ADJOINT LINEAR SYSTEMS ON NORMAL LOG SURFACES

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Abstract. We prove a Reider type theorem for separating any cluster by an adjoint system to a pseudoeffective divisor on a normal surface. As a corollary we get a Reider type theorem for adjoint linear systems (to a nef $\mathbb{Q}$-divisor) on normal log surfaces. This theorem is new even for smooth surfaces.

0. Introduction

Let X be a normal projective surface over $\mathbb{C}$ and L a pseudoeffective Weil divisor on X such that K_X+L is Cartier. It is well known that every such L has the Zariski decomposition $L=P+N$ into the nef part P and the negative part N. This paper is devoted to the study of properties of the linear system $|K_X+L|$ in terms of P. More precisely, fix a cluster ζ (i.e., a 0-dimensional subscheme of X). We are interested when the restriction map $H^0(K_X+L)\to \mathcal{O}_\zeta(K_X+L)$ is surjective. Without loss of generality we can assume that $H^0(K_X+L)\to \mathcal{O}_{\zeta'}(K_X+L)$ is surjective for any subcluster $\zeta'\subsetneq\zeta$. Then either $H^0(K_X+L)\to\mathcal{O}_\zeta(K_X+L)$ is onto or ζ is Gorenstein. In this last case we consider an invariant δ_ζ equal to $4\deg\zeta$ plus a correction term vanishing for a locally complete intersection in a Gorenstein surface (see Definition 1.12). One can interpret δ_ζ as a local second Chern class of a certain vector bundle associated to ζ (see 2.1). Now we have the following theorem (this is a simplified version of Theorem 3.2).

Theorem 0.1. *Under the above notation, assume that $P^2>\delta_\zeta$. If the restriction map $H^0(K_X+L)\to\mathcal{O}_\zeta(K_X+L)$ is not surjective then there exists a curve D containing ζ such that*
1. $H^0(\mathcal{O}_D(K_X+L))\to\mathcal{O}_\zeta(K_X+L)$ is not surjective,
2. for every subcurve C of D
$$(L-D)C\le\frac{1}{4}\delta_\zeta,$$
3. the following inequality is satisfied:
$$PD\le\frac{1}{2}\delta_\zeta\cdot\frac{1}{1+\sqrt{1-\delta_\zeta/P^2}}\le\frac{1}{2}\delta_\zeta.$$

Let M be a $\mathbb{Q}$-divisor on a normal surface X such that $K_X+\lceil M\rceil$ is Cartier. As a corollary to Theorem 0.1 we get the criterion for the surjectivity of the restriction map $H^0(K_X+\lceil M\rceil)\to\mathcal{O}_\zeta(K_X+\lceil M\rceil)$.

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Theorem 0.2. *If*
(1) $M^2 > \delta_\zeta \cdot \beta^2$, *where* $\beta \geq 1$, *and*
(2) $MC \geq \frac{1}{2}\delta_\zeta \cdot \frac{\beta}{\beta+\sqrt{\beta^2-1}}$ *for every curve* C *on* X,
then $H^0(K_X + \lceil M \rceil) \to \mathcal{O}_\zeta(K_X + \lceil M \rceil)$ *is surjective.*

This is a log version of the Reider type theorem obtained in [La2]. For a more precise version, with condition (2) relaxed, see Theorem 4.1.

This formulation bridges the two most common forms of generalization of Reider's results. Assume X is a smooth surface (or has at most Du Val singularities). Then on the one hand we get a separation of degree s clusters when $M^2 > 4s$ and $MC \geq 2s$ for all curves C (this theorem, in case M is Cartier, is due to Beltrametti, Francia and Sommese), and on the other hand when $M^2 > (s+1)^2$ and $MC \geq s+1$ for all curves C (take $\beta = \frac{s+1}{2\sqrt{s}}$). This second form is useful when considering multiples of an ample divisor.

Instead of assuming that $K_X + \lceil M \rceil$ is Cartier in Theorem 0.2 it is sufficient to assume that $K_X + \lceil M \rceil$ is Cartier in the support of ζ. This, together with some results by Kollár and Alexeev, implies a birational boundedness of minimal log surfaces of general type (see Theorem 5.4). The bound is effective thanks to [AM] but unrealistically high and hence we do not state a precise bound.

Theorems 0.1 and 0.2 would be useless without a better knowledge of δ_ζ. Since δ_ζ is additive, we can assume that ζ is supported on a single point x. Here we provide a few basic properties of this invariant.

(0.3.1) $\delta_\zeta \geq 0$. Moreover, if $\zeta' \subset \zeta$ then $\delta'_\zeta \leq \delta_\zeta$.
(0.3.2) If (X, x) is smooth then $\delta_\zeta = 4 \deg \zeta$. If (X, x) is a Du Val singularity then $\delta_\zeta \leq 4 \deg \zeta$.
(0.3.3) $\delta_x = 4/|\pi_1(X, x)|$, where $\pi_1(X, x)$ is the local fundamental group around x. In particular, $\delta_x = 0$ if (X, x) is not a quotient singularity.
(0.3.4) If $f\colon (\tilde{X}, E) \to (X, x)$ is any resolution and D is a divisor such that $f_*\mathcal{O}_{\tilde{X}}(-D) \subset \mathcal{I}_\zeta$ then for any effective f-exceptional $\mathbb{Q}$-divisor G we have

$$\delta_\zeta \leq -(K_{\tilde{X}} + D + G)^2.$$

In particular, if (X, x) is a rational singularity (non smooth) and $m_x^k \subset \mathcal{I}_\zeta$ then

$$\delta_\zeta \leq k^2(\operatorname{emb}\dim_x X - 1).$$

(0.3.5) If $\pi\colon (Y, y) \to (X, x)$ is a proper map, étale in codimension 1 and $\mathcal{I}_{\zeta'} = \pi^*\mathcal{I}_\zeta/\text{torsion}$ (i.e., ζ' is a scheme-theoretic preimage of ζ), then

$$\delta_{\zeta'} = \deg \pi \cdot \delta_\zeta.$$

The above properties are proved in 1.1.2. More precise information about δ_ζ for a degree 2 cluster associated to the tangent vector is provided in [La2].

In particular, we get an effective criterion for spannedness (see Proposition 4.3) and very ampleness (by Proposition 4.4.8, [La2]) of $K_X + \lceil M \rceil$ for a nef $\mathbb{Q}$-divisor on any normal surface. Further applications include very precise results on smooth surfaces (by (0.3.2)) and effective k-jet ampleness on surfaces with at most rational singularities (by (0.3.4)).

Results about spannedness of $K_X + \lceil M \rceil$ were previously known (the first such result appeared in [EL] for surfaces with at most Du Val singularities; for the best recent result see Theorem 1, [Ka] with a worse bound for M^2) but very ampleness was known only for smooth surfaces (see Theorem 1, [Maş]; the bound for MC is slightly worse than in our theorem). Apparently nothing was known about separating of higher jets of $K_X + \lceil M \rceil$ even for smooth surfaces. As we already remarked the case $M = \lceil M \rceil$ of Theorem 0.2 was proved in [La2] by using Bogomolov's instability, but even in this case Theorem 0.1 is new.

As we already mentioned Theorem 0.2 follows from Theorem 0.1. Theorem 0.1 is proved in the following steps. First we construct a reflexive sheaf $\mathcal{E}$ as an extension of $\mathcal{I}_\zeta \mathcal{O}(K_X + L)$ by ω_X (generalised Serre's construction). Using Kawamata's covering trick we compute an asymptotic growth of the number of sections of $\hat{S}^{2n}\mathcal{E}_{norm}(nN)$. Then the Mumford-Mehta-Ramanathan theorem yields P-instability of $\mathcal{E}$. Clearly, this idea comes from Miyaoka's proof of Bogomolov's instability theorem (see [Mi], Section 4). The idea behind passing to covering is that for a finite cover f of X the sheaf $\mathcal{E}$ is P-semistable if and only if $(f^*\mathcal{E})^{**}$ is f^*P-semistable. Now we can use P-instability of $\mathcal{E}$ to get the suspected "bad" curve D. To bound $(L-D)C$ for a subcurve C of D we perform an elementary transformation of $\mathcal{E}$ with respect to C. Finally we use obtained inequalities and instability of $\mathcal{E}$ to get a precise bound for PD.

The structure of the paper is as follows. In Section 1 we recall some facts and notions we need in the proof. In Section 2 we introduce a certain invariant of a saturated inclusion. This invariant is very important in the proof of Theorem 0.1. Section 3 is devoted to the statement and proof of a more refined version of Theorem 0.1. In Section 4 we prove a version of Theorem 0.2 and give an example how it can be applied to more special cases. Finally, in Section 5 we prove a birational boundedness of minimal log surfaces of general type.

Notation and conventions

We use Mumford's intersection of divisors on a normal surface: if D_1 and D_2 are Weil divisors on a normal surface X then $D_1D_2 = f^*D_1 f^*D_2$, where f is any resolution of singularities and f^*D is the unique $\mathbb{Q}$-divisor of the form $f^{-1}D +$ exceptional part having zero intersection with each irreducible component of the exceptional set of f. Furthermore we use the following notation:

$\mathcal{F}\hat{\otimes}\mathcal{G} = (\mathcal{F}\otimes\mathcal{G})^{**}$,
$\hat{S}^n\mathcal{F} = (S^n\mathcal{F})^{**}$,
$\mathcal{F}(D) = \mathcal{F}\hat{\otimes}\mathcal{O}(D)$.

1. Preliminaries

1.1. Let (X,x) be a germ of a normal surface and let $f\colon(\tilde{X},E)\to(X,x)$ be a good resolution of (X,x), i.e., such that the exceptional curve E has only normal crossings.

Definition 1.1.1. *For a vector bundle $\mathcal{F}$ on $\tilde{X}$ we define*

(1) the modified Euler characteristic (see [Wa])

$$\chi(x,\mathcal{F}) = \dim (f_*\mathcal{F})^{**}/f_*\mathcal{F} + \dim R^1 f_*\mathcal{F},$$

(2) the first Chern class $c_1(x,\mathcal{F})$ as the unique exceptional $\mathbb{Q}$-divisor such that for any exceptional curve C we have $c_1(x,\mathcal{F})\cdot C = \deg \mathcal{F}|_C$,

Now assume that $\operatorname{rk}\mathcal{F} = 2$ *and define*

(3) the second RR Chern class $c_2'(x, \mathcal{F})$ *as a real number such that*

$$\liminf_{n\to\infty} \frac{\chi(x, S^{2n}\mathcal{F}(-n\det\mathcal{F}))}{n^3} = \frac{1}{3}(4c_2'(x,\mathcal{F}) - c_1(x,\mathcal{F})^2).$$

(4) the RR anomaly $a'(x, \mathcal{F})$ *by setting*

$$a'(x,\mathcal{F}) = \chi(x,\mathcal{F}) - 2\cdot\chi(x,\mathcal{O}_{\tilde{X}}) + \frac{1}{2}c_1(x,\mathcal{F})(c_1(x,\mathcal{F}) - K_{\tilde{X}}) - c_2'(x,\mathcal{F}).$$

Let us remark that $a'(x, \mathcal{F})$ depend only on the isomorphism class of the reflexive sheaf $(f_*\mathcal{E})^{**}$ at the singularity (X, x).

Definition 1.1.2. *Let* ζ *be a Gorenstein cluster supported on a point* x. *We define an invariant* $\delta_{\zeta,x}$ *by*

$$\delta_{\zeta,x} = 4(a'(x,\mathcal{E}_\zeta) + \deg\zeta),$$

where $\mathcal{E}_\zeta$ *is a unique class of a reflexive sheaf in* $\operatorname{Ext}^1(\mathcal{I}_\zeta, \omega_X)$.

For a general Gorenstein cluster ζ we set $\delta_{\zeta,x} = \delta_{\zeta',x}$, where ζ' is a part of ζ supported on x.

We already listed several properties of δ_ζ (see (0.3.1)–(0.3.5)). Property (0.3.1) follows from Corollary 2.8. (0.3.2) follows from the fact that $a(x, \mathcal{E}) \le 0$ for any reflexive sheaf at a Du Val singularity (with equality if and only if $\mathcal{E}$ is locally free). (0.3.3) is Theorem 4.3.1 in [La2] and (0.3.5) is a special case of Lemma 2.2. Let us sketch a proof of (0.3.4):

Take $\mathcal{F} \in \operatorname{Ext}^1(\mathcal{O}_{\tilde{X}}, \omega_{\tilde{X}}(D))$ such that $(f_*\mathcal{F})^{**} \in \operatorname{Ext}^1(\mathcal{I}_\zeta, \omega_X)$. Then $\delta_\zeta = -4c_2(x, \mathcal{F})$. Pass to the finite covering $g\colon (\tilde{Y}, F) \to (\tilde{X}, E)$ such that g^*G has integral coefficients and take the kernel $\mathcal{G}$ of the natural map $g^*\mathcal{F} \to \mathcal{O}_{g^*G}$. It is easy to compute that $4c_2(x,\mathcal{F})\cdot\deg g = 4c_2(x, g^*\mathcal{F}) = 4c_2(x, \mathcal{G}(g^*G)) \ge (g^*(K_{\tilde{X}} + D + G))^2$. Q.E.D.

1.1.3. Let X be a global projective surface and $\mathcal{E}$ a reflexive sheaf on X. We set $a'(\mathcal{E}) = \sum_{x\in\operatorname{Sing}X} a'(x,\mathcal{E})$ and $\delta_\zeta = \sum_{x\in\operatorname{Supp}\zeta}\delta_{\zeta,x}$.

Take any good resolution of singularities $f\colon Y \to X$ and any vector bundle $\mathcal{F}$ such that $(f_*\mathcal{F})^{**} = \mathcal{E}$. Then the number $c_2'\mathcal{E} = c_2\mathcal{F} - \sum_{x\in X} c_2'(x,\mathcal{F})$ depends only on an isomorphism class of the reflexive sheaf $\mathcal{E}$ and it is called *the second RR Chern class* of $\mathcal{E}$.

We have the following Riemann–Roch type formula for a rank 2 reflexive sheaf $\mathcal{E}$ on a normal surface X:

$$\chi(X,\mathcal{E}) = 2\cdot\chi(\mathcal{O}_X) + \frac{1}{2}((c_1\mathcal{E})^2 - K_X\cdot c_1\mathcal{E}) - c_2'\mathcal{E} + a'(\mathcal{E}).$$

Lemma 1.1.4. *Let* $\pi\colon X' \to X$ *be a finite morphism of normal surfaces and* $\mathcal{E}$ *a reflexive sheaf on* X. *Then*

$$c_2'((\pi^*\mathcal{E})^{**}) = \deg\pi\cdot c_2'\mathcal{E}.$$

This is a globalization of Theorem 4.14, [La3].

Lemma 1.1.5. *(see [Ha], Corollary 1.7) Let $\pi\colon X' \to X$ be a finite morphism of normal surfaces and $\mathcal{E}$ a reflexive sheaf on X'. Then $\pi_*\mathcal{E}$ is reflexive.*

1.2. Let us recall a few basic facts about a position of a cluster with respect to a linear system.

Definition 1.2.1. *A cluster ζ is in special position (or has the Cayley–Bacharach property) with respect to a linear system $|L|$ if $H^0(\mathcal{I}_{\zeta'}(L)) = H^0(\mathcal{I}_\zeta(L))$ for every $\zeta' \subset \zeta$ with $\deg \zeta' = \deg \zeta - 1$.*

Definition 1.2.2. *Let $\mathcal{L}$ be a line bundle on a Cohen–Macaulay curve C. A cluster ζ is in very special position with respect to $|\mathcal{L}|$ if and only if there exists an injection $\varphi\colon \mathcal{I}_\zeta\mathcal{L} \to \omega_C$, which is not induced by any $\varphi'\colon \mathcal{I}_{\zeta'}\mathcal{L} \to \omega_C$ for a subcluster $\zeta' \subsetneq \zeta$.*

Remark 1.2.3. If ζ is in very special position with respect to $|\mathcal{L}|$, then $\varphi|_{C'}$ is generically an isomorphism for any subcurve C' of C. In particular, if C is a curve on a smooth surface X and $\mathcal{L}$ is a line bundle on X then

$$\chi(\omega_C|_{C'}) \geq \chi(\mathcal{I}_{\zeta, C'}\mathcal{L}|_{C'}) = \chi(\mathcal{O}_{C'}) + C' \cdot c_1\mathcal{L} - \deg(\zeta \cap C'),$$

i.e.,

$$(c_1\mathcal{L} - K_X - C)C' \leq \deg(\zeta \cap C').$$

Theorem 1.2.4. *(see [La1], Theorem 1.2 and [La2], Theorem 1.5) Let X be a normal projective surface, ζ a cluster and L a Cartier divisor on X. Then the following conditions are equivalent:*

1. There exists an extension

$$0 \to \omega_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_\zeta \cdot \mathcal{O}(L) \to 0$$

such that $\mathcal{E}$ is a reflexive sheaf.

2. ζ is Gorenstein and it is in special position with respect to $|L|$.

Theorem 1.3. *(Mumford-Mehta-Ramanathan, see [Mi], Theorem 2.5) Let X be a normal projective variety of dimension $n \geq 2$ and $\mathcal{E}$ a torsion free sheaf. Let H_1, ..., H_{n-1} be ample Cartier divisors. Then, for sufficiently large integers m_1, ..., m_{n-1}, the sheaf $\mathcal{E}$ is $(H_1, ..., H_{n-1})$-semistable if and only if $\mathcal{E}|_C$ is semistable, where C is a general complete intersection curve of $|m_iH_i|$'s.*

Theorem 1.4. *(see, e.g., [Ma], Theorem 2.6) Let $\mathcal{E}$ be a reflexive sheaf on a normal variety X and H an ample Cartier divisor. Then $\mathcal{E}$ is H-semistable if and only if $\hat{S}^n\mathcal{E}$ is H-semistable.*

Lemma 1.5. *Let L be a pseudoeffective Weil divisor on a normal surface X. Let $L = P + N$ be the Zariski decomposition of L. Then for $k \geq 0$*

$$h^0(kL) = \frac{1}{2}P^2k^2 + \mathrm{O}(k) \quad \textit{and} \quad h^1(kL) = -\frac{1}{2}N^2k^2 + \mathrm{O}(k).$$

Proof. Let (Y, L') be the minimal model of the pair (X, L) and $f\colon X \to Y$ the associated morphism. The divisor L' is nef and by the construction of the Zariski decomposition $P = f^*L'$ and N is contracted by f (see the proof of Corollary 7.5, [Sa]). Therefore $f_*\mathcal{O}_X(kL) = \mathcal{O}_Y(kL')$ for $k \geq 0$ (see Theorem 6.2, [Sa]). Hence $h^0(X, kL) = h^0(Y, kL') = \frac{1}{2}(L')^2k^2 + \mathrm{O}(k) = \frac{1}{2}P^2k^2 + \mathrm{O}(k)$. Since $\chi(X, nL) = \frac{1}{2}L^2k^2 + \mathrm{O}(k)$ and $h^2(X, kL) = h^0(K_X - kL) = 0$ for $k \gg 0$ we get the lemma.

2. Invariants of saturated inclusions

Let $\mathcal{E}$ be a rank 2 reflexive sheaf on a germ of a normal surface (X,x). Let $\mathcal{L}$ be a rank 1 reflexive sheaf on (X,x) and suppose we have an inclusion $\mathcal{L}\hookrightarrow\mathcal{E}$ with a torsion free cokernel $\mathcal{M}$.

Definition 2.1. *We define an invariant $\delta_x(\mathcal{L}\to\mathcal{E})$ by setting*

$$\delta_x(\mathcal{L}\to\mathcal{E})=a'(x,\mathcal{E})-a(x,\mathcal{L})-a(x,\mathcal{M}^{**})+h^0(\mathcal{M}^{**}/\mathcal{M}).$$

By Proposition 2.11, [La3] we know that $\delta_x(\mathcal{L}\to\mathcal{E})\geq 0$. If (X,x) is a germ of smooth surface then $\delta_x(\mathcal{L}\to\mathcal{E})=h^0(\mathcal{M}^{**}/M)$ measures how far is the cokernel of the map from being locally free. Note also that if $\mathcal{E}_\zeta\in\operatorname{Ext}^1(\mathcal{I}_\zeta,\omega_X)$ is the unique class of a reflexive sheaf then $\delta_\zeta=4\delta_x(\omega_X\to\mathcal{E}_\zeta)$.

Here we give a very useful interpretation of $\delta_x(\mathcal{L}\to\mathcal{E})$. Let $(\tilde{X},E)\to(X,x)$ be a good resolution of singularity (X,x) such that on $\tilde{X}$ there exists a sequence of vector bundles

$$0\to\tilde{\mathcal{L}}\longrightarrow\tilde{\mathcal{E}}\longrightarrow\tilde{\mathcal{M}}\to 0$$

such that $(\tilde{\mathcal{L}}\to\tilde{\mathcal{E}})|_{\tilde{X}-E}=(\mathcal{L}\to\mathcal{E})|_{X-x}$. Such a resolution always exists (see the proof of Proposition 2.11, [La3]). Then we have

$$\chi(x,\tilde{\mathcal{E}})=\chi(x,\tilde{\mathcal{L}})+\chi(x,\tilde{\mathcal{M}})-h^0(\operatorname{coker}(\mathcal{E}\to\mathcal{M}^{**})).$$

After a simple computation one gets

$$\delta_x(\mathcal{L}\to\mathcal{E})=-c_2'(\tilde{\mathcal{E}}\otimes\tilde{\mathcal{M}}^{-1}).\tag{2.1.1}$$

In particular,

$$4\delta_x(\mathcal{L}\to\mathcal{E})\leq-(c_1(x,\tilde{\mathcal{L}})-c_1(x,\tilde{\mathcal{M}}))^2.\tag{2.1.2}$$

This interpretation implies immediately the following lemmas:

Lemma 2.2. *Let $\pi\colon(Y,y)\to(X,x)$ be a finite proper map. Then*

$$\delta_y((\pi^*\mathcal{L})^{**}\to(\pi^*\mathcal{E})^{**})=\deg\pi\cdot\delta_x(\mathcal{L}\to\mathcal{E}).$$

Lemma 2.3. *For any divisor D we have*

$$\delta_x(\mathcal{L}(D)\to\mathcal{E}(D))=\delta_x(\mathcal{L}\to\mathcal{E}).$$

2.4. Now assume that (X,x) is a germ of a normal (quasiprojective) surface and fix a sequence

$$0\to\mathcal{L}\longrightarrow\mathcal{E}\longrightarrow\mathcal{M}\to 0$$

together with a curve C passing through x. *An elementary transformation* $\mathcal{E}[-C]$ of $\mathcal{E}$ with respect to C is defined as a kernel of the surjection $\mathcal{E}\to\mathcal{M}\otimes\mathcal{O}_C/\mathcal{O}_C$-Torsion. It is easy to show that $\mathcal{E}[-C]$ is a rank 2 reflexive sheaf and we have an inclusion $i'\colon\mathcal{L}\to\mathcal{E}[-C]$ induced by $i\colon\mathcal{L}\to\mathcal{E}$.

Example 2.4.1. Suppose that X is smooth at x and write $\mathcal{M}$ in the form $\mathcal{I}_\zeta\mathcal{M}^{**}$ for some cluster ζ. Then $\delta(\mathcal{L}\to\mathcal{E})=\deg\zeta$ and $\delta(\mathcal{L}\to\mathcal{E}[-C])=\deg\zeta-\deg(\zeta\cap C)$. In particular, we have $\delta_x(\mathcal{L}\to\mathcal{E})\geq\delta_x(\mathcal{L}\to\mathcal{E}[-C])$. This inequality holds also for a general normal singularity (X,x). To prove it we will need the following lemma

Lemma 2.5. *For any globally generated rank r vector bundle $\mathcal{F}$ on $\tilde{X}$ we have*

$$\dim R^1f_*S^n\mathcal{F}=\mathrm{O}(n^{r-1}).$$

Proof. Take an effective exceptional cycle Z such that $R^1f_*\mathcal{O}_{\tilde{X}}(-Z)=0$. Since $\mathcal{F}$ is globally generated we have $R^1f_*\mathcal{F}(-Z)=0$. Hence from the sequence

$$0\to\mathcal{F}(-Z)\longrightarrow\mathcal{F}\longrightarrow\mathcal{F}|_Z\to 0$$

we get $\dim R^1f_*\mathcal{F}=H^1(Z,\mathcal{F}|_Z)$. Since $S^n\mathcal{F}$ is also globally generated we have $\dim R^1f_*S^n\mathcal{F}=h^1(Z,S^n\mathcal{F}|_Z)$. If $\mathcal{F}$ is a rank r vector bundle then for generic r sections of $\mathcal{F}$ the induced map $\mathcal{O}_Z^r\to\mathcal{F}|_Z$ is generically onto. Then the induced map on symmetric powers $S^n(\mathcal{O}_Z^r)\to S^n\mathcal{F}|_Z$ is surjective outside the finite set of points. Therefore $h^1(Z,S^n\mathcal{F}|_Z)\leq\operatorname{rk}S^n(\mathcal{O}_Z^r)\cdot h^1(Z,\mathcal{O}_Z)$ and $\dim R^1f_*S^n\mathcal{F}=\mathrm{O}(n^{r-1})$. Q.E.D.

Theorem 2.6.

$$\delta_x(\mathcal{L}\to\mathcal{E})\geq\delta_x(\mathcal{L}\to\mathcal{E}[-C])$$

Proof. First we consider the following special case. Assume that there exists a good resolution of singularities $(\tilde{X},E)\to(X,x)$ such that $D=f^*C$ is a Weil divisor and we have the following commutative diagram on $\tilde{X}$:

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & \tilde{\mathcal{M}}|_D \arrow[r] \arrow[u] & \tilde{\mathcal{M}}|_D \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \tilde{\mathcal{L}} \arrow[r, "j"'] \arrow[u] & \tilde{\mathcal{E}} \arrow[r] \arrow[u] & \tilde{\mathcal{M}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \tilde{\mathcal{L}} \arrow[r, "j'"'] \arrow[u] & \tilde{\mathcal{E}}[-D] \arrow[r] \arrow[u] & \tilde{\mathcal{M}}(-D) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

such that $j|_{\tilde{X}-E}=i|_{X-x}$ and $j'|_{\tilde{X}-E}=i'|_{X-x}$ and $\tilde{\mathcal{M}}$ is a line bundle.

We will show that under such assumptions we have

$$\chi(x,S^n\tilde{\mathcal{E}})\leq\chi(x,S^n(\tilde{\mathcal{E}}[-D]))+\mathrm{O}(n^2). \tag{2.6.1}$$

Using this inequality and Proposition 4.18, [La3], one can see that

$$\chi(x,S^{2n}\tilde{\mathcal{E}}\otimes\det\tilde{\mathcal{E}}^{-n})\leq\chi(x,S^{2n}(\tilde{\mathcal{E}}[-D])\otimes\det\tilde{\mathcal{E}}[-D]^{-n})+\mathrm{O}(n^2).$$

Since $c_1(x,\tilde{\mathcal{E}})=c_1(\tilde{\mathcal{E}}[-D])$ we get

$$c_2'(x,\tilde{\mathcal{E}}\otimes\tilde{\mathcal{L}}^{-1})\le c_2'(x,\tilde{\mathcal{E}}[-D]\otimes\tilde{\mathcal{L}}^{-1}).$$

But $\delta_x(\mathcal{L}\to\mathcal{E})=-c_2'(x,\tilde{\mathcal{E}}\otimes\tilde{\mathcal{L}}^{-1})$ and $\delta_x(\mathcal{L}\to\mathcal{E}[-C])=-c_2'(x,\tilde{\mathcal{E}}[-D]\otimes\tilde{\mathcal{L}}^{-1})$, so the theorem follows in this case.

Note that (2.6.1) is equivalent to the similar inequality for $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}[-D]$ twisted by the same line bundle on $\tilde{X}$ (see Proposition 4.18, [La3]). Therefore we can assume that all the bundles $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}}[-D]$, $\tilde{\mathcal{L}}$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}(-K_{\tilde{X}})$ and $\tilde{\mathcal{M}}(-K_{\tilde{X}})$ are globally generated.

We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0\to & S^{n-k-1}\tilde{\mathcal{E}}\otimes\tilde{\mathcal{L}}^{k+1} & \to & S^{n-k}\tilde{\mathcal{E}}\otimes\tilde{\mathcal{L}}^{k} & \to & \tilde{\mathcal{M}}^{n-k}\otimes\tilde{\mathcal{L}}^{k} & \to 0\\
 & \uparrow & & \uparrow & & \uparrow & \\
0\to & S^{n-k-1}(\tilde{\mathcal{E}}[-D])\otimes\tilde{\mathcal{L}}^{k+1} & \to & S^{n-k}(\tilde{\mathcal{E}}[-D])\otimes\tilde{\mathcal{L}}^{k} & \to & \tilde{\mathcal{M}}^{n-k}(-(n-k)D)\otimes\tilde{\mathcal{L}}^{k} & \to 0\\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Using it one can see that
(2.6.2)

$$\dim\operatorname{coker}(H^1_E(\tilde{X},S^n(\tilde{\mathcal{E}}[-D]))\to H^1_E(\tilde{X},S^n\tilde{\mathcal{E}}))\le\sum_{k=0}^{n}h^1_E(\mathcal{O}_{kD}\otimes\tilde{\mathcal{M}}^k\otimes\tilde{\mathcal{L}}^{n-k}).$$

Claim 2.6.3. *The set*

$$\{h^1_E(\mathcal{O}_D(A))\colon\ A-K_{\tilde{X}}\ \textit{is } f\textit{-nef}\}$$

is finite.

To prove the claim note that $H^1(\mathcal{O}_{\tilde{X}}(A))=H^1(\mathcal{O}_{\tilde{X}}(A-D))=0$ (D is f-trivial) and therefore

$$\begin{aligned}h^1_E(\mathcal{O}_D(A))&=h^1_E(\mathcal{O}_{\tilde{X}}(A-D))-h^1_E(\mathcal{O}_{\tilde{X}}(A))=\chi(x,\mathcal{O}_{\tilde{X}}(A))-\chi(x,\mathcal{O}_{\tilde{X}}(A-D))\\&=a(x,A)-a(x,A-D).\end{aligned}$$

But the set $\{a(x,\mathcal{N})\colon\mathcal{N}$ is a line bundle$\}$ is bounded by Corollary 4.13, [La3], which proves the claim.

Now, we can use the claim and the sequences

$$0\to\mathcal{O}_{(k-1)D}(-D)\longrightarrow\mathcal{O}_{kD}\longrightarrow\mathcal{O}_D\to 0$$

to show that

$$h^1_E(\mathcal{O}_{kD}\otimes\tilde{\mathcal{M}}^k\otimes\mathcal{L}^{n-k})=\mathrm{O}(k).$$

Therefore by (2.6.2)

$$\dim\operatorname{coker}(H^1_E(\tilde{X},S^n(\tilde{\mathcal{E}}[-D]))\to H^1_E(\tilde{X},S^n\tilde{\mathcal{E}}))=\mathrm{O}(n^2).$$

Note that we have the following commutative diagram

$$\begin{array}{ccccccc}
0\longrightarrow & H^0(\tilde X, S^n\tilde{\mathcal{E}}) & \longrightarrow & H^0(\tilde X - E, S^n\tilde{\mathcal{E}}) & \longrightarrow & H^1_E(\tilde X, S^n\tilde{\mathcal{E}}) & \longrightarrow 0\\
& \uparrow & & \uparrow & & \uparrow & \\
0\longrightarrow & H^0(\tilde X, S^n(\tilde{\mathcal{E}}[-D])) & \longrightarrow & H^0(\tilde X - E, S^n(\tilde{\mathcal{E}}[-D])) & \longrightarrow & H^1_E(\tilde X, S^n(\tilde{\mathcal{E}}[-D])) & \longrightarrow 0\\
& \uparrow & & \uparrow & & & \\
& 0 & & 0 & & &
\end{array}$$

from which follows that

$$\begin{aligned}
&\dim\operatorname{coker}((f_*S^n(\tilde{\mathcal{E}}[-D]))^{**}/S^n(\tilde{\mathcal{E}}[-D]) \to (f_*S^n\tilde{\mathcal{E}})^{**}/S^n\tilde{\mathcal{E}})\\
&\le \dim\operatorname{coker}(H^1_E(\tilde X, S^n(\tilde{\mathcal{E}}[-D])) \to H^1_E(\tilde X, S^n\tilde{\mathcal{E}})) = \mathrm{O}(n^2).
\end{aligned} \tag{2.6.4}$$

On the other hand by Lemma 2.5 we know that

$$\dim R^1 f_* S^n(\tilde{\mathcal{E}}[-D]) = \mathrm{O}(n) \tag{2.6.5}$$

and

$$\dim R^1 f_* S^n\tilde{\mathcal{E}} = \mathrm{O}(n). \tag{2.6.6}$$

Now the inequality (2.6.1) follows from inequality (2.6.4) and equalities (2.6.5) and (2.6.6).

Now we reduce the general case to the special one in which we proved the theorem. Let $f\colon(\tilde X, E)\to(X,x)$ be any good resolution. Let us take a generically finite proper covering $\tilde\pi : (\tilde Y, F) \to (\tilde X, E)$ from a smooth surface $\tilde Y$ such that $\tilde\pi^* f^* C$ is a Weil divisor. Take the Stein factorization of $f\tilde\pi\colon(\tilde Y, F)\to(X,x)$. We get a finite covering $\pi\colon(Y,y)\to(X,x)$ together with a resolution of singularities $\tilde f\colon(\tilde Y, F)\to(Y,y)$. By Lemma 2.2 to prove the theorem it is sufficient to prove that

$$\delta_x((\pi^*\mathcal{L})^{**}\xrightarrow{j}(\pi^*\mathcal{E})^{**}) \ge \delta_x((\pi^*\mathcal{L})^{**}\xrightarrow{j'}(\pi^*\mathcal{E}[-C])^{**}),$$

whereas we know that $\tilde f^*(\pi^* C) = \tilde\pi^* f^* C$ is a Weil divisor.

Therefore we can assume that $D = f^*C$ is a divisor. Clearly, we can also assume that $\mathcal{L} = \omega_X$ (see Lemma 2.3). Now take any line bundle $\tilde{\mathcal{M}}$ on $(\tilde X, E)$ such that $f_*\tilde{\mathcal{M}} \subset \operatorname{coker} i$ and $f_*\tilde{\mathcal{M}}(-D) \subset \operatorname{coker} i'$ and both these inclusions have finite cokernels. We can achieve this by taking any $\tilde{\mathcal{M}}'$ such that $(f_*\tilde{\mathcal{M}}') = (\operatorname{coker} i)^{**}$ and twisting it by a sufficiently f-ample exceptional divisor. Since we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}^1(\tilde{\mathcal{M}}, \omega_{\tilde X}) & \twoheadrightarrow & \mathrm{Ext}^1(f_*\tilde{\mathcal{M}}, \omega_X) & \hookleftarrow & \mathrm{Ext}^1(\operatorname{coker} i, \omega_X) & \ni & \mathcal{E}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\mathrm{Ext}^1(\tilde{\mathcal{M}}(-D), \omega_{\tilde X}) & \twoheadrightarrow & \mathrm{Ext}^1(f_*\tilde{\mathcal{M}}(-D), \omega_X) & \hookleftarrow & \mathrm{Ext}^1(\operatorname{coker} i', \omega_X) & \ni & \mathcal{E}[-C]
\end{array}$$

we can lift $\mathcal{E}$ to $\tilde{\mathcal{E}} \in \mathrm{Ext}^1(\tilde{\mathcal{M}}, \omega_{\tilde X})$ and then $\tilde{\mathcal{E}}[-D] \to \tilde{\mathcal{E}}$ maps to $\mathcal{E}[-C] \to \mathcal{E}$. Q.E.D.

Lemma 2.7. *Let R be a local ring and let $I \subset I' \subset R$ be ideals such that $A = R/I$ and $B = R/I'$ are 0-dimensional Gorenstein rings. Then there exists a nonzero $f \in R$ such that $fI' = I \cap (f)$.*

Proof. By Theorem 21.23c, [E], an ideal $J = (0 :_A I'A)$ is principal. Let f be any lift of the generator of J to R. Since $f \cdot I'A = 0$ in A we have $fI' \subset I \cap (f)$. Now take any element $x = fg$ in $I \cap (f)$. Then $f \cdot gI = 0$ in A and $g \in (0 :_A J) = I'A$ (since $(0 :_A (0 :_A J)) = J$ for any ideal J in a 0-dimensional Gorenstien local ring A; see also Theorem 21.23a, [E]). Therefore $g \in I'$ and $x \in fI'$, Q.E.D.

Corollary 2.8. *Let (X, x) be a germ of a normal surface and let $\zeta' \subset \zeta$ be Gorenstein clusters supported on x. Then $\delta_{\zeta'} \le \delta_\zeta$.*

Proof. By Lemma 2.7 there exists an element $f \in \mathcal{O}_{X,x}$ such that $f\mathcal{I}_{\zeta'} = \mathcal{I}_\zeta \cap (f)$. Let C be the zero set of f. Then the following sequence is exact:

$$0 \to \mathcal{I}_{\zeta'}(-C) \to \mathcal{I}_\zeta \to \mathcal{I}_\zeta \otimes \mathcal{O}_C/(\mathcal{O}_C - \text{torsion}) \to 0.$$

In particular, if $\mathcal{E}_\zeta$ is a reflexive sheaf lying in a sequence

$$0 \to \omega_X \to \mathcal{E}_\zeta \to \mathcal{I}_\zeta \to 0$$

and we make an elementary transformation with respect to C then we get a reflexive sheaf $\mathcal{E}_{\zeta'} \in \mathrm{Ext}^1(\mathcal{I}_{\zeta'}(-C), \omega_X) \simeq \mathrm{Ext}^1(\mathcal{I}_{\zeta'}, \omega_X)$. Now by Theorem 2.6 we get the required inequality, Q.E.D.

3. The main theorem

Definition 3.1. *Let us set*

$$\delta_{\zeta,C} = \sum_{x \in C} (\delta_{\zeta,x} - 4\delta_x(\omega_X \to \mathcal{E}[-C])).$$

Remark 3.1.2. Note that $0 \le \delta_{\zeta,C} \le \sum_{x\in C} \delta_{\zeta,x} \le \delta_\zeta$ and $\delta_{\zeta,C} = 0$ if $\zeta \cap C = \emptyset$. However, it is important to subtract also contributions from the local elementary transformations of $\mathcal{E}_\zeta$. For instance, if X is smooth at $\operatorname{Supp}\zeta$ then

$$\delta_{\zeta,C} = \deg(\zeta \cap C),$$

whereas $\sum_{x\in C} \delta_{\zeta,x}$ is usually larger.

Theorem 3.2. *Let L be a pseudoeffective Weil divisor and ζ a Gorenstein cluster on a normal surface X. Let $L = P + N$ be the Zariski decomposition of L. Assume that $K_X + L$ is Cartier and $P^2 > \delta_\zeta$. Then ζ is in special position with respect to $K_X + L$ if and only if there exists a curve D containing ζ and such that ζ is in very special position with respect to $|\mathcal{O}_D(K_X + L)|$. We can choose D such that for every subcurve C of D*

$$(L - D)C \le \frac{1}{4}\delta_{\zeta,C}$$

and

$$PD \le \frac{1}{2}\delta_\zeta \cdot \frac{1}{1 + \sqrt{1 - \delta_\zeta/P^2}} \le \frac{1}{2}\delta_\zeta.$$

Moreover, if $\delta_\zeta = 0$ *then* $D \leq N$.

Proof. Let ζ be in special position with respect to K_X+L. By the Serre construction (see Theorem 1.2.4) there exists a reflexive sheaf $\mathcal{E} \in \operatorname{Ext}^1(\mathcal{I}_\zeta\mathcal{O}(K_X+L), \omega_X)$. By Lemmas 3.3 and 3.4 the sheaf $\mathcal{E}$ is not P-stable. Let $\mathcal{M}$ be a maximal P-destabilizing divisorial subsheaf of $\mathcal{E}$. By definition $0 \leq P(2c_1\mathcal{M} - (2K_X+L))$, i.e., $P(c_1\mathcal{M} - K_X) \geq P^2 > 0$. Therefore $\operatorname{Hom}(\mathcal{M}, \omega_X) = 0$ and we can apply Lemma 3.1, [La2]. So we get an effective curve D such that ζ is in very special position with respect to $|\mathcal{O}_D(K_X+L)|$ and $\mathcal{M} = \mathcal{O}(K_X+L-D)$. By P-instability of $\mathcal{E}$ we have $(L-2D)P \geq 0$.

Let C be a subcurve of D. Let $\mathcal{E}[-C]$ be the kernel of the natural map $\mathcal{E} \to \mathcal{I}_\zeta\mathcal{O}_C(K_X+L)$. It is a rank 2 reflexive sheaf and we have an inclusion $i' \colon \mathcal{M} \hookrightarrow \mathcal{E}[-C]$ induced by $i \colon \mathcal{M} \hookrightarrow \mathcal{E}$. Moreover, we also have another inclusion $j' \colon \omega_X \hookrightarrow \mathcal{E}[-C]$ induced by $j \colon \omega_X \hookrightarrow \mathcal{E}$ and all of the above inclusions have torsion free cokernels.

Note that we have a sequence

$$0 \to \operatorname{coker} i' \longrightarrow \operatorname{coker} i \longrightarrow \mathcal{I}_\zeta\mathcal{O}_C(K_X+L) \to 0$$

from which one can compute that

$$h^0((\operatorname{coker} i)^{**}/\operatorname{coker} i) - h^0((\operatorname{coker} i')^{**}/\operatorname{coker} i') = \chi(K_X+D) - \chi(K_X+D-C) - \chi(\mathcal{I}_\zeta\mathcal{O}_C(K_X+L)).$$

Now note that by the Riemann–Roch theorem on C we have

$$\chi(\mathcal{I}_\zeta\mathcal{O}_C(K_X+L)) = \chi(\mathcal{O}_C) + (K_X+L)C - \deg(\zeta \cap C),$$

and by the Riemann–Roch theorem on X we have

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) = -\frac{1}{2}C(K_X+C) - a(-C).$$

Therefore

$$\begin{aligned}
&\delta(\mathcal{M} \to \mathcal{E}) - \delta(\mathcal{M} \to \mathcal{E}[-C]) \\
&= a'(\mathcal{E}) - 2a(-D) + h^0((\operatorname{coker} i)^{**}/\operatorname{coker} i) - (a'(\mathcal{E}[-C]) - a(-D) \\
&- a(-(D-C)) + h^0((\operatorname{coker} i')^{**}/\operatorname{coker} i')) \\
&= a'(\mathcal{E}) - a'(\mathcal{E}[-C]) + a(-(D-C)) - a(-D) + \chi(K_X+D) \\
&- \chi(K_X+D-C) - \chi(\mathcal{I}_\zeta\mathcal{O}_C(K_X+L)) \\
&= a'(\mathcal{E}) - a'(\mathcal{E}[-C]) - \chi(\mathcal{I}_\zeta\mathcal{O}_C(K_X+L)) + \frac{1}{2}(K_XC - C^2) + CD \\
&= a'(\mathcal{E}) - a'(\mathcal{E}[-C]) + C(D-L) + a(-C) - \deg(\zeta \cap C).
\end{aligned}$$

In much the same way as above one can compute that

$$h^0((\operatorname{coker} j)^{**}/\operatorname{coker} j) - h^0((\operatorname{coker} j')^{**}/\operatorname{coker} j') = \deg(\zeta \cap C)$$

and

$$\begin{aligned}\frac{1}{4}\delta_\zeta - \delta(\omega_X \to \mathcal{E}[-C]) &= \delta(\omega_X \to \mathcal{E}) - \delta(\omega_X \to \mathcal{E}[-C]) \\ &= a'(\mathcal{E}) - a'(\mathcal{E}[-C]) + a(-C) + \deg \zeta - \deg(\zeta \cap C).\end{aligned}$$

But we also have

$$\frac{1}{4}\delta_\zeta - \delta(\omega_X \to \mathcal{E}[-C]) = \sum_{x\in C}(\delta_x(\omega_X \to \mathcal{E}) - \delta_x(\omega_X \to \mathcal{E}[-C])) = \frac{1}{4}\delta_{\zeta,C},$$

since at the points $x \notin \zeta \cap C$ the reflexivization of the cokernel of j' is locally free and $\delta_x(\omega_X \to \mathcal{E}) = \delta_x(\omega_X \to \mathcal{E}[-C])$. Therefore

$$\delta(\mathcal{M} \to \mathcal{E}) - \delta(\mathcal{M} \to \mathcal{E}[-C]) + (L-D)C = \frac{1}{4}\delta_{\zeta,C}.$$

and by Theorem 2.6 we get

$$(L-D)C \leq \frac{1}{4}\delta_{\zeta,C}.$$

Claim 3.2.1. *For any effective $\mathbb{Q}$-divisor $F \leq D$ we have*

$$(L-D)F \leq \frac{1}{4}\delta_\zeta.$$

To prove this write $F = \sum f_i C_i + \sum g_j C_j$ as a sum of irreducible divisors, where $(L-D)C_i \leq 0$ and $(L-D)C_j > 0$. Then

$$(L-D)F \leq \sum g_j C_j (L-D) \leq \sum \lceil g_j \rceil C_j (L-D)$$

and the claim follows from the fact that $\sum \lceil g_j \rceil C_j$ is a subcurve of D and $\delta_{\zeta,C} \leq \delta_\zeta$ for any subcurve C of D.

To prove the second part of the theorem write $N - D = N' - D'$, where N' and D' are effective $\mathbb{Q}$-divisors without common irreducible components. Then by Claim 3.2.1 we get

$$(P - D')D' \leq (P + N' - D')D' = (L-D)D' \leq \frac{1}{4}\delta_\zeta.$$

By the Hodge index theorem

$$PD' - \frac{1}{4}\delta_\zeta \leq (D')^2 \leq \frac{(PD')^2}{P^2}$$

and hence

$$\left(P^2 - 2PD'\right)^2 \geq P^2(P^2 - \delta_\zeta).$$

We already know that $PD = PD' \leq P^2$, so

$$PD \leq \frac{1}{2}\left(P^2 - \sqrt{P^2(P^2 - \delta_\zeta)}\right) = \frac{1}{2}\delta_\zeta \cdot \frac{1}{1 + \sqrt{1 - \delta_\zeta / P^2}}.$$

Finally note that if $\delta_\zeta = 0$ then $PD' = 0$ and $(D')^2 \geq PD' = 0$, so by the Hodge index theorem $D' = 0$, i.e., $D \leq N$. Q.E.D.

Remarks.

(1) As a special case of this theorem for $\zeta = \emptyset$ one gets the Kawamata-Viehweg vanishing theorem for surfaces: $H^1(K_X + \lceil M \rceil) = 0$ for any nef and big $\mathbb{Q}$-divisor M. In fact, our theorem in this case is much stronger (especially for singular surfaces).

(2) If X is smooth then the inequality $(L - D)C \leq 1/4\delta_{\zeta,C} = \deg(\zeta \cap C)$ follows from the fact that ζ is in very special position with respect to $K_X + L$ (see Remark 1.2.3).

(3) In the proof of Theorem 3.2 we used only the fact that $K_X + L$ is Cartier in the support of ζ and we do not need to assume that $K_X + L$ is Cartier everywhere.

Lemma 3.3. *Let $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{I}_\zeta \mathcal{O}(K_X + L), \omega_X)$ be a reflexive sheaf. Set $\mathcal{F} = \mathcal{E}(-K_X)$. Then*

$$h^0(\hat{S}^{2n}\mathcal{F}(-nP)) \geq \frac{1}{6}(P^2 - \delta_\zeta)n^3 + \mathrm{o}(n^3).$$

Proof. Let $f: \tilde{X} \to X$ be a log resolution of the pair (X, N). There exists a finite covering $\tilde{\pi}: \tilde{Y} \to \tilde{X}$ from a smooth surface $\tilde{Y}$ such that $\tilde{\pi}^* f^* N$ has integral coefficients (see Theorem 1.1.1, [KMM]). Let us take the Stein factorization of $\tilde{\pi} f$. We get a normal surface Y and two maps $g: \tilde{Y} \to Y$ and $\pi: Y \to X$ such that g is birational and π is finite. Moreover, $\pi^* N = g_* \tilde{\pi}^* f^* N$ is an effective divisor with integral coefficients. Set $\tilde{\mathcal{E}} = (\pi^* \mathcal{E})^{**}$.

Now let us note that on Y we have the following sequence

$$0 \to \mathcal{O}_Y(\pi^* K_X) \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{I}_{\zeta'} \mathcal{O}_Y(\pi^* K_X + \pi^* L) \to 0$$

where ζ' is some cluster and $4\delta(\mathcal{O}_Y(\pi^* K_X) \to \tilde{\mathcal{E}}) = \deg \pi \cdot \delta_\zeta$ (see Lemma 2.2). Let $\mathcal{G}$ be the kernel of the natural map $\tilde{\mathcal{E}} \to \mathcal{I}_{\zeta'} \mathcal{O}_{\pi^* N}(\pi^* K_X + \pi^* L)$ (if $\pi^* N = 0$ set $\mathcal{G} = \tilde{\mathcal{E}}$). Set $\mathcal{H} = \mathcal{G}(-\pi^* K_X)$ and $\tilde{\mathcal{F}} = (\pi^* \mathcal{F})^{**}$. By definition we get an induced inclusion $\mathcal{O}_Y \to \mathcal{H}$ with a torsion free cokernel. It is easy to see that $c_2' \mathcal{H} = \delta(\mathcal{O}_Y \to \mathcal{H})$. By the Serre duality theorem we get

$$h^2(\hat{S}^{2n}\mathcal{H}(-n\pi^* P)) = h^0(\hat{S}^{2n}\mathcal{H}(-n\pi^* P)\hat{\otimes}\omega_Y) = h^0(\hat{S}^{2n}\mathcal{H}(-n\pi^* P)) + \mathrm{O}(n^2).$$

Therefore

$$h^0(\hat{S}^{2n}\tilde{\mathcal{F}}(-n\pi^* P)) \geq h^0(\hat{S}^{2n}\mathcal{H}(-n\pi^* P)) \geq \frac{1}{2}\chi(\hat{S}^{2n}\mathcal{H}(-n\pi^* P)) + \mathrm{O}(n^2)$$
$$\geq \frac{1}{6}((\pi^* P)^2 - 4\delta(\mathcal{O}_Y \to \mathcal{H}))n^3 + \mathrm{o}(n^3).$$

By Theorem 2.6 we have $4\delta(\mathcal{O}_Y \to \mathcal{H}) \leq 4\delta(\mathcal{O}_Y \to \tilde{\mathcal{F}}) = \deg \pi \cdot \delta_\zeta$. By the projection formula:

$$\pi_* \hat{S}^{2n}\tilde{\mathcal{F}}(-n\pi^* P) = (\hat{S}^{2n}\mathcal{F}(-nP))\hat{\otimes}\pi_* \mathcal{O}_Y.$$

For any fixed divisorial sheaf $\mathcal{L}$ on X we have

$$h^0(\hat{S}^{2n}\mathcal{F}(-nP)\hat{\otimes}\mathcal{L}) - h^0(\hat{S}^{2n}\mathcal{F}(-nP)) = \mathrm{O}(n^2).$$

Since $\pi_*\mathcal{O}_Y$ is a rank $\deg \pi$ reflexive sheaf one gets

$$h^0((\hat{S}^{2n}\mathcal{F}(-nP))\hat{\otimes}\pi_*\mathcal{O}_Y) \leq \deg \pi \cdot h^0(\hat{S}^{2n}\mathcal{F}(-nP)) + \mathrm{O}(n^2).$$

Hence

$$h^0(\hat{S}^{2n}\mathcal{F}(-nP)) \geq \frac{1}{\deg \pi} h^0(\hat{S}^{2n}\tilde{\mathcal{F}}(-n\pi^* P)) + \mathrm{O}(n^2) \geq \frac{1}{6}(P^2 - \delta_\varsigma)n^3 + \mathrm{o}(n^3).$$

Q.E.D.

Lemma 3.4. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf with $c_1\mathcal{F} = L$. If*

$$h^0(\hat{S}^{2n}\mathcal{F}(-nP)) \geq an^3 + \mathrm{o}(n^3)$$

and $a > 0$ then $\mathcal{F}$ is not P-stable.

Proof. Let H' be a small ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and set $H = P + H'$. Then for sufficiently large and divisible n we have an exact sequence

$$0 \to \hat{S}^{2n}\mathcal{F}(-nH) \longrightarrow \hat{S}^{2n}\mathcal{F}(-nP) \longrightarrow \hat{S}^{2n}\mathcal{F}(-nP)|_C \to 0 \tag{3.4.1}$$

where $C \in |nH'|$ is a general curve.

If $\mathcal{F}|_C$ is not semistable for $n \gg 0$ then by the Mumford-Mehta-Ramanathan theorem (see Theorem 1.3) $\mathcal{F}$ is not H'-semistable. If this happens for any small H' then $\mathcal{F}$ cannot be P-stable.

Therefore we can assume that for some small H' the bundle $\mathcal{F}|_C$ is semistable for $n \gg 0$. Then $\hat{S}^{2n}\mathcal{F}(-nP)|_C$ is also semistable for any $n \gg 0$. Hence

$$h^0(\hat{S}^{2n}\mathcal{F}(-nP)|_C) \leq \mathrm{rk}\, \hat{S}^{2n}\mathcal{F}(-nP)|_C + \deg \hat{S}^{2n}\mathcal{F}(-nP)|_C = (2n+1)(n^2NH'+1).$$

Hence by (3.4.1) we get

$$\begin{aligned} h^0(\hat{S}^{2n}\mathcal{F}(-nH)) &\geq h^0(\hat{S}^{2n}\mathcal{F}(-nP)) - h^0(\hat{S}^{2n}\mathcal{F}(-nP)|_C) \\ &\geq (a - 2NH')n^3 + o(n^3) > 0 \end{aligned}$$

for $n \gg 0$ (since NH' is small by assumption).

Let us note that

$$c_1(\hat{S}^{2n}\mathcal{F}(-nH)) = (2n+1)n(N - H')$$

and $(N - H')P = -H'P < 0$. Therefore we can choose an ample $\mathbb{Q}$-Cartier divisor H'' close to P and such that $(N - H')H'' < 0$.

By Theorem 1.4 the sheaf $\mathcal{F}$ is H''-semistable if and only if $\hat{S}^{2n}\mathcal{F}(-nH)$ is H''-semistable. But

$$h^0(\hat{S}^{2n}\mathcal{F}(-nH)) > 0,$$

whereas

$$c_1(\hat{S}^{2n}\mathcal{F}(-nH))H'' < 0,$$

so $\hat{S}^{2n}\mathcal{F}(-nH)$ is not H''-semistable. It follows that $\mathcal{F}$ is not P-stable. Q.E.D.

Remark. If P in the above lemma is ample then by restriction to the general curve $C \in |kP|$ for $k \gg 0$ one gets that $\hat{S}^{2n}\mathcal{F}(-nP)|_C$ is a degree 0 vector bundle with a nonzero section (otherwise $h^0(\hat{S}^{2n}\mathcal{F}(-nP)(-sC)) = 0$ for any positive s and we have a contradiction with Serre's vanishing theorem). Therefore by the Mumford-Mehta-Ramanathan theorem $\mathcal{F}$ is not P-semistable. Unfortunately these arguments do not work if P is only nef and big and we need another proof.

4. Applications to $\mathbb{Q}$-divisors on log surfaces

Theorem 4.1. *Let X be a normal projective surface. Let M be a $\mathbb{Q}$-divisor on X such that $K_X + \lceil M \rceil$ is Cartier and let ζ be a Gorenstein cluster on X. If*
1. $M^2 > \delta_\zeta \cdot \beta^2$, where $\beta \geq 1$, and
2. $MC \geq \frac{1}{2}\delta_{\zeta,C} \cdot \frac{\beta}{\beta+\sqrt{\beta^2-1}}$ for all irreducible, reduced curves C,
then ζ is not with special position with respect to $|K_X + \lceil M \rceil|$.

Proof. Set $L = \lceil M \rceil$, $B = L - M$ and let $L = P + N$ be the Zariski decomposition of L. Assume that ζ is in special position with respect to $|K_X + L|$. Since $P^2 \geq M^2 > \delta_\zeta$ we can apply Theorem 3.2. Let D be the curve satisfying all the assertions of this theorem.

Write $B - D = B' - D'$, where B' and D' are effective $\mathbb{Q}$-divisors without common irreducible components. Note that $D' \neq 0$, since $[B] = 0$. Write $D' = \sum a_i C_i$ as a sum of irreducible components C_i. Set

$$\gamma = \min(\delta_\zeta, \sum a_i \delta_{\zeta, C_i}).$$

Since $D' \leq D$ is an effective $\mathbb{Q}$-divisor we can use Claim 3.2.1 to get

$$(M - D')D' \leq (M + B' - D')D' = (L - D)D' \leq \frac{1}{4}\gamma,$$

that is

$$MD' - \frac{1}{4}\gamma \leq (D')^2.$$

By the Hodge index theorem we have

$$(D')^2 \leq \frac{(PD')^2}{P^2} \leq \frac{1}{4}\delta_\zeta.$$

Therefore

$$MD' \leq \frac{1}{4}(\delta_\zeta + \gamma) < \frac{1}{2}M^2. \tag{4.1.1}$$

Now apply the Hodge index theorem once more to get

$$MD' - \frac{1}{4}\gamma \leq (D')^2 \leq \frac{(MD')^2}{M^2}.$$

Hence

$$\left(\frac{1}{2}M^2 - MD'\right)^2 \geq \frac{1}{4}M^2(M^2 - \gamma). \tag{4.1.2}$$

If $\gamma = 0$ then $MD' = 0$. Hence by the Hodge index theorem $(D')^2 < 0$ though $MD' \leq (D')^2$, a contradiction. Therefore we can assume that $\gamma > 0$. Then by (4.1.1) and (4.1.2) we get

$$\begin{aligned} MD' &\leq \frac{1}{2}\left(M^2 - \sqrt{M^2(M^2-\gamma)}\right) = \frac{1}{2}\gamma \cdot \frac{M^2}{M^2 + \sqrt{M^2(M^2-\gamma)}} \\ &< \frac{1}{2}\gamma \cdot \frac{\beta}{\beta + \sqrt{\beta^2 - 1}}. \end{aligned}$$

But by assumption we have

$$MD' = \sum a_i MC_i \geq \frac{1}{2}\left(\sum a_i \delta_{\zeta, C_i}\right) \cdot \frac{\beta}{\beta + \sqrt{\beta^2 - 1}},$$

a contradiction. Q.E.D.

Remark 4.2.
(1) Note that we used $[B] = 0$ only to show that $D' \neq 0$.
(2) Let $f: Y \to X$ be a resolution of singularities of $X - \operatorname{Supp} \zeta$. If the map $H^0(Y, \mathcal{I}_{f^{-1}(\zeta)}\mathcal{O}(K_Y + \lceil f^*M \rceil)) \to \mathcal{O}_{f^{-1}(\zeta)}(K_Y + \lceil f^*M \rceil)$ is surjective then the map $H^0(X, \mathcal{I}_\zeta \mathcal{O}(K_X + \lceil M \rceil)) \to \mathcal{O}_\zeta(K_X + \lceil M \rceil)$ is also surjective. This follows from the fact that $H^1(X, \mathcal{I}_\zeta \mathcal{O}(K_X + \lceil M \rceil)) = H^1(X, f_* \mathcal{I}_{f^{-1}(\zeta)}\mathcal{O}(K_Y + \lceil f^*M \rceil)) \subset H^1(Y, \mathcal{I}_{f^{-1}(\zeta)}\mathcal{O}(K_Y + \lceil f^*M \rceil)) = 0$.

Finally we give a more special example of application of Theorem 0.2.

Proposition 4.3. *Let X be a normal projective surface. Let M be a $\mathbb{Q}$-divisor on X such that $K_X + \lceil M \rceil$ is Cartier. If $M^2 > 4/|\pi_1(X, x)|$ then $|K_X + \lceil M \rceil|$ is globally generated at x unless (X, x) is smooth or of type A_n and there exists a curve C such that*

$$MC \leq \frac{2}{|\pi_1(X, x)|} \cdot \frac{1}{1 + \sqrt{1 - \frac{4}{|\pi_1(X,x)| M^2}}} < \frac{2}{|\pi_1(X, x)|}.$$

Proof. By Remark 4.2, (2) we can assume that X is smooth off x. The assertions is nontrivial only at quotient singularities of type different to A_n. But then $s = \min\{n : nC \text{ is Cartier for any } C \text{ on } X\} < |\pi_1(X, x)|$, since the class group of the singularity (X, x) is equal to the abelianization of $\pi_1(X, x)$.

Applying Theorem 3.2 we get a curve D such that for any its subcurve C

$$(L - D)C \leq \frac{1}{|\pi_1(X, x)|} < \frac{1}{s}$$

and $2PD < P^2$, where P is the positive part of $\lceil M \rceil$. But $(L - D) \cdot (sC) \in \mathbb{Z}$, so it follows that $(L - D)C \leq 0$. As in the proof of Theorem 3.2 write $N - D = N' - D'$. Then

$$(P - D')D' \leq (L - D)D' \leq 0.$$

Therefore

$$PD' \leq (D')^2 \leq \frac{(PD')^2}{P^2}.$$

If $PD' = 0$ then $(D')^2 = 0$ and we have a contradiction with the Hodge index theorem. Otherwise we get $P^2 \leq PD' = PD$, a contradiction with $2PD < P^2$. Q.E.D.

5. Birational boundedness of log surfaces of general type

Definition 5.1. *Let a subset $\mathcal{C} \subset [0, 1]$ satisfies a descending chain condition and assume that $1 \in \mathcal{C}$. Then we define $S(\mathcal{C})$ as a set of normal projective surfaces with boundary (S, Δ) such that*

1. *(S, Δ) is log canonical,*
2. *coefficients of Δ belong to $\mathcal{C}$,*
3. *$K_S + \Delta$ is nef and big.*

All the definitions and notation used from now on is explained in [Ko]. For any $\mathcal{C}$ one can define a certain invariant $\epsilon_1(\mathcal{C}) > 0$ such that for any covering family $\{C_t\}$ of curves and $(S, \Delta) \in S(\mathcal{C})$ we have $(K_S + \Delta + C_t)C_t \geq \epsilon_1(\mathcal{C})$ (see Complement 5.7.4, [Ko]). Moreover, from the proof of Complement 5.7.4, [Ko], one can see that $\epsilon_1(\mathcal{C}) \geq \min(\min(\sum \operatorname{Diff} \mathcal{C} \cap \mathbb{R}_+), \min(\sum(\operatorname{Diff} \mathcal{C} - 2) \cap \mathbb{R}_+))$, so we get an effectively computable bound on $\epsilon_1(\mathcal{C})$.

Theorem 5.2. *(Theorem 6.4, [Ko]) Let $(S, \Delta) \in S(\mathcal{C})$ and set $(K_S + \Delta)^2 = d^2$. Let $\{C_t\}$ be a covering family of curves. Then*

$$(K_S + \Delta)C_t \geq d\left(\sqrt{\epsilon_1(\mathcal{C}) + \frac{1}{4}} - \frac{1}{2}\right) \geq (\sqrt{483} - 21)d\epsilon_1(\mathcal{C}) \geq 0.977d\epsilon_1(\mathcal{C}).$$

Proof. By the Hodge index theorem

$$((K_S + \Delta)C_t)^2 \geq C_t^2 d^2.$$

On the other hand

$$C_t^2 \geq \epsilon_1(\mathcal{C}) - (K_S + \Delta)C_t.$$

Now the first inequality follows by a simple computation. Second inequality follows from the fact that $\epsilon_1(\mathcal{C}) \leq \frac{1}{42}$ for any $\mathcal{C}$ (see Remark 5.15, [Ko]). Q.E.D.

Theorem 5.3. *(Theorem 4.12, [AM]) Let $(S, \Delta) \in S(\mathcal{C})$. Then there exists an effectively computable bound $c(\mathcal{C})$ such that*

$$(K_S + \Delta)^2 \geq c(\mathcal{C}).$$

Theorem 5.4. *Let $(S, \Delta) \in S(\mathcal{C})$. Then there exists an effectively computable integer $N(\mathcal{C})$ such that for any $n > N(\mathcal{C})$ a rational map defined by the linear system $|K_S + \lceil n(K_S + \Delta)\rceil|$ is birational. In particular, $h^0(K_S + \lceil n(K_S + \Delta)\rceil) \geq 3$.*

Remark. As a corollary we see that there exists $N \in \mathbb{N}$ such that for any log canonical surface with K_S nef and big the map defined by NK_S is birational and the plurigenus $P_N(S) = h^0(NK_S) \geq 3$. This implies Conjecture 5.5, [Bl].

Proof. Take $N(\mathcal{C}) = \max(\sqrt{\frac{4}{c(\mathcal{C})}}, \frac{2.05}{\epsilon_1(\mathcal{C})\sqrt{c(\mathcal{C})}})$. Then for $n > N(\mathcal{C})$

$$(n(K_S + \Delta))^2 > 4.$$

Take two very general points of S and assume that they are not separated by $|K_S + \lceil n(K_S + \Delta)\rceil|$. Then by Theorem 0.2 there exists an irreducible curve C passing through a general point of S and such that $n(K_S + \Delta)C < 2$. But by the countability of the number of irreducible components of the Hilbert scheme any irreducible curve passing through a very general point occurs as a fiber in a covering family of curves. Hence by Theorem 5.2 we get

$$n(K_S + \Delta)C > N(\mathcal{C}) \cdot 0.977d\epsilon_1(\mathcal{C}).$$

Since the last term is bigger than 2, we get a contradiction. Q.E.D.

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References

[AM] V. Alexeev, S. Mori, *Bounding singular surfaces of general type*, preprint, `http://www.math.utah.edu/~kollar/`.

[Bl] R. Blache, *Riemann–Roch theorem for normal surfaces and applications*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 307–340.

[EL] L. Ein, R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds*, J. Amer. Math. Soc. **6** (1993), 875–903.

[E] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150** (1995), Springer–Verlag, New York.

[Ha] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[Ka] T. Kawachi, *Effective base point freeness on normal surfaces*, Manuscripta Math. **101** (2000), 23–38.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model program*, Adv. Stud. Pure Math. **10** (1987), 283–360.

[Ko] J. Kollár, *Log surfaces of general type; some conjectures*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math. **162** (1994), 261–275.

[La1] A. Langer, *Adjoint linear systems on normal surfaces I*, J. Algebraic Geom. **8** (1999), 41–66.

[La2] ———, *Adjoint linear systems on normal surfaces II*, J. Algebraic Geom. **9** (2000), 71–92.

[La3] ———, *Chern classes of reflexive sheaves on normal surfaces*, to appear in Math. Z.

[Ma] M. Maruyama, *The theorem of Grauert-Müllich-Spindler*, Math. Ann. **255** (1981), 317–333.

[Maş] V. Maşek, *Very ampleness of adjoint linear systems on smooth surfaces with boundary*, Nagoya Math. J. **153** (1999), 1–29.

[Mi] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai, Adv. Stud. Pure Math. **10** (1987), 449–476.

[Rdr] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127** (1988), 309—316.

[Sa] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51** (1984), 877-887.

[Wa] J. Wahl, *Second Chern class and Riemann–Roch for vector bundles on resolutions of surface singularities*, Math. Ann. **279** (1988), 583–598.

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