D.

3. Covering lemma and uniformization of log canonical singularities

The following theorem is similar in statement and proof to [3, Theorem 17] (see also [5, Theorem 1-1-1]).

Theorem 3.1. *(see [3, Theorem 17]) Let (X,D) be a log canonical variety with a fractional boundary such that each prime component of $D-[D]$ is $\mathbb{Q}$-Cartier. Then there exists a finite morphism $\pi\colon (Y,\tilde D)\to (X,D)$ such that $R=K_Y+\tilde D-\pi^*(K_X+D)$ is an effective divisor and $(Y,\tilde D)$ is a log canonical variety with reduced boundary. In particular, we have a natural inclusion*

$$\widehat{\pi^*\Omega}_X(\log D)\hookrightarrow \widehat{\Omega}_Y(\log \tilde D).$$

Moreover, we can assume that $\pi(\operatorname{Supp} R)\cap \operatorname{Sing}(X,D)=\emptyset$.

Proof. Write $D-[D]=\sum_{i=1}^n(1-\frac{1}{q_i})D_i$, where D_i are distinct prime divisors. We will construct π inductively as a sequence of ramified cyclic coverings π_i. Since D_1

is $\mathbb{Q}$-Cartier there exists a non-zero integer m divisible by q_1 such that mD_1 is a Cartier divisor. Let us take a very ample divisor H' on X such that $mH' - mD_1$ is very ample. Choose a general divisor $H \in |mH' - mD_1|$ such that $(X, D + (1 - \frac{1}{q_1})H)$ is a log canonical pair.

This is always possible since by Bertini's theorem we can take smooth H and such that H intersects $\operatorname{Supp} D$ transversally in the smooth locus of X and $\operatorname{Supp} D$. Now take as π_1 a normalized q_1-th cyclic covering associated to the divisor $mD_1 + H \in |mH'|$. Then $K_Y + \pi_1^{-1}([D]) + \sum_{i>1}(1 - \frac{1}{q_i})\pi_1^* D_i = \pi_1^*(K_X + D + (1 - \frac{1}{k})H)$. Therefore $(Y, \pi_1^*(D - (1 - \frac{1}{q_1})D_1))$ is a log canonical surface. In general, $\pi_1^* D_i$ may be reducible but it is reduced and it consists of a disjoint sum of irreducible components. Hence we may apply the above process for $\pi_1^* D_2$ (which is still $\mathbb{Q}$-Cartier). Repeating the above procedure, we get the required morphism. The last part of the theorem follows from the Logarithmic Ramification Formula. Q.E.D.

Remark 3.2. If X is a surface with at most quotient singularities then there exists a generically finite morphism $f \colon (Y, \tilde{D}) \to (X, D)$ from a smooth surface Y such that $(Y, \tilde{D})$ is log canonical and

$$\widehat{\pi^* \Omega}_X(\log D) \hookrightarrow \widehat{\Omega}_Y(\log \tilde{D}).$$

More precisely, there exists a finite morphism $\pi \colon Z \to X$ which locally factorizes through $\pi_\alpha \colon X_\alpha \to U_\alpha \subset X$, where X_α is smooth and $K_{X_\alpha} + \pi_\alpha^{-1}(D - [D]) = \pi_\alpha^*(K_X + D|_{U_\alpha})$. To construct π one can use Theorem 3.3. Then the morphism f is constructed by resolving the singularities of $(Z, \pi^*(D - [D]))$.

This is the main idea of a Q-variety structure on the pair (X, D), which was used in [11] to define Chern numbers of (X, D). The Chern numbers defined in this way agree with our Chern numbers. Equality of c_1^2 is obvious whereas equality of c_2 follows from the existence of asymptotic Riemann–Roch theorem in both cases (cf. the proof of [8, Theorem 5.1]). We show it also by an explicit calculation in Section 6.

Here we want to show an "algebraic" uniformization of log canonical surface singularities with boundary whose proof does not depend so heavily on a classification:

Theorem 3.3. *(see [15] and [6, Section 3.1]) Let (X, D, x) be a log canonical surface singularity with fractional boundary. Then there exists a surface $(Y, \tilde{D}, y)$ with reduced boundary and a proper finite map $\pi \colon (Y, \tilde{D}, y) \to (X, D, x)$ such that*

$$\widehat{\Omega}_Y(\log \tilde{D}) = \widehat{\pi^* \Omega}_X(\log D).$$

In particular, π is branched with ramification index m over an irreducible curve C if and only if C appears in D with coefficient $(1 - \frac{1}{m})$. Furthermore, we can choose $(Y, \tilde{D}, y)$ such that

1. *if (X, x) is a quotient singularity then Y is smooth,*
2. *if (X, x) is not a quotient singularity then $D = \tilde{D} = 0$ and Y is either a simple elliptic singularity or a cusp.*

Proof. To construct π we take a log canonical index 1 covering, i.e., a cyclic cover given by

$$\operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X([i(K_X + D)]) \to X,$$

where m is the smallest positive integer such that $m(K_X+D)$ is Cartier. If (X,x) is a quotient singularity then we compose it with a local smoothing covering of the obtained singularity (which is also a quotient singularity). Q.E.D.

4. Logarithmic Ramification Formula

Theorem 4.1. *(Logarithmic Ramification Formula; see [5, Lemma 0-2-13]) Let $f\colon X\to Y$ be a generically surjective morphism between smooth varieties. Let D and B be effective normal crossing divisors on X and Y, respectively. If $f^{-1}(B)\subset D$ then there exists a natural inclusion*

$$f^*\Omega_Y(\log B)\hookrightarrow\Omega_X(\log D).$$

Theorem 4.2. *Let (X,D) be a log canonical surface with reduced boundary. Let $f\colon(\tilde X,E)\to(X,x)$ be a good resolution of the pair (X,D) and $\pi\colon(Y,y)\to(X,x)$ a finite proper morphism. Consider the commutative diagram*

$$\begin{array}{ccc}(\tilde Y,F) & \xrightarrow{g} & (Y,y)\\ \downarrow{\scriptstyle\tilde\pi} & & \downarrow{\scriptstyle\pi}\\ (\tilde X,E) & \xrightarrow{f} & (X,x)\end{array}$$

obtained by the normalization of $\tilde X$ in the function field of Y. Then for any positive integer n

$$g_*\tilde\pi^*(S^n\Omega_{\tilde X}(\log(f^{-1}D+E)))=\hat S^n\widehat{\pi^*\Omega}_X(\log D).$$

Proof. To prove the theorem it is sufficient to check it after a further pull back. Hence by Theorem 3.3 we can assume that (X,x) is either a smooth germ, a simple elliptic singularity or a cusp.

In the first case the assertion follows directly from Theorem 4.1.

In the remaining cases $D=0$ and it is sufficient to prove the theorem for one fixed resolution of (X,x).

Assume that (X,x) is simple elliptic. Let f be the minimal resolution. Then $K_{\tilde X}+E\simeq\mathcal O_{\tilde X}$ and on $\tilde X$ we have a non-split exact sequence

$$0\to\mathcal O_{\tilde X}\longrightarrow\Omega_{\tilde X}(\log E)\longrightarrow\mathcal O_{\tilde X}\to 0 \tag{4.2.1}$$

obtained by lifting an Euler derivation of the local ring $\mathcal O_{X,x}$ to $\tilde X$ and contracting with it. Passing to a resolution of singularities of $\tilde Y$ we can assume that $\tilde Y$ is smooth and $\tilde\pi$ is generically finite. Taking the pull back of the sequence (4.2.1) to $\tilde Y$ we see that for any g-exceptional curve F_i and any line bundle A of negative degree on F_i we have

$$h^0(\tilde\pi^*S^n\Omega_{\tilde X}(\log E)\otimes\mathcal O_{F_i}(A))=0.$$

Therefore we can use the formal function theorem to show that the cohomology group $H^1_F(\tilde\pi^*S^n\Omega_{\tilde X}(\log E))$ vanishes (cf. the proof of [17, Lemma 3.13]). In particular, $g_*(\tilde\pi^*S^n\Omega_{\tilde X}(\log E))$ is reflexive and the assertion follows.

If (X,x) is a cusp then we take a resolution $f\colon(\tilde X,E)\to(X,x)$ such that E is a cycle of smooth rational curves (if in the minimal resolution the exceptional curve is nodal we blow it up twice to get the required resolution). By assumption $(E-E_i)E_i = 2$ for any component E_i of E. Since $(K_{\tilde{X}}+E_i)E_i = -2$ by the genus formula we get $(K_{\tilde{X}}+E)E_i = 0$. Hence the residue map

$$\Omega_{\tilde{X}}(\log E)|_{E_i} \to \mathcal{O}_{E_i}$$

leads to $\Omega_{\tilde{X}}(\log E)|_{E_i} \simeq \mathcal{O}_{E_i}^2$. Similarly as above we can pull back $S^n\Omega_{\tilde{X}}(\log E)$ to $\tilde{Y}$ and use the formal function theorem to get the required assertion. Q.E.D.

Corollary 4.3. *(Logarithmic Ramification Formula for reduced log canonical surfaces) Let $f\colon (X,D) \to (X',D')$ be a birational morphism of log canonical surfaces with reduced boundary. Assume that $f^{-1}(D') \subset D$. Then there exists a natural inclusion*

$$(f^*\widehat{\Omega}_{X'}(\log D'))^{**} \hookrightarrow \widehat{\Omega}_X(\log D).$$

Corollary 4.4. *(Bogomolov's lemma for reduced log canonical surfaces) (see [1, Theorem 4] and [2, Corollary 6.9]) Let (X,D) be a log canonical surface with reduced boundary and $\pi\colon Y \to X$ a finite morphism from a normal surface. If a rank 1 reflexive sheaf $\mathcal{L}$ is contained in $\widehat{\pi^*\Omega_X(\log D)}$ then $\kappa(Y,\mathcal{L}) \le 1$.*

Proof. Let $f\colon \tilde{X} \to X$ be a log resolution of (X,D) and let $g\colon \tilde{Y} \to Y$ be a resolution of Y dominating the normalization of the main component of $Y \times_X \tilde{X}$. Let $\tilde{\pi}\colon \tilde{Y} \to \tilde{X}$ be the induced map. By Lemma 4.2 $\mathcal{F} = \tilde{\pi}^*\Omega_{\tilde{X}}(\log\lceil f^{-1}D\rceil + E)$ contains $(g^*\mathcal{L})^{**}$. Take a rank 1 subsheaf $\mathcal{M}$ of $\mathcal{F}$ containing $(g^*\mathcal{L})^{**}$ and such that $\mathcal{Q} = \mathcal{F}/\mathcal{M}$ is torsion free. Changing $\tilde{Y}$ we can assume that $\mathcal{Q}$ is a line bundle. It follows that $\mathcal{M}^{\otimes n}$ is a subline bundle of $S^n\mathcal{F}$ with a locally free quotient $\mathcal{Q}_n$. Now we can use the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(\tilde{Y},\mathcal{M}^{\otimes n}) & \longrightarrow & H^0(\tilde{Y},S^n\mathcal{F}) & \longrightarrow & H^0(\tilde{Y},\mathcal{Q}_n) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^0(\tilde{Y}-F,\mathcal{M}^{\otimes n}) & \longrightarrow & H^0(\tilde{Y}-F,S^n\mathcal{F}) & \longrightarrow & H^0(\tilde{Y}-F,\mathcal{Q}_n)
\end{array}$$

to deduce that $H^0(\tilde{Y},\mathcal{M}^{\otimes n}) = H^0(\tilde{Y}-F,\mathcal{M}^{\otimes n}) = H^0(Y,\mathcal{L}^{\otimes n})$. Since $\kappa(\tilde{Y},\mathcal{M}) \le 1$ by [1, Theorem 4], we also get $\kappa(Y,\mathcal{L}) \le 1$, Q.E.D.

Conjecture 4.5. *Let (X,D) be a log canonical variety with reduced boundary. Let $f\colon \tilde{X} \to X$ be a log resolution of the pair (X,D) and E the exceptional divisor of f. Then*

$$f_*(\Omega_{\tilde{X}}(\log(f^{-1}D+E))) = \widehat{\Omega}_X(\log D).$$

The conjecture implies the Bogomolov–Sommese vanishing theorem for log canonical varieties (see [2, Corollary 6.9]).

Conjecture 4.6. *(cf. [17, Conjecture 4.4]) Let $f\colon (\tilde{X},E) \to (X,x)$ be a good resolution of a surface (X,D,x) with reduced boundary. Let $\mathcal{F} = \Omega_{\tilde{X}}(\log(f^{-1}D+E))$ and let $c_1(x,\mathcal{F}) = P+N$ be the Zariski decomposition. Then*

$$\dim(f_*S^n\mathcal{F})^{**}/f_*S^n\mathcal{F} = -\frac{n^3}{6}P^2 + \mathrm{O}(n^2),$$

$$\dim R^1f_*S^n\mathcal{F} = -\frac{n^3}{6}(N^2 - c_2(x,\mathcal{F})) + \mathrm{O}(n^2).$$

By Theorem 4.2 Conjecture 4.6 is true for log canonical surfaces with reduced boundary. In fact, in this case $\dim(f_*S^n\mathcal{F})^{**}/f_*S^n\mathcal{F} = 0$.

5. Generalized Bogomolov–Miyaoka–Yau inequality

This section is devoted to the proof of Theorem 0.1 and its corollaries. The notation and assumptions in this section are as in Theorem 0.1.

Lemma 5.1. *Let $\pi\colon Y \to X$ be a finite morphism. If $h^0((\pi^*\mathcal{F})^{**}(-C)) \neq 0$ and $L(C - \frac{1}{2}\pi^*N) > 0$ for some nef divisor L then*

$$C\pi^*P \leq c_2((\pi^*\mathcal{F})^{**}) - \frac{1}{4}(\pi^*N)^2.$$

Proof. For simplicity of notation assume that $Y = X$ (in general the proof is the same). Replacing $\mathcal{O}(C)$ by a larger subsheaf of $\mathcal{F}$ we can assume that $\mathcal{O}(C)$ is saturated in $\mathcal{F}$ (i.e., $\mathcal{F}/\mathcal{O}(C)$ is torsion free). Then by [9, Proposition 2.12]

$$c_2\mathcal{F} \geq C(c_1\mathcal{F} - C) = PC + \frac{1}{4}N^2 - \left(C - \frac{1}{2}N\right)^2. \tag{5.1.1}$$

Since $\mathcal{O}(C) \hookrightarrow \mathcal{F} \subset \Omega_X(\log D)$, we have $\kappa(X, C - \frac{1}{2}N) \leq \kappa(X, C) \leq 1$ by Bogomolov's Lemma 4.4. In particular, if $L(C - \frac{1}{2}N) > 0$ for some nef L then $(C - \frac{1}{2}N)^2 \leq 0$. Hence $PC \leq c_2\mathcal{F} - \frac{1}{4}N^2$ by (5.1.1). Q.E.D.

Lemma 5.2. *(Branched covering trick, see [12, Lemma 11]) Let X be a normal surface and $\mathcal{E}$ a rank 2 reflexive sheaf on X. Let s be a section of $\hat{S}^n\mathcal{E}(-D)$. Then there exists a finite morphism π from a normal surface Y to X such that π^*s splits into the product $s_1 \ldots s_n$, where s_i is a section of $(\pi^*\mathcal{E})^{**}(-D_i)$.*

The proof is almost the same as the proof of [12, Lemma 11].

Lemma 5.3. *If $h^0(\hat{S}^n\mathcal{F}(-C)) \neq 0$ and $L(C - \frac{n}{2}N) > 0$ for some nef divisor L then*

$$PC \leq n\left(c_2\mathcal{F} - \frac{1}{4}N^2\right).$$

Proof. Take a section $s \in H^0(\hat{S}^n\mathcal{F}(-C))$ and a finite morphism $\pi\colon Y \to X$ splitting s into the product of $s_i \in H^0((\pi^*\mathcal{F})^{**}(-D_i))$ (see Lemma 5.2). Note that $\sum D_i = \pi^*C$ and $(\sum D_i - \frac{n}{2}\pi^*N)\pi^*L = \deg\pi(C - \frac{n}{2}N)L > 0$.

If $(D_i - \frac{1}{2}\pi^*N)\pi^*L > 0$ or $D_i\pi^*P = (D_i - \frac{1}{2}\pi^*N)\pi^*P > 0$ then by Lemma 5.1

$$D_i\pi^*P \leq c_2((\pi^*\mathcal{F})^{**}) - \frac{1}{4}(\pi^*N)^2 = \deg\pi(c_2\mathcal{F} - \frac{1}{4}N^2).$$

Note that $(D_j - \frac{1}{2}\pi^*N)\pi^*L > 0$ for some j. Hence $D_i\pi^*P = 0 \leq \deg\pi(c_2\mathcal{F} - \frac{1}{4}N^2)$ also for the remaining indices i. Summing all these inequalities and dividing by $\deg\pi$, we get the required inequality. Q.E.D.

5.4. Proof of Theorem 0.1.
Since $(c_1\mathcal{F})^2 = P^2 + N^2$, the asserted inequality is equivalent to

$$\frac{1}{3}P^2 \leq c_2\mathcal{F} - \frac{1}{4}N^2.$$

Assume that $\frac{1}{3}P^2 > c_2\mathcal{F} - \frac{1}{4}N^2$.

Let us fix an ample Cartier divisor H. If

$$h^0\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - naP - H\right)\right) \neq 0$$

for some $a \geq 1/3$ then $\frac{1}{3}nP^2 \leq naP^2 < P(naP + H) \leq n(c_2\mathcal{F} - \frac{1}{4}N^2)$ by Lemma 5.3. We can therefore assume that

$$h^0\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - naP - H\right)\right) = 0$$

for all $a \geq 1/3$.

Now we have

$$h^2\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - naP - H\right)\right) = h^0\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - n(1-a)P + H\right)\hat{\otimes}\omega_X\right)$$
$$= h^0\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - n(1-a)P - H\right)\right) + \mathrm{O}(n^2)$$

In particular, for $a = \frac{1}{3}$, our assumption gives

$$\chi\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - \frac{n}{3}P - H\right)\right) \leq \mathrm{O}(n^2).$$

Since

$$\chi\left(X, \hat{S}^n\mathcal{F}\left(-\frac{n}{2}N - \frac{n}{3}P - H\right)\right) = \frac{1}{6}\left(\frac{1}{4}N^2 - c_2\mathcal{F} + \frac{1}{3}P^2\right)n^3 + \mathrm{O}(n^2),$$

we have

$$\frac{1}{3}P^2 \leq c_2\mathcal{F} - \frac{1}{4}N^2$$

as required. Q.E.D.

Define a function $f\colon \mathbb{Q}_+ \to \mathbb{Q}_+$ by $f(\frac{p}{q}) = \frac{\max(p,q-1)}{q}$, where p and q are relatively prime positive integers. Let (X, D) be a surface with boundary. Write $D = \sum d_i D_i$, where the D_i are distinct prime divisors. Then we set $D^{(f)} = \sum f(d_i)D_i$. The divisor $D^{(f)}$ is a round up of D to a fractional boundary divisor (see [11, 6.1]). The following corollary generalizes Corollary 0.2.

Corollary 5.5. *Let (X, D) be a surface with boundary such that $(X, D^{(f)})$ is log canonical. Assume that $K_X + D$ is pseudoeffective and let $K_X + D = P + N$ be its Zariski decomposition. Then*

$$3c_2(X, D) + \frac{1}{4}N^2 \geq c_1^2(X, D) = (K_X + D)^2.$$

Proof. It follows immediately from Theorems 0.1 and 3.1. Q.E.D.

Remark 5.6.
1. It is obvious from the definitions that $\kappa(X, D) \geq 0$ implies $K_X + D$ pseudoeffective. The converse is also true for log canonical surfaces (X, D), but it is much harder. (It is an important step in the proof of log abundance for surfaces.) Corollary 5.5 was known for a log canonical surface (X, D) with fractional boundary if $\kappa(X, D) = 2$ (see [7, Theorem 12]) or $\kappa(X, D) \geq 0$ (see [11, Theorem 0.1.(i)]).

Note that if (X_m, D_m) is a log minimal model of any pair (X', D') such that $K_{X'} + D' = f^*(K_X + D)$ for some $f\colon (X', D') \to (X, D)$ then $c_1^2(X_m, D_m) = P^2$. This follows from the fact that the pull back of $K_{X_m} + D_m$ to X' is a positive part of the Zariski decomposition of $K_{X'} + D'$ (see the proof of [Corollary 7.5, Sa2]).

Therefore [11, Theorem 0.1.(i)] is equivalent to Corollary 5.5 for surfaces with fractional boundary and his formulation seems to be unnecessarily complicated. The second part of [11, Theorem 0.1] follows from the first one by Proposition 2.7.
2. Looking at our proof we see that even if we want to prove the Bogomolov–Miyaoka–Yau inequality for surfaces with at most quotient singularities we need to use Chern classes of reflexive sheaves in a more general setting. There are two steps of our proof at which we need to go outside the category of orbifolds or Q-varieties. Both of them are related to covering tricks (see Theorem 3.1 and Lemma 5.2).

Proposition 5.7. *(see [14, Theorem 6.1]) Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X. Assume that $c_1\mathcal{E}$ is nef and $c_1\mathcal{F}$ of any torsion free quotient $\mathcal{F}$ of $\mathcal{E}$ is pseudoeffective. Let S be a general complete intersection of* $(\dim X - 2)$ *globally generated divisors. Then*

$$c_2((\mathcal{E}|_S)^{**}) \geq 0.$$

Corollary 5.8. *(see [14, Theorem 1.1]) Let (X, D) be a log canonical variety with reduced boundary. Assume that $K_X + D$ is nef and X is not uniruled. Let S be a general complete intersection of $r = \dim X - 2$ globally generated divisors $H_1, \ldots, H_r$. Then*

$$3c_2(\widehat{\Omega}_X(\log D)|_S) \geq c_1^2(\widehat{\Omega}_X(\log D)|_S) = (K_X + D)^2 H_1 \ldots H_r \geq 0.$$

The proof is essentially the same as in [14] and it follows from Theorem 0.1 and the fact that $(S, D|_S)$ has only log canonical singularities.

6. Local Chern classes of (X, D) and computation of $c_2(\widehat{\Omega}_X(\log D))$

Definition 6.1. *(cf. [11, Definition 1.7]) Let (X, D, x) be a germ of a log canonical surface with fractional boundary. If (X, D, x) is Kawamata log terminal and π is the map defined in Theorem 3.3, then $\frac{1}{\deg \pi}$ is called the* local orbifold Euler number *of (X, D, x) and denoted by $e_{orb}(x; X, D)$. Otherwise we set $e_{orb}(x; X, D) = 0$.*

Definition 6.2. *(cf. [7, Theorem 12] and [11, Definition 1.8]) Let (X, D) be a log canonical surface with fractional boundary and let Y be a constructible subset of X. Write $D - [D] = \sum (1 - \frac{1}{m_i}) D_i$. Then we define the* orbifold Euler number *of Y by*

$$e_{orb}(Y; X, D) = e_{top}(Y \cap (X - \operatorname{Supp} D \cup \operatorname{Sing} X)) + \sum \frac{1}{m_i} e_{top}(\{y \in Y\colon \dim_y(Y \cap (D_i - \operatorname{Sing}(X, D))) = 1\}) + \sum_{x \in Y \cap \operatorname{Sing}(X,D)} e_{orb}(x; X, D).$$

The number $e_{orb}(X; X, D)$ will be denoted by $e_{orb}(X, D)$. It is clear from the definition that the orbifold Euler number is, similarly to the topological Euler characteristic, additive over disjoint subsets.

Lemma 6.3. *Let (X, D) be a germ of smooth surface with a normal crossing divisor D. Let $f\colon (\tilde{X}, E) \to (X, x)$ be the blow up of X at x. Then*

$$c_2(x, \Omega_{\tilde{X}}(\log(f^{-1}D + E))) = \begin{cases} -1, & \text{if } D = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We distinguish three cases according to the number of components of D (passing through x).

If D has two smooth components passing through x then using local coordinates one can easily check that $\Omega_{\tilde{X}}(\log(f^{-1}D + E)) = f^*(\Omega_X(\log D)) \simeq \mathcal{O}^2_{\tilde{X}}$ and the assertion follows.

If D has only one component passing through x, then we have the following short exact sequence (see the proof of [17, Definition–Proposition 2.9])

$$0 \to f^*(\Omega_X(\log D)) \longrightarrow \Omega_{\tilde{X}}(\log(f^{-1}D + E)) \longrightarrow \omega_E(f^{-1}D|_E) \to 0.$$

The last term in this sequence is isomorphic to $\mathcal{O}_E(E)$. Therefore by [17, Theorem 2.7]

$$c_2(x, \Omega_{\tilde{X}}(\log(f^{-1}D + E))) = c_2(x, \Omega_{\tilde{X}}(\log f^{-1}D)) - \chi(\mathcal{O}_E(E)) - \frac{1}{2}E(K_{\tilde{X}} - E) = 0.$$

If $D = 0$ then using local coordinates one can check that $\Omega_{\tilde{X}}(\log E) \simeq \mathcal{O}(E) \oplus \mathcal{O}(E)$. Hence $c_2(x, \Omega_{\tilde{X}}(\log E)) = -1$. Q.E.D.

Definition–Corollary 6.4. *(cf. [17, Definition–Proposition 2.9]) Let (X, D, x) be a germ of a surface with reduced boundary. Let $f\colon (\tilde{X}, E) \to (X, x)$ be a non-trivial log resolution of the pair (X, D). Then the number*

$$c_2(x, X, D) = c_2(x, \Omega_{\tilde{X}}(\log(f^{-1}D + E)))$$

does not depend on the choice of f.

Proof. It follows from the fact that for any two log resolutions of (X, D) we can find a third one dominating both of them. By Lemma 6.3 $c_2(x, \Omega_{\tilde{X}}(\log(E + f^{-1}D)))$ does not change under a single blow up so the corollary follows. Q.E.D.

Corollary 6.5. *If (X, D) is a log canonical surface with reduced boundary then*

$$c_2(x, X, D) = \begin{cases} -\dfrac{1}{|\pi_1(X, x)|}, & \text{if } D = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $D = 0$ then the result is known (see [17, Theorem 2.11]). Otherwise, (X, x) is a quotient singularity. Let $\pi\colon (Y, y) \to (X, x)$ be its local smoothing covering. By the base change and the minimal resolution of singularities we get the following commutative diagram

$$\begin{array}{ccc} (\tilde{Y}, F) & \xrightarrow{\ g\ } & (Y, y) \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ (\tilde{X}, E) & \xrightarrow{\ f\ } & (X, x) \end{array}$$

in which $\tilde{Y}$ is smooth. Set $\tilde{D} = \pi^* D$. Then by the Logarithmic Ramification Formula

$$\Omega_{\tilde{Y}}(\log(g^{-1}\tilde{D} + F)) = \tilde{\pi}^*\Omega_{\tilde{X}}(\log(f^{-1}D + E)).$$

Therefore the assertions follow from the "naturality" of Chern classes. Q.E.D.

Corollary 6.6. *(see [4, Lemma 2.6] and [10, Theorem 10.8]) If (X,D) is a log canonical surface with reduced boundary then*

$$c_2(\widehat{\Omega}_X(\log D)) = e_{orb}(X,D).$$

Proof. Let $f\colon(\tilde X,E)\to X$ be a log resolution of the pair (X,D). For simplicity we can assume that f is an isomorphism outside of $\operatorname{Sing}(X,D)$. Then we have $e_{top}(X-(D\cup\operatorname{Sing}X)) = e_{top}(\tilde X-(E\cup f^{-1}D)) = c_2(\Omega_{\tilde X}(\log(E+f^{-1}D)))$. Since

$$f_*\Omega_{\tilde X}(\log(f^{-1}D+E)) = \widehat{\Omega}_X(\log D)$$

(actually we only need equality of reflexivizations), we have

$$c_2(\widehat{\Omega}_X(\log D)) = c_2(\Omega_{\tilde X}(\log(f^{-1}D+E))) - \sum_{x\in\operatorname{Sing}(X,D)} c_2(x,\Omega_{\tilde X}(\log(f^{-1}D+E))).$$

Therefore the asserted equality follows from Corollary 6.5. Q.E.D.

Definition-Proposition 6.7. *Let (X,D,x) be a germ of a log canonical surface with fractional boundary. Let $\pi\colon(Y,\tilde D,y)\to(X,D,x)$ be a finite proper morphism from a log canonical surface with reduced boundary such that $\pi^*(K_X+D)=K_Y+\tilde D$. Then the number*

$$c_2(x,X,D) = \frac{c_2(y,Y,\tilde D)}{\deg\pi}$$

depends only on (X,D,x).

In particular, if (X,D,x) is not Kawamata log terminal then by Corollary 6.5 $c_2(x,X,D)=0$. Note that a map satisfying the assumptions of the definition exists for every (X,D,x) (see Theorem 3.3). From the definition it follows that $c_2(x,X,D) = -e_{orb}(x;X,D)$.

Proposition 6.8. *(see [Me2, Theorem 3.5]) Let (X,D) be a log canonical surface with fractional boundary. Then*

$$c_2(\widehat{\Omega}_X(\log D)) = e_{orb}(X,D).$$

Proof. By Theorem 3.1 there exists a finite morphism $\pi\colon(Y,\tilde D)\to(X,D)$ such that $\pi^*(K_X+D)\le K_Y+\tilde D$. Let $B_1,\dots,B_k$ denote irreducible components of the branch locus of π not contained in $\operatorname{Supp}D$. We can assume that the ramification indices e_i over a fixed component B_i are the same. Set $F = D+\sum(1-\frac{1}{e_i})B_i$. Then

$$K_Y+\tilde D = \pi^*(K_X+F)$$

and hence (X,F) is log canonical. The proposition will be proved once we show the following assertions:

1. $c_2(X,F) = e_{orb}(X,F)$,
2. $c_2(X,F) - c_2(X,D) = e_{orb}(X,F) - e_{orb}(X,D)$.

To prove 1 note that by Corollary 6.6

$$c_2(X,F) = \frac{1}{\deg\pi}c_2(Y,\tilde D) = \frac{1}{\deg\pi}e_{orb}(Y,\tilde D).$$

Therefore we need only to show that $e_{orb}(Y,\tilde D) = \deg\pi\cdot e_{orb}(X,F)$. This follows immediately from Definition 6.2 and Definition–Proposition 6.7.

Note that $\widehat{\pi^*\Omega_X(\log F)} = \widehat{\Omega}_Y(\log\tilde D)$. Therefore the second equality follows by the Riemann–Roch theorem from the appropriate exact sequence relating $\widehat{\pi^*\Omega_X(\log D)}$ and $\widehat{\pi^*\Omega_X(\log F)}$. Q.E.D.

Appendix: Bogomolov's instability

Theorem A.1. *(Bogomolov's instability theorem) Let $\mathcal{E}$ be an H-semistable rank 2 vector bundle on a smooth surface X. Then $4c_2\mathcal{E} \geq c_1^2\mathcal{E}$.*

Proof. Assume that $4c_2\mathcal{E} < c_1^2\mathcal{E}$. Since

$$\chi(X, S^{2n}\mathcal{E}(-nc_1\mathcal{E})) = \frac{1}{3}(c_1^2\mathcal{E} - 4c_2\mathcal{E})n^3 + \mathrm{O}(n^2)$$

and $h^2(X, S^{2n}\mathcal{E}(-nc_1\mathcal{E})) = h^0(X, S^{2n}\mathcal{E}(-nc_1\mathcal{E})) + \mathrm{O}(n^2)$ by the Serre duality theorem, we get

$$h^0(X, S^{2n}\mathcal{E}(-nc_1\mathcal{E})) = \frac{1}{6}(c_1^2\mathcal{E} - 4c_2\mathcal{E})n^3 + \mathrm{O}(n^2).$$

In particular, there exists a non-zero section $s \in H^0(X, S^{2n}\mathcal{E}(-nc_1\mathcal{E}))$ for a large n. We can find a covering $\pi: Y \to X$ decomposing π^*s into the product $s_1 \dots s_n$, where $s_i \in H^0(\pi^*\mathcal{E}(-D_i))$ (see Lemma 5.2). Let $\mathcal{O}(D_i')$ be a saturation of $\mathcal{O}(D_i)$ in $\pi^*\mathcal{E}$. Then

$$D_i'(c_1\mathcal{E} - D_i') \leq c_2(\pi^*\mathcal{E}). \tag{A.1.1}$$

Since $\pi^*\mathcal{E}$ is π^*H-semistable (see, e.g., [14, Proposition 3.2]) we have $D_i\pi^*H \leq D_i'\pi^*H \leq \frac{1}{2}c_1(\pi^*\mathcal{E})\cdot\pi^*H$. Because $\sum D_i = \frac{1}{2}nc_1(\pi^*\mathcal{E})$, we get $D_i\pi^*H = \frac{1}{2}c_1(\pi^*\mathcal{E})\cdot \pi^*H$ and $D_i = D_i'$. By the Hodge index theorem we have

$$(D_i - \frac{1}{2}c_1(\pi^*\mathcal{E}))^2 \leq 0.$$

Therefore by $(A.1.1)$

$$\frac{1}{4}c_1^2\mathcal{E} - c_2\mathcal{E} \leq \frac{1}{\deg \pi}\left(\frac{1}{4}c_1^2(\pi^*\mathcal{E}) + D_i(D_i - c_1(\pi^*E))\right) \leq 0,$$

a contradiction. Q.E.D.

The idea of the above proof is that Bogomolov's inequality for a semistable but not stable vector bundle follows from the inequalities for vector bundles of lower rank. Therefore it is sufficient to prove Bogomolov's inequality for bundles, which remain stable under each finite covering. We prove that if this happens for a rank 2 vector bundle $\mathcal{E}$ then $h^0(S^{2n}\mathcal{E}(-nc_1\mathcal{E})) = 0$ for each $n \geq 1$.

Added in proof. Recently the author generalized Theorem 0.1 to any log canonical surface with boundary (see `math.AG/0012180`). The only difference in the proof is a generalization of Corollary 4.4 to any log canonical surface with (non-reduced) boundary (cf. Conjecture 4.5).

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