TORIC INVARIANT THEORY

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Introduction

The topic of this survey are quotients with respect to actions of subgroups of the big torus on a toric variety. This setting is of particular interest due to two important features of the toric category: The description of toric varieties by combinatorial data is a powerful tool for explicit constructions, and on the other hand, the class of toric varieties is large enough to reflect many general principles of algebraic geometry.

Following [10], we base our investigation on a comprehensive concept of quotient including most of the classically studied notions as special cases, namely: A *quotient* for a class $\mathfrak{M}$ of morphisms $X \to X'$ from a fixed variety X to varieties X' is a morphism $X \to Y$ of $\mathfrak{M}$ such that any further morphism of $\mathfrak{M}$ factors uniquely through $X \to Y$.

If for example we consider the class of morphisms that are constant on orbits of a given group action we obtain the notion of a categorical quotient as introduced by Mumford [26, Section 0]. Other examples are provided by comparing a category with a full subcategory: A quotient for the class of all morphisms from X to objects in this subcategory can be viewed as a *reduction* of X to the subcategory.

In general, the problem whether or not a given class of morphisms has a quotient in the above sense is hard, however in the toric setting we have a general existence result: In Section 1, we prove that quotients in the category of toric varieties exist for any reasonable class $\mathfrak{M}$ of toric morphisms. Here reasonable means that $\mathfrak{M}$ allows to split off generically finite affine morphisms and that $\mathfrak{M}$ contains a certain kind of product, for the precise formulation see Definition 1.2.

This result suggests using the following strategy when asking for a solution to the quotient problem in a larger category: The first step is to consider the analoguous quotient problem in the toric category and to look for an explicit construction of the quotient in terms of combinatorial data. The second step consists in analyzing whether the constructed toric object even satisfies the universal property in the larger category.

Our basic construction in the spirit of the first step is presented in Section 2, where we describe an algorithm for the construction of the *toric quotient*, that means a quotient for the class of toric morphisms that are constant on the orbits of a subgroup action on a given toric variety. We also treat the analogous problem in the nonseparated setting since that naturally occurs in applications.

In Section 3, we consider the classical notions of good and geometric quotients in the framework of Toric Invariant Theory. This setting was studied by several authors, see for example [22], [18] and [32]. Both good and geometric quotients turn out to be special cases of toric quotients as constructed in Section 2.

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In Sections 4 and 5 we apply our strategy to the problem of finding quotients for subgroup actions on toric varieties in the categories of quasiprojective and divisorial varieties. In both cases we obtain necessary and sufficient conditions for the existence of such a quotient, and if the conditions are fulfilled we have an explicit construction for the quotient involving the abovementioned toric quotient and a reduction step.

Finally, in Sections 6 and 7, we consider categorical quotients for subgroup actions on toric varieties, or in our terminology we discuss the quotient problem in the categories of algebraic varieties and prevarieties. To illustrate the difficulties we give several counterexamples, but we also present criteria for the existence of categorical quotients.

Throughout the whole article we work over an algebraically closed field $\mathbb{K}$. A brief review of the main facts on toric varieties and prevarieties as well as their combinatorial description can be found in the appendix where we have also introduced most of the notation that we use throughout the text.

Contents

1. Quotients for toric morphism classes

In this section, we first introduce the basic general concept of quotient that will reappear in different specializations in the succeeding sections. Then we give a general existence result for the category of toric varieties. Following Białynicki-Birula [10], we formulate our definition in terms of classes of morphisms:

Let $\mathfrak{C}$ be any category, $X \in \mathrm{Obj}(\mathfrak{C})$ an object and $\mathfrak{M}$ a subclass of the class $\mathrm{Mor}(X, ?)$ of morphisms from X to objects X' of $\mathfrak{C}$.

Definition 1.1. A *quotient* for the class $\mathfrak{M}$ is a morphism $p\colon X \to Y$ belonging to $\mathfrak{M}$ such that for every further morphism $f\colon X \to X'$ of $\mathfrak{M}$ there is a unique morphism $f'\colon Y \to X'$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \\ & Y \arrow[ur, "f'"'] & \end{tikzcd}$$

In the whole survey, $\mathfrak{C}$ will always be a subcategory of the category of prevarieties over a fixed algebraically closed field $\mathbb{K}$; for example, $\mathfrak{C}$ may be the category of all (separated) varieties over $\mathbb{K}$, or $\mathfrak{C}$ may consist of all quasiprojective varieties over $\mathbb{K}$. Note that [10, Definition 4.1] is even more general, but for our purposes the above version will do.

If $\mathfrak{M}$ consists of all morphisms that are constant on the classes of some equivalence relation, then we obtain the usual notion of a categorical quotient. However, there are other cases of interest: If $\mathfrak{M}$ consists of all morphisms from X to the objects X' of a full subcategory $\mathfrak{C}'$ of $\mathfrak{C}$, then a quotient for $\mathfrak{M}$ is a *reduction* of X to the subcategory $\mathfrak{C}'$.

While in general the problem of existence of quotients is quite delicate, the toric category allows existence statements. The reason is an intrinsic finiteness of the toric category due to its combinatorial description in terms of finite fans. We will show below, that a quotient always exists provided the class $\mathfrak{M}$ of morphisms is reasonable in the following sense:

Definition 1.2. Let $\mathfrak{C}$ be the category of toric varieties and let X be a toric variety. We call a nonempty subclass $\mathfrak{M}$ of $\mathrm{Mor}(X, ?)$ *admissible*, if it has the following properties:

(i) If a morphism $f \in \mathfrak{M}$ admits a decomposition $f = f' \circ f''$ into toric morphisms f', f'' such that f' is affine and generically finite then f'' belongs to $\mathfrak{M}$.

(ii) If $f_i \colon X \to X_i$ are two morphisms of $\mathfrak{M}$ then there exists a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dd, "f_{12}"] \arrow[dr, "f_2"] & \\ X_1 & & X_2 \\ & X_{12} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

of toric morphisms such that $f_{12} \colon X \to X_{12}$ is dominant and belongs to $\mathfrak{M}$.

Note that the second property can often be verified by taking as X_{12} the normalization of the closure of the image of the product morphism $f_1 \times f_2$. For example, this applies to the following important cases:

Remark 1.3. Let $\mathfrak{C}$ be the category of toric varieties, and let X be a toric variety. Then following subclasses of $\mathrm{Mor}(X, ?)$ are admissible:

(i) The class consisting of all toric morphisms of $\mathrm{Mor}(X, ?)$ that are constant on the orbits of the action of a given subtorus $H \subset T_X$.

(ii) The class consisiting of all toric morphisms $X \to X'$ to quasiprojective toric varieties X'.

(iii) The class consisting of all toric morphisms $X \to X'$ to divisorial toric varieties X'.

The respective notions of quotients arising from the above classes, are discussed in detail in Sections 2, 4 and 5. As announced, the main result of this section is a general existence statement:

Theorem 1.4. *Let $\mathfrak{C}$ be the category of toric varieties, and let X be a toric variety. Then every admissible subclass of* $\mathrm{Mor}(X, ?)$ *has a quotient.*

Proof. By Property 1.2 (i), we may assume that X arises from a fan Δ in a lattice N and that all toric morphisms in question arise from maps of fans. The proof then consists of three reduction steps. In the first one, we establish the following decomposition property of morphisms of $\mathfrak{M}$:

Let $f\colon X \to X'$ be an element of $\mathfrak{M}$. Then there is a commutative diagram of toric morphisms:

(1.4.1)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f''"'] & & X' \\ & X'' \arrow[ur, "f'"'] & \end{tikzcd}$$

where $f''\colon X \to X''$ is an element of $\mathfrak{M}$ whose associated lattice homomorphism $F''\colon N \to N''$ is surjective.

To verify this property, we first reduce to the case that f is dominant. Property 1.2 (ii) applied to $f_1 := f_2 := f$ yields a factorization of f through a dominant morphism f_{12} belonging to $\mathfrak{M}$. Replacing f with f_{12} we may hence assume that f is dominant. In terms of the lattice homomorphism $F\colon N \to N'$ associated to f, this means that the image $N'' := F(N)$ is a sublattice of finite index in N'.

Now consider the fans Δ in N and Δ' in N' of the toric varieties X and X' respectively. Let Δ'' denote the fan in N'' having the same cones in $N''_{\mathbb{Q}} = N'_{\mathbb{Q}}$ as Δ', and let $F''\colon \Delta \to \Delta''$ be the canonical map of fans. The inclusion $N'' \subset N'$ induces a map of fans $F'\colon \Delta'' \to \Delta'$, and the toric morphisms $f'\colon X'' \to X'$ and $f''\colon X \to X''$ associated to F' and F'' respectively satisfy $f = f' \circ f''$. Moreover, since f' is finite, Property 1.2 (ii) tells us that f'' belongs to $\mathfrak{M}$. This proves the decomposition 1.4.1.

Because of 1.4.1, we can choose a toric morphism $q\colon X \to Z$ in $\mathfrak{M}$ corresponding to a map of fans $Q\colon \Delta \to \Sigma$ with the following properties:

- $q(X)$ is of maximal dimension among all images of elements of $\mathfrak{M}$,
- the underlying lattice homomorphism $Q\colon N \to L$ is surjective.

Our second claim is that every morphism $f\colon X \to X'$ of $\mathfrak{M}$ factors through a morphism g of $\mathfrak{M}$ whose associated lattice homomorphism is $Q\colon N \to L$.

To check this, apply Property 1.2 (ii) to the above $q\colon X \to Z$ and a given $f\colon X \to X'$ of $\mathfrak{M}$. Then the resulting toric morphisms determine a commutative diagram of lattice homomorphisms:

$$\begin{tikzcd} & N \arrow[dl, "Q"'] \arrow[dr, "F"] \arrow[dd, "F''"] & \\ L & & N' \\ & N'' \arrow[ul, "Q'"] \arrow[ur, "F'"'] & \end{tikzcd}$$

Since the toric morphism $f''\colon X \to X''$ of $\mathfrak{M}$ associated to F'' is dominant, and $L_{\mathbb{Q}}$ is of maximal dimension, we obtain $\dim(N''_{\mathbb{Q}}) = \dim(L_{\mathbb{Q}})$. Consequently, Q' is injective. Since Q is surjective, we even have that Q' is an isomorphism of lattices.

So $Q'\colon N'' \to L$ also defines an isomorphism of fans between the fan Δ'' in N'' associated to X'' and the fan Δ''' in L consisting of the cones $Q'(\tau)$, $\tau \in \Delta''$. Let X''' denote the toric variety associated to Δ''' and $g\colon X \to X'''$ the toric morphism induced by $Q' \circ F''$. By construction f factors through g, and since f'' belongs to $\mathfrak{M}$, by Property 1.2 (i) the same is true for g. This proves our second claim.

Now we are left with analyzing elements of $\mathfrak{M}$ whose associated lattice homomorphism is $Q\colon N \to L$. As before, let Δ be the fan of X in N, and consider the finite set R of rays in L consisting of all onedimensional cones among $Q(\varrho)$, $\varrho \in \Delta^{(1)}$. Our third claim is the following decomposition property:

Given any fan Δ' in L such that Q defines a map of fans $\Delta \to \Delta'$ and the associated toric morphism $X \to X'$ belongs to $\mathfrak{M}$, then there is a commutative diagram of maps of fans

(1.4.2)
$$\begin{tikzcd} \Delta \arrow[rr, "Q"] \arrow[dr, "Q"'] & & \Delta' \\ & \Sigma' \arrow[ur, "\mathrm{id}_L"'] & \end{tikzcd}$$

where the rays of the fan Σ' all belong to R and the toric morphism $X \to Z'$ associated to $\Delta \to \Sigma'$ is a member of $\mathfrak{M}$.

To verify this claim, we have to construct the fan Σ'. This is done as follows: For every maximal cone $\tau' \in \Delta'$ define a cone

$$\sigma' := \mathrm{conv}(\varrho;\ \varrho \in R,\ \varrho \subset \tau').$$

The union over all faces of these cones is a fan Σ' in L. Clearly the rays of Σ' are elements of R. Moreover, the identity of L defines a map of fans $\Sigma' \to \Delta'$. The associated toric morphism $Z' \to X'$ is birational, and we have a commutative diagram of toric morphisms

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & X' \\ & Z' \arrow[ur] & \end{tikzcd}$$

Since for every maximal cone $\tau' \in \Delta'$ its associated maximal cone of Σ' satisfies $\tau' \cap |\Sigma'| = \sigma'$, the toric morphism $Z' \to X'$ is affine. So, by Property 1.2 (i), the toric morphism $X \to Z'$ belongs to $\mathfrak{M}$, and the desired decomposition property 1.4.2 is verified.

Now, since R is finite, there are only finitely many maps of fans $Q_i\colon \Delta \to \Sigma_i$ with the properties of the left diagonal map in 1.4.2, say $Q_1, \ldots, Q_r$. Let $q_i\colon X \to Z_i$ denote the associated toric morphisms. We can conclude from Claims 2 and 3, that every element of $\mathfrak{M}$ factors through one of these q_i.

The rest of the proof is easy. Applying recursively Property 1.2 (ii) to the toric morphisms $q_i\colon X \to Z_i$, we obtain a toric morphism $p\colon X \to Y$ in $\mathfrak{M}$ such that each q_i factors through p. So $p\colon X \to Y$ is the desired quotient. □

Using essentially the same arguments as above, one can prove an analogue of Theorem 1.4 for the nonseparated case. More precisely, the notion of an admissible class carries over to the category of toric prevarieties, and one has:

Theorem 1.5. *Let $\mathfrak{C}$ be the category of toric prevarieties, and let X be a toric prevariety. Then every admissible subclass of* $\mathrm{Mor}(X, ?)$ *has a quotient.*

Having shown existence of quotients already, there are mainly two problems arising this context. The first one is to find an explicit description of the quotient, since our proof for the existence is not constructive:

Problem 1.6. For important admissible classes $\mathfrak{M}$, find an explicit description of the quotient in terms of combinatorial data, or even a concrete algorithmic construction.

The second problem concerns extension to larger categories. For example, given an action of a subgroup $H \subset T_X$ on a toric variety X, we can view the class of all toric morphisms from X that are constant on H-orbits as a subclass of all morphisms from X to arbitrary varieties that are constant on H-orbits.

Considering such an extension of categories leads to the question, whether the quotient even satisfies the analogous universal property in the larger category:

Problem 1.7. Let X be a toric variety and $\mathfrak{M}$ a class of toric morphisms admitting a quotient $X \to Y$. Given a class $\mathfrak{M}'$ of morphisms from X in a larger category, so that $\mathfrak{M}$ is contained in $\mathfrak{M}'$, when is $X \to Y$ also a quotient for $\mathfrak{M}'$?

2. Construction of toric quotients

The starting point for all further constructions in Toric Invariant Theory is the construction of quotients for subgroup actions in the toric category. We begin with a discussion of the separated case, more precisely we consider the following type of quotient in the category $\mathfrak{C}$ of toric varieties:

Definition 2.1. Let X be a toric variety, and let $H \subset T_X$ be a closed subgroup. A *toric quotient* for the action of H on X is a quotient for the class of all H-invariant toric morphisms $X \to X'$ to toric varieties X'.

Note that by an *H-invariant* morphism we mean in this article a morphism that is constant on the orbits of H. In the sequel we refer to the induced action of a closed subgroup $H \subset T_X$ of the big torus of a toric (pre-)variety X for short as a *subgroup action* on X.

It follows from the general existence theorem 1.4 that subtorus actions always admit toric quotients, and this statement can easily be extended to arbitrary subgroup actions. The most remarkable feature of the toric quotient is that there is even an algorithm for its construction, compare [1, Theorems 1.4, 1.5 and 2.3]:

Theorem 2.2. *Every subgroup action on a toric variety admits a toric quotient and the toric quotient can be constructed algorithmically from combinatorial data.*

We now describe this explicit construction of toric quotients in terms of fans. So, let X arise from a fan Δ in a lattice N. We first need to describe the closed subgroups of the big torus $T_X \subset X$, compare [12, III.8.2]:

Remark 2.3. Let $M := \operatorname{Hom}(N, \mathbb{Z})$. Then $T_X = \operatorname{Spec}(\mathbb{K}[M])$, and there is a one-to-one correspondence

$$\begin{array}{rcl} \{\text{sublattices of } M\} & \to & \{\text{closed subgroups of } T_X\} \\ K & \mapsto & \displaystyle\bigcap_{u \in K} \ker(\chi^u). \end{array}$$

Here the sublattices of full rank correspond to finite subgroups, and the primitive sublattices correspond to subtori.

Now, fix a sublattice $K \subset M$, and let $H \subset T_X$ denote the corresponding subgroup. As we have to deal with H-invariant toric morphisms $X \to X'$, we have to describe them by our combinatorial data:

Remark 2.4. Let Δ' be a fan in a lattice N'. A map $F\colon \Delta \to \Delta'$ of fans determines an H-invariant toric morphism $f\colon X \to X'$ if and only if the dual lattice homomorphism $F^*\colon M' \to M$ satisfies $F^*(M') \subset K$.

The first step in the construction of the toric quotient is to reduce to the case that H is a subtorus. For this, write H as a product H_0H_1 with a torus H_0 and a finite group H_1. Dividing X by the finite group H_1 is easy:

Remark 2.5. Let $K_1 \subset M$ denote the sublattice corresponding to the finite subgroup $H_1 \subset T_X$. Then one has:

(i) The inclusion $K_1 \subset M$ induces a lattice homomorphism $P_1\colon N \to N_1$ of full rank, and the cones $P_1(\sigma)$, $\sigma \in \Delta$ form a fan Δ_1 in N_1.
(ii) The toric morphism $p_1\colon X \to X_1$ associated to the map of fans $P_1\colon \Delta \to \Delta_1$ is a toric quotient for the action of H_1 on X.

Thus, replacing X with the above X_1 and $H \subset T_X$ with $p_1(H) \subset T_{X_1}$, we can assume in the sequel that H is a subtorus. Let $L \subset N$ be the sublattice consisting of all $v \in N$ with $\langle K, v\rangle = 0$. The translation of the defining universal property of the toric quotient into the language of fans is the following:

Remark 2.6. A map of fans $P\colon \Delta \to \Sigma$ defines a toric quotient for the action of H on X if and only if $L \subset \ker(P)$ and for any further map $F\colon \Delta \to \Delta'$ of fans with $L \subset \ker(F)$, there is a unique map $F'\colon \Sigma \to \Delta'$ with $F = F' \circ P$.

Given a map of fans $P\colon \Delta \to \Sigma$ with the universal property of the above remark, we call Σ the *quotient fan* of Δ by the sublattice $L \subset N$. This quotient fan can be constructed by the following procedure, see [1, proof of Theorem 1.5]:

Algorithm 2.7. Let Δ be a fan in a lattice N, and let $L \subset N$ be a primitive sublattice.

Initialization: Set $N_1 := N/L$, let $P_1\colon N \to N_1$ be the projection, and let S consist of all set-theoretically maximal cones among the $P_1(\sigma)$, $\sigma \in \Delta$.

Loop: While there are $\tau_1, \tau_2 \in S$ such that $\tau_1 \cap \tau_2$ is not a face of τ_2 do the following:

- let ϱ_2 denote the minimal face of τ_2 containing $\tau_1 \cap \tau_2$,
- replace S with the set of maximal cones of $S \cup \{\tau_1 + \varrho_2\}$.

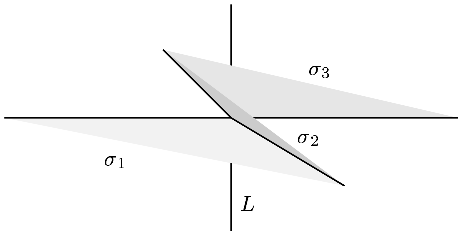
Strictification: Let $L_1 \subset N_1$ be the maximal sublattice contained in all cones of S, set $N_2 := N_1/L_1$, and let $P_2\colon N_1 \to N_2$ be the projection.

Output: The fan Σ having $P_2(\tau)$, $\tau \in S$, as its maximal cones and the map $P := P_2 \circ P_1$ of the fans Δ and Σ.

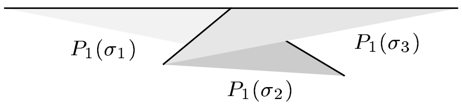
Let us demonstrate by means of a concrete example how the above algorithm works:

Example 2.8. Consider the lattice $\mathbb{Z}^3$, the sublattice $L := \mathbb{Z}e_3$ generated by the third canonical basis vector, and the fan Δ in $\mathbb{Z}^3$ having the following maximal cones

$$\sigma_1 := \operatorname{cone}(-e_1, e_1 - e_2), \qquad \sigma_2 := \operatorname{cone}(e_1 - e_2, e_3 - e_1 - e_2),$$
$$\sigma_3 := \operatorname{cone}(e_3 - e_1 - e_2, e_1).$$



Then, in the initialization, $P_1 \colon \mathbb{Z}^3 \to \mathbb{Z}^2$ is the projection onto the first two coordinates. After projecting the maximal cones, we are in the following situation:



Thus, running through the loop, we obtain a single cone, namely the lower half plane. So after strictification we end up with a single ray in $\mathbb{Z}$ which is the unique maximal cone of the quotient fan.

In several situations it is useful to consider also the nonseparated analogue of the toric quotient, since this may have better properties, see Example 2.10 given below. The precise notion is the following quotient in the category $\mathfrak{C}$ of toric prevarieties:

Definition 2.9. Let X be a toric prevariety, and let $H \subset T_X$ be a closed subgroup. A *toric prequotient* for the action of H on X is a quotient for the class $\mathfrak{M}$ of all H-invariant toric morphisms $X \to X'$ to toric prevarieties X'.

Example 2.10. Consider the toric variety $X := \mathbb{K}^2 \setminus \{0\}$, and the onedimensional subtorus H of the big torus $T_X = (\mathbb{K}^*)^2$ given by

$$H := \{(t, t^{-1});\ t \in \mathbb{K}^*\}.$$

The action of H is free, i.e., each orbit map $t \mapsto t{\cdot}x$ is a closed embedding. The toric quotient of this action is the following morphism:

$$X \to \mathbb{K}, \qquad (z, w) \mapsto zw.$$

To obtain a toric prequotient, let $Y := \mathbb{K} \cup_{\mathbb{K}^*} \mathbb{K}$ denote the affine line with doubled zero. Then the maps

$$X_z \to \mathbb{K},\ (z,w) \mapsto zw \quad \text{and} \quad X_w \to \mathbb{K},\ (z,w) \mapsto zw$$

glue together to the desired toric prequotient $X \to Y$. In contrast to the toric quotient, this map is affine and separates all the H-orbits.

Similar to the separated situation, we have the following result for toric prequotients, compare [4, Theorem 7.3]:

Theorem 2.11. *Every subgroup action on a toric prevariety admits a toric prequotient and the toric prequotient can be constructed algorithmically from combinatorial data.*

Let us sketch this algorithmic construction of toric prequotients. The reduction to the case of a subtorus action, that is dividing by finite subgroups of the big torus, is completely analogous to Remark 2.5.

For the construction of a toric prequotient of a toric prevariety X by a subtorus $H \subset T_X$, let X arise from a system of fans $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ in a lattice N and let $L \subset N$ be the primitive sublattice corresponding to $H \subset T_X$.

Remark 2.12. A map $(P, \mathfrak{p})\colon \mathcal{S} \to \mathcal{T}$ of systems of fans defines a toric prequotient if and only if $L \subset \ker(P)$ and for any further map $(F, \mathfrak{f})\colon \mathcal{S} \to \mathcal{S}'$ of systems of fans with $L \subset \ker(F)$, there is a unique map of systems of fans $(F', \mathfrak{f}')\colon \mathcal{T} \to \mathcal{S}'$ such that $(F, \mathfrak{f}) = (F', \mathfrak{f}') \circ (P, \mathfrak{p})$.

To construct a map of systems of fans with the properties of the above remark, one can assume that the system $\mathcal{S}$ of fans is *affine*, that means every fan Δ_{ii} is the fan of faces of a single cone $\sigma(i)$. Then the construction is done by the following procedure, see [4, proof of Theorem 7.5]:

Algorithm 2.13. Let $\mathcal{S}$ be an affine system of fans in a lattice N, and let $L \subset N$ be a primitive sublattice.

Initialization: Set $N_1 := N/L$ and let $P_1\colon N \to N_1$ be the projection. For $i \in I$ set $\tau(i) := P_1(\sigma(i))$. For $i, j \in I$ let S_{ij} be the set of faces of the cones $P_1(\varrho)$, $\varrho \in \Delta_{ij}^{\max}$.

Loop 1: While there are $i, j \in I$, $\varrho \in S_{ij}^{\max}$, where "max" refers to the face relation, with $\varrho \not\preccurlyeq \tau(i)$ do the following:

- Let ϱ_i denote the face of $\tau(i)$ with $\varrho^\circ \subset \varrho_i^\circ$.
- Replace $\tau(j)$ with $\operatorname{conv}(\tau(j) \cup \varrho_i)$ and replace S_{jj} with the set of faces of $\operatorname{conv}(\tau(j) \cup \varrho_i)$.
- Remove $\{\varrho';\ \varrho' \prec \varrho\}$ from S_{ij} and S_{ji} and instead add $\{\varrho';\ \varrho' \prec \varrho_i\}$ to both sets.

Loop 2: While there are $i, j, k \in I$ and $\varrho \in S_{ij} \cap S_{jk}$ such that $\varrho \notin S_{ik}$, replace S_{ki} and S_{ik} with $S_{ik} \cup \{\varrho'; \varrho' \prec \varrho\}$.

Strictification: Let $L_1 \subset N_1$ be the maximal sublattice of the cones $\tau(i)$, $i \in I$, let $N_2 := N_1/L_1$, and let $P_2\colon N_1 \to N_2$ be the projection.

Output: The system of fans $\mathcal{T} = (\Sigma_{ij})_{i,j\in I}$, where $\Sigma_{ij} := \{P_2(\tau);\ \tau \in S_{ij}\}$, and the canonical map $(P, \mathfrak{p})\colon \mathcal{S} \to \mathcal{T}$ of systems of fans, where $P = P_2 \circ P_1$.

Finally, let us observe that for a given subgroup action on a toric variety the toric quotient and the toric prequotient are in a natural way related to each other. The connecting notion is the following quotient in the category of toric prevarieties:

Definition 2.14. A *toric separation* of a toric prevariety X is a quotient for the class of toric morphisms $X \to X'$ to toric varieties X'.

Note that this type of quotient is an example for a reduction to a full subcategory. We have the following result, see [4, Theorem 4.1]:

Theorem 2.15. *Every toric prevariety admits a toric separation, and the toric separation can be constructed algorithmically from the combinatorial data.*

Basically the construction here is an application of Algorithm 2.7 to the set of maximal cones of a system of fans. As mentioned, the toric separation is the link between toric prequotients and toric quotients:

Remark 2.16. Let X be a toric variety, and let $H \subset T_X$ be a closed subgroup. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Z \\ & Y \arrow[ur] & \end{tikzcd}$$

where $X \to Y$ is the toric prequotient, $X \to Z$ is the toric quotient, and $Y \to Z$ is the toric separation.

3. Good and geometric quotients

In this section, we discuss classical quotients in the context of subgroup actions on toric varieties. First we consider good quotients. This concept appears implicitly in Mumford's book on Geometric Invariant Theory [26], and was later explicitly formulated by Seshadri [29]:

Definition 3.1. Let a reductive group G act morphically on a variety X. A *good quotient* for this action is an affine G-invariant morphism $p\colon X \to Y$ such that $p^*\colon \mathcal{O}_Y \to p_*(\mathcal{O}_X)^G$ is an isomorphism.

Good quotients also fit into the framework of our general Definition 1.1: A good quotient is a quotient for the class of invariant affine morphisms and vice versa. This follows from the criterion [27, Proposition 3.12] and the well known properties of good quotients listed below:

Proposition 3.2. *Let $p\colon X \to Y$ be a good quotient for a G-variety X. Then we have:*

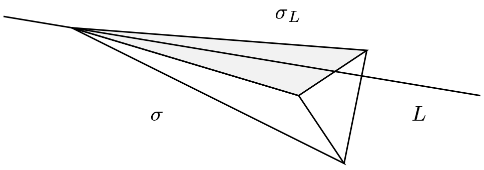
(i) *If A, B are disjoint closed G-invariant subsets of X, then their images $p(A)$, $p(B)$ are disjoint closed subsets of Y.*
(ii) *Y carries the quotient topology with respect to $p\colon X \to Y$.*
(iii) *Every fibre $p^{-1}(y)$ contains precisely one closed G-orbit and this orbit is in the closure of any further G-orbit in $p^{-1}(y)$.*

As good quotients are locally modeled on the affine case, we begin our discussion with affine toric varieties. So, let σ be a cone in a lattice N, and let K be a sublattice of the dual lattice $M := \mathrm{Hom}(N, \mathbb{Z})$. Denote by X the affine toric variety defined by σ, and let $H \subset T_X$ be the subgroup corresponding to $K \subset M$.

Remark 3.3. (i) The algebra of invariants $\mathcal{O}(X)^H$ of the action $H \times X \to X$ is the semigroup algebra $\mathbb{K}[K \cap \sigma^\vee]$.
(ii) The toric morphism $X \to Y$ corresponding to the inclusion $\mathbb{K}[K \cap \sigma^\vee] \subset \mathbb{K}[M \cap \sigma^\vee]$ is a good quotient for the action of H on X.

The explicit calculation of the *Hilbert basis*, i.e., the minimal system of generators of $K \cap \sigma^\vee$ is an interesting interface between Toric Invariant Theory and Integer Programming, see [31, Section 1.4]. There are still open problems; for example, finding degree bounds for the monomomials corresponding to the members of the Hilbert basis of $K \cap \sigma^\vee$ in the case $X = \mathbb{K}^n$, see [33] and [15].

Though Remark 3.3 already gives the good quotient, it is instructive also to construct it as a map of fans. For this, we assume that H is a torus, which due to Remark 2.5 is no loss of generality. Let $L \subset N$ be the sublattice consisting of all $v \in N$ with $\langle K, v \rangle = 0$, and let $\sigma_L \prec \sigma$ denote the minimal face with $\sigma \cap L \subset \sigma_L$.



Let $\widehat{L} := N \cap \mathrm{lin}(\sigma_L)$, and consider the projection $P\colon N \to N/\widehat{L}$. Then the image $\tau := P_{\mathbb{Q}}(\sigma)$ is a strictly convex cone in $N/\widehat{L}$. Note that passing from L to $\widehat{L}$ corresponds to the strictification step in Algorithm 2.7. Denoting by Y the affine toric variety associated to τ, we have:

Remark 3.4. The toric morphism $p\colon X \to Y$ associated to $P\colon N \to N/\widehat{L}$ is the good quotient for the action of H on X.

Now, let Δ be a fan in N, and let X be the associated toric variety. The action of a subtorus $H \subset T_X$ admits a good quotient if and only if the constructions of Remark 3.4 glue together in a saturated manner. More precisely, denoting by $L \subset N$ the primitive sublattice determined by $H \subset T_X$, we have, compare [32, Theorem 4.1] and [18, Theorem 4.7]:

Proposition 3.5. *In the above notation, the following statements are equivalent:*

(i) *The action of H on X admits a good quotient.*
(ii) *The toric quotient for the action of H on X is an affine morphism.*
(iii) *The output $P\colon \Delta \to \Sigma$ of Algorithm 2.7 induces a bijection between the respective sets of maximal cones.*
(iv) *Any two different maximal cones of Δ can be separated by an L-invariant linear form on N.*

Moreover, if one of these statements holds, then toric and good quotient coincide.

Here we say that a linear form u on the lattice N *separates* two cones σ_1, σ_2 in N if the restriction $u|_{\sigma_1}$ is nonnegative, $u|_{\sigma_2}$ is nonpositive and $\ker(u) \cap \sigma_i$ equals $\sigma_1 \cap \sigma_2$ for $i = 1, 2$.

We now discuss an important example of good quotients, namely a canonical presentation of a given toric variety as a good quotient space of a subgroup action on an open toric subvariety of $\mathbb{K}^n$:

Example 3.6. *Cox's Construction* [14]. Let Δ be a fan in a lattice N, and denote by $\Delta^{(1)}$ the set of its rays. Suppose that the primitive vectors v_ϱ of the rays $\varrho \in \Delta^{(1)}$ generate the vector space $N_{\mathbb{Q}}$. Consider the lattice homomorphism

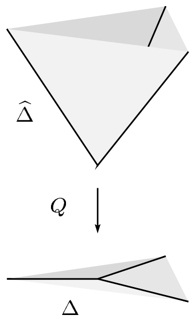
$$Q\colon \mathbb{Z}^{\Delta^{(1)}} \to N, \qquad e_\varrho \mapsto v_\varrho,$$

where for each $\varrho \in \Delta^{(1)}$, by $e_\varrho \in \mathbb{Z}^{\Delta^{(1)}}$ we mean the canonical basis vector corresponding to ϱ. Let $\Delta^{\max}$ be the set of maximal cones of Δ. For each cone $\sigma \in \Delta^{\max}$ set

$$\widehat{\sigma} := \mathrm{cone}(e_\varrho;\ \varrho \in \sigma^{(1)}).$$

The above cones generate a subfan $\widehat{\Delta}$ of the fan of faces of the positive quadrant in $\mathbb{Q}^{\Delta^{(1)}}$. The toric variety $\widehat{X}$ associated to $\widehat{\Delta}$ is an open subset of $\mathbb{C}^n$, and the toric morphism $q\colon \widehat{X} \to X$ determined by $Q\colon \mathbb{Z}^{\Delta^{(1)}} \to N$ is a good quotient for the action of the subgroup $\ker(Q) \subset T_{\widehat{X}}$ on $\widehat{X}$.

The picture below shows how Cox's construction looks like in the classical case when X is the projective plane. Note that here $\widehat{X}$ equals $\mathbb{K}^3 \setminus \{0\}$.



Of particular interest is the case when the good quotient even parametrizes all the orbits. Formally, this amounts to the following notion: Suppose that a reductive group G acts morphically on a variety X.

Definition 3.7. A *geometric quotient* for the action of G on X is a good quotient $p\colon X \to Y$ separating the G-orbits.

Note that for us a geometric quotient is by definition an affine morphism. This is not always required in the literature. However, if G acts properly on a variety X, i.e., the map $G \times X \to X \times X$ sending (g, x) to $(g \cdot x, x)$ is proper, then affineness is no extra requirement, see [26, Proposition 0.7]:

Proposition 3.8. *Let G act properly on a variety X. A G-invariant morphism $p\colon X \to Y$ is a geometric quotient if and only if it separates orbits and $p^*\colon \mathcal{O}_Y \to p_*(\mathcal{O}_X)^G$ is an isomorphism.*

We come back to subgroup actions on toric varieties. The very first examples of geometric quotients for subgroup actions are the toric quotients by finite subgroups presented in Remark 2.5.

To treat subtorus actions, let the toric variety X arise from a fan Δ in a lattice N. Suppose that $H \subset T_X$ is a subtorus, and denote by $L \subset N$ the associated sublattice. Let $P\colon N \to N/L$ be the projection. Then we have, see [18, Theorem 5.6]:

Proposition 3.9. *In the above notation, the following two statements are equivalent:*

- (i) *The action of H on X admits a geometric quotient.*
- (ii) *The projected cones $P(\sigma)$, $\sigma \in \Delta$, form a fan in N/L, and the restriction $P\colon |\Delta| \to |\Sigma|$ is injective.*

Moreover, if one of these statements holds, then $P\colon \Delta \to \Sigma$ defines a geometric quotient.

In view of Cox's construction 3.6, where every toric variety is presented as a good quotient space of a subgroup action on some open toric subset of $\mathbb{K}^n$, one can ask if there is something similar involving geometric quotients. In fact there is: If a toric variety has *enough invariant effective Cartier divisors* in the sense of [21], then it occurs as a geometric quotient space of a quasiaffine toric variety.

Example 3.10. *Kajiwara's Construction* [21]. Let Δ be a fan in lattice N, and let $\mathrm{SF}(\Delta)$ be the lattice of support functions on Δ, i.e., the kernel of the lattice homomorphism

$$\bigoplus_{i\in I} M/(M\cap\sigma_i^{\perp}) \to \bigoplus_{(i,j)\in I^2} M/(M\cap(\sigma_i\cap\sigma_j)^{\perp}), \qquad (u_i)_{i\in I}\mapsto(u_i-u_j)_{(i,j)\in I^2},$$

where the set I indexes the maximal cones of Δ. Suppose that for every maximal cone σ there is a support function $h_\sigma=(u_i)_{i\in I}\in\mathrm{SF}(\Delta)$ such that for every $i\in I$ we have

$$u_i|_{\sigma_i}\geq 0, \qquad \ker(u_i)\cap\sigma_i=\sigma\cap\sigma_i.$$

Consider the canonical inclusion $M\to\mathrm{SF}(\Delta)$, and let $Q\colon\widehat{N}\to N$ be the dual lattice homomorphism. Let $\widehat{\sigma}\subset\widehat{N}_{\mathbb{Q}}$ denote the dual of the cone in $\mathrm{SF}(\Delta)$ generated by the nonnegative support functions. Then $\widehat{\sigma}$ is strictly convex, and the above support functions h_σ cut out the maximal cones of a subfan $\widehat{\Delta}$ of the fan of faces of $\widehat{\sigma}$.

Note that the toric variety $\widehat{X}$ associated to $\widehat{\Delta}$ is quasiaffine. The lattice homomorphism $Q\colon\widehat{N}\to N$ is a map of the fans $\widehat{\Delta}$ and Δ. The toric morphism $q\colon\widehat{X}\to X$ determined by $Q\colon\widehat{N}\to N$ is a geometric quotient for the action of the subgroup $\ker(q)\subset T_{\widehat{X}}$ on $\widehat{X}$.

Note that for projective toric varieties, equivariant affine cones provide further presentations as quotients. For a unifying treatment of the various quotient presentations of toric varieties in the spirit of Examples 3.6 and 3.10 etc., we refer the reader to [8].

We conclude this section with a few words about the nonseparated setting. As the basic Example 2.10 shows, it may happen that an action on a variety admits neither a geometric nor a good quotient, but that there is a nonseparated substitute. The precise definitions are:

Definition 3.11. Let a reductive group G act morphically on a prevariety X.

(i) A *good prequotient* for the G-action is an affine G-invariant morphism $p\colon X\to Y$ of prevarietes such that $p^*\colon\mathcal{O}_Y\to p_*(\mathcal{O}_X)^G$ is an isomorphism.
(ii) A *geometric prequotient* for the G-action is a good quotient $p\colon X\to Y$ separating the G-orbits.

By definition, every good quotient is also a good prequotient. Moreover, good prequotients also have the properties of Proposition 3.2. Concerning geometric prequotients of torus actions, there is a basic characterization of existence due to Sumihiro [30, Corollary 3]:

Proposition 3.12. *For an algebraic torus action $T\times X\to X$ an a normal prevariety X the following statements are equivalent:*

(i) *Every orbit $T\cdot x$, $x\in X$, is closed in X.*
(ii) *There exists a geometric prequotient $X\to X/T$.*

Now, let X be the toric prevariety arising from an affine system of fans $\mathcal{S}=(\Delta_{ij})_{i,j\in I}$ in a lattice N. Recall that here affine means that each Δ_{ii} is the fan of faces of a single cone $\sigma(i)$. Fix a subtorus $H\subset T_X$, and let $L\subset N$ be the corresponding sublattice. We obtain, see [4, Theorem 6.7]:

Proposition 3.13. *In the above notation, the following statements are equivalent:*

(i) *The action of H on X admits a good prequotient.*
(ii) *Each $\tau \in \Delta_{ij}^{\max}$ is cut out from $\sigma(i)$ by an L-invariant linear form on N.*

4. Quasiprojective quotients

Let us now consider the action of a subgroup $H \subset T_X$ on a quasiprojective toric variety X. Our goal is to characterize when this action admits a quotient in the category of quasiprojective varieties. More precisely, we are interested in the following type of quotient:

Definition 4.1. Let $\mathfrak{C}$ be the category of quasiprojective varieties. A *quasiprojective quotient* for the action of H on X is a quotient for the class of all H-invariant morphisms $X \to Z$ to quasiprojective varieties Z.

Following our general philosophy, we first look for an analogous object in the toric setting, and then take this as a candidate for the general problem. This leads to the following notion of quotient:

Definition 4.2. Let $\mathfrak{C}$ be the category of quasiprojective toric varieties. A *toric quasiprojective quotient* for the action of H on X is a quotient for the class of all H-invariant toric morphisms $X \to X'$ to quasiprojective toric varieties X'.

Proposition 4.3. *Every subgroup action on a quasiprojective toric variety admits a toric quasiprojective quotient.*

Proof. Let X be a quasiprojective toric variety, and let $H \subset T_X$ be a subgroup. Write $H = H_0 H_1$ with a torus H_0 and finite group H_1. Then the class $\mathfrak{M}_0$ of all H_0-invariant toric morphisms $X \to X'$ to quasiprojective toric varieties is admissible in the sense of Definition 1.2.

According to Theorem 1.4, there is a quotient $p_0 \colon X \to Y_0$ for the class $\mathfrak{M}_0$. Consider the finite subgroup $p_0(H_1) \subset T_{Y_0}$. This action admits a geometric quotient $Y_0 \to Y$ with a quasiprojective toric variety Y. The composition $X \to Y_0 \to Y$ is the desired toric quasiprojective quotient for the action of H on X. □

The following result characterizes existence of quasiprojective quotients, compare [2, Theorem 2]:

Theorem 4.4. *For a subgroup action $H \subset T_X$ on a quasiprojective toric variety X the following statements are equivalent:*

(i) *There exists a quasiprojective quotient $p \colon X \to Y$ for the action of H on X.*
(ii) *The toric quasiprojective quotient $p' \colon X \to Y'$ is surjective.*

If one of these statements holds, then the toric quasiprojective quotient $p' \colon X \to Y'$ is also a quasiprojective quotient.

The crucial step in the proof of this statement is a decomposition result for H-invariant morphisms $f \colon X \to Z$ to quasiprojective varieties Z: Each such f is of the form $f = h \circ g$ with an H-invariant dominant toric morphism $g \colon X \to X'$ to a projective toric variety X' and a rational map $h \colon X' \to Z$ which is defined near the image $g(X)$, see [2, Proposition 2.2].

In order to decide whether or not a given subtorus action on a quasiprojective toric variety has a quasiprojective quotient, we are left with the problem of constructing the toric quasiprojective quotient. The first naïve idea is to take the toric quotient for the action of H on X constructed in Section 2. But the toric quotient space is in general no longer quasiprojective:

Example 4.5. Let X be any complete nonprojective toric variety like [28, p. 84], and consider Cox's construction $q\colon \widehat{X} \to X$, see Example 3.6. Then $\widehat{X}$ is quasiaffine and $q\colon \widehat{X} \to X$ is a good and hence toric quotient for the action of $\ker(q) \subset T_{\widehat{X}}$ on $\widehat{X}$.

To overcome this problem, we need a reduction. Formally, in the sense of Definition 1.1, this is the following type of quotient:

Definition 4.6. Let $\mathfrak{C}$ be the category of toric varieties. A *toric quasiprojective reduction* of a toric variety X is a quotient for the class $\mathfrak{M}$ of all toric morphisms $X \to X'$ to quasiprojective toric varieties X'.

Again the general existence theorem 1.4 tells us that toric quasiprojective reductions always exist. Moreover, from the respective universal properties one obtains for the action of a subgroup $H \subset T_X$ on a quasiprojective toric variety X:

Remark 4.7. The composition of the toric quotient $X \to Y''$ for the H-action and the toric quasiprojective reduction $Y'' \to Y'$ gives the toric quasiprojective quotient $X \to Y'$.

In the sequel, we sketch the construction of the toric quasiprojective reduction presented in [2, Section 1]. The main idea is to use the characterization of projectivity in terms of strictly convex support functions. However, since we also consider noncomplete toric varieties we work with a more adapted notion.

Suppose that the toric variety X is defined by a fan Δ in a lattice N. We say that a finite family $\mathfrak{U} := (u_i)_{i \in I}$ of linear forms $u_i \in M := \operatorname{Hom}(N, \mathbb{Z})$ is Δ-*convex*, if for every $\sigma \in \Delta^{\max}$ there is an $i(\sigma)$ such that

$$u_{i(\sigma)}|_\sigma \ \leq \ u_i|_\sigma \quad \text{for all } i \in I.$$

Note that the linear forms $u_{i(\sigma)}$ fit together to a piecewise linear function on the support $|\Delta|$. Examples of Δ-convex families arise from toric morphisms to projective toric varieties:

Remark 4.8. Let the map of fans $F\colon \Delta \to \Delta'$ define a toric morphism $f\colon X \to X'$ to a projective toric variety X'. Then any choice of a strictly convex support function h' on Δ' gives rise to a Δ-convex family with $u_{i(\sigma)} := h' \circ F|_\sigma$ for $\sigma \in \Delta^{\max}$.

Every Δ-convex family $\mathfrak{U}$ defines a *quasifan* in N, that means a finite set of convex cones satisfying all the axioms of a fan apart from being strictly convex: Consider the convex hull $P_{\mathfrak{U}} \subset M_{\mathbb{Q}}$. Then each face P' of the polytope $P_{\mathfrak{U}}$ defines a convex cone

$$\tau_{P'} := \{v \in N_{\mathbb{R}};\ p'(v) \leq p(v) \text{ for all } (p', p) \in P' \times P_{\mathfrak{U}}\}$$

in the lattice N, and these cones form a quasifan $\mathfrak{S}_{\mathfrak{U}}$ in N. Call a subset $R \subset \Delta^{(1)}$ of the rays of Δ *indecomposable*, if for every Δ-convex family $\mathfrak{U}$ the set R is contained in some maximal cone of $\mathfrak{S}_{\mathfrak{U}}$. The main observation is, see [2, Remark 1.2]:

Remark 4.9. There is a quasifan $\mathfrak{S}$ in N such that the maximal cones of $\mathfrak{S}$ are precisely the cones that are generated by the rays of Δ belonging to a maximal indecomposable family.

This quasifan $\mathfrak{S}$ gives us the candidate for the toric quasiprojective reduction. Let L denote the maximal sublattice contained in all cones of $\mathfrak{S}$. Denote by $Q\colon N \to N/L$ the projection and consider the fan $\Sigma := \{Q(\sigma);\ \sigma \in \mathfrak{S}\}$. Then we have, see [2, Proposition 1.3]:

Proposition 4.10. *The toric morphism $q\colon X \to Y$ associated to the map of fans $Q\colon \Delta \to \Sigma$ is the toric quasiprojective reduction of the toric variety X.*

Note that up to determining the maximal indecomposable subsets of the fan Δ, this Proposition is constructive. Hence, by Remark 4.7, the explicit construction of the toric quotient and the above consideration give us at least some recipe to decide whether or not there exists a quasiprojective quotient for a given subgroup action on a quasiprojective toric variety.

However, the reduction step used to stay in the quasiprojective category in general affects good properties of the toric quotient, see the examples discussed in [2, Section 3]. Thus it is desirable to have some a priori criteria ensuring that the toric quotient space of a subgroup action on a quasiprojective toric variety is again quasiprojective.

We conclude this section with a systematic discussion of the above problem in the special case of geometric quotients $p\colon X \to Y$ for subtorus actions $H \subset T_X$ on quasiaffine toric varieties X:

If H is onedimensional and the quotient space Y is complete, then the H-action defines a w.l.o.g. positive grading of the algebra $\mathcal{O}(X)$ and the quotient space is nothing but $\mathrm{Proj}(\mathcal{O}(X))$. As soon as we drop either of these two assumptions, the orbit space in general is no longer quasiprojective, see [6, Theorem 2]:

Theorem 4.11. (i) *There is a twodimensional subtorus action on a fivedimensional quasiaffine toric variety with a complete nonprojective geometric quotient space.*
(ii) *There is a onedimensional subtorus action on a fivedimensional quasiaffine toric variety with a nonquasiprojective orbit space.*

A case of particular interest are subsets of $\mathbb{K}^n$. If $X \subset \mathbb{K}^n$ is a toric open subset admitting a geometric quotient $q\colon X \to Y$ for an action of a subtorus $H \subset (\mathbb{K}^*)^n$, then the quotient space Y is a simplicial toric variety with at most $n - \dim(H)$ invariant prime divisors. We have:

Theorem 4.12. *Let $H \subset (\mathbb{K}^*)^n$ be a subtorus, and let $X \subset \mathbb{K}^n$ be a toric subset admitting a geometric quotient $X \to Y$. The following table indicates when Y is quasiprojective:*

	Y complete, H acts freely	*Y complete*	*Y arbitrary*
$\dim(H) = 1$	$+$	$+$	$+$, [11, Ex. 5A]
$\dim(H) = 2$	$+$, [23, Thm. 2]	$+$, [6, Prop. 2.1]	$-$, [11, Ex. 5B]
$\dim(H) = 3$	$+$, [24, Thm. 1]	$-$, [6, Prop. 1.2]	$-$
$\dim(H) = 4$	$-$, [28, p. 84]	$-$	$-$

Here the top row and the left column specify the assumptions on the action, "$+$" stands for "the quotient space Y is quasiprojective", and "$-$" means that Y is not necessarily quasiprojective.

5. Divisorial quotients

In this section, we present an analogue of Theorem 4.4 in the more general framework of divisorial varieties. Recall from [13] that an irreducible variety X

is *divisorial* if it admits an open cover of affine sets $X \setminus \operatorname{Supp}(D)$, where D is an effective Cartier divisor. For example, every $\mathbb{Q}$-factorial variety is divisorial.

The class of divisorial varieties was introduced by Borelli as a generalization of the class of quasiprojective varieties. The idea was to extend the concept of a single ample divisor to finite families of divisors. This allows to carry over many constructions from the quasiprojective to the divisorial case, see, e.g. [9, II.2].

Concerning quotients, divisorial varieties behave more naturally than quasiprojective ones. For geometric quotients of torus actions we obtain, see [5, Proposition 1.1]:

Proposition 5.1. *If an algebraic torus action $T \times X \to X$ on a normal divisorial variety X admits a geometric quotient $X \to Y$, then the quotient variety Y is divisorial.*

We turn to divisorial toric varieties. The first observation is that a toric variety is divisorial if it has enough effective invariant Cartier divisors in the sense of Kajiwara [21]. This is a consequence of the following more general fact, see [5, Proposition 1.1]:

Proposition 5.2. *Let the algebraic torus T act morphically on a normal variety X. If X is divisorial, then there exist T-invariant Cartier divisors $D_1, \ldots, D_r$ such that the sets $X \setminus \operatorname{Supp}(D_i)$ are affine and cover X.*

Together with Kajiwara's construction presented in Example 3.10, the above two Propositions give the following characterization of divisorial toric varieties, compare also the corresponding general statement [19, Theorem 3.1]:

Proposition 5.3. *A toric variety X is divisorial if and only if there is a quasiaffine toric variety $\widehat{X}$ and a subtorus action on $\widehat{X}$ with geometric quotient $\widehat{X} \to X$.*

Note that this statement has no analogue in the setting of quasiprojective toric varieties: Every quasiprojective toric variety is a geometric quotient space of a quasiaffine toric variety by a onedimensional subtorus action, but according to Theorem 4.11 (ii), the converse does not hold.

Let us now consider the action of a closed subgroup $H \subset T_X$ on a divisorial toric variety X. Analogous to the previous section, we consider the following two types of quotients:

Definition 5.4. (i) Let $\mathfrak{C}$ be the category of divisorial varieties. A *divisorial quotient* for the action of H on X is a quotient for the class $\mathfrak{M}$ of all H-invariant morphisms $X \to Z$ to divisorial varieties Z.

(ii) Let $\mathfrak{C}$ be the category of divisorial toric varieties. A *toric divisorial quotient* for the action of H on X is a quotient for the class $\mathfrak{M}^t$ of all H-invariant toric morphisms $X \to X'$ to divisorial toric varieties X'.

The existence of the second type of quotient is established in a similar way as for the corresponding quasiprojective notion:

Proposition 5.5. *Every subgroup action on a divisorial toric variety admits a toric divisorial quotient.*

Proof. Write $H = H_0 H_1$ with a torus H_0 and a finite group H_1. Then the class of H_0-invariant toric morphisms $X \to X'$ is admissible in the sense of Definition 1.2 and hence admits a quotient $p\colon X \to Y_0$, with a divisorial toric variety Y_0.

Consider Kajiwara's construction $q\colon \widehat{Y}_0 \to Y_0$ as in Example 3.10, and choose a finite group $\widehat{H}_1 \subset T_{\widehat{Y}_0}$ with $q(\widehat{H}_1) = p(H_1)$. Then there is a commutative diagram of geometric quotients

$$\begin{tikzcd} \widehat{Y}_0 \arrow[r] \arrow[d, "q"'] & \widehat{Y}_0/\widehat{H}_1 \arrow[d] \\ Y_0 \arrow[r] & Y_0/p(H_1) \end{tikzcd}$$

Since $\widehat{Y}_0/\widehat{H}_1$ is again quasiaffine, we can use Proposition 5.1, to conclude that $Y := Y_0/p(H_1)$ is a divisorial toric variety. Thus the composition $X \to Y_0 \to Y$ is the desired toric divisorial quotient. $\square$

The answer to the question when the toric divisorial quotient is also a quotient for the substantially larger class of all H-invariant morphisms $X \to Z$ to divisorial varieties Z is given by the following result, compare [5, Corollary 6.3]:

Theorem 5.6. *For a subgroup action $H \subset T_X$ on a divisorial toric variety X the following statements are equivalent:*

(i) *There exists a divisorial quotient $p\colon X \to Y$ for the action of H on X.*
(ii) *The toric divisorial quotient $p'\colon X \to Y'$ is surjective.*

If one of these statements holds, then the toric divisorial quotient $p'\colon X \to Y'$ is also a divisorial quotient.

As in the quasiprojective case the essential part of the proof is a decomposition result relating arbitrary H-invariant morphisms $X \to Z$ to H-invariant toric morphisms $X \to X'$, see [5, Lemma 5.1].

Lemma 5.7. *Let $f\colon X \to Z$ be an H-invariant morphism to a divisorial variety Z. Then there exists a dominant H-invariant toric morphism $g\colon X \to X'$ to a divisorial toric variety X', an open subset $U \subset X'$ with $g(X) \subset U$ and a morphism $h\colon U \to Z$ such that $f = h \circ g$.*

The divisorial decomposition lemma is deeper than its quasiprojective counterpart [2, Proposition 2.2]. There are two important ingredients for the proof. The first one is an embedding result which provides the link to the toric setting, compare [19, Theorem 3.2]:

Proposition 5.8. *Every divisorial variety admits a closed embedding into a smooth toric prevariety.*

The second ingredient is a lifting result, which reduces the study of morphisms to the quasiaffine case, compare [5, Lemma 4.1]:

Lemma 5.9. *Let $f\colon X_1 \to X_2$ be a regular map of divisorial toric prevarieties such that $f(X_1)$ intersects the big torus of X_2. Then there exists a commutative diagram*

$$\begin{tikzcd} \widehat{X}_1 \arrow[r, "\widehat{f}"] \arrow[d, "q_1"'] & \widehat{X}_2 \arrow[d, "q_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

where $\widehat{X}_1$, $\widehat{X}_2$ are quasiaffine toric varieties, $q_i\colon \widehat{X}_i \to X_i$ are geometric prequotients for free subtorus actions on $\widehat{X}_i$ and $\widehat{f}\colon \widehat{X}_1 \to \widehat{X}_2$ is a regular map.

These two results basically reduce the decomposition problem to the case of quasiaffine toric varieties. That case can be handled using certain facts on existence of good quotients. For the details of the proof we refer to [5].

In order to apply the criterion 5.6, it is important to have an explicit construction of the toric divisorial quotient. Given a subgroup $H \subset T_X$ acting on a divisorial toric variety X, this construction splits into two steps:

The first step is to determine the toric quotient $p\colon X \to Y$ as described in Section 2. In general, the toric quotient space Y is not divisorial:

Remark 5.10. The threedimensional complete toric variety Y presented in [16, Example 3.5] has trivial Picard group and hence is not divisorial. However, by Cox's construction 3.6, it is a good and hence toric quotient space of a quasiaffine toric variety.

So, in the second step one has to construct a *toric divisorial reduction*, that means a quotient $Y \to Z$ for the class of all toric morphisms $Y \to Y'$ to divisorial toric varieties Y'. By Theorem 1.4, such a reduction always exists.

Remark 5.11. The composition of the toric quotient $X \to Y$ and the toric divisorial reduction $Y \to Z$ is the toric divisorial quotient for the action of the subgroup $H \subset T_X$ on X.

Let us now summarize the explicit construction of the toric divisorial reduction in terms of combinatorial data presented in [5, Sections 2 and 3]. The main tool for this is a generalization of the concept of convex support functions on a fan:

A *support map* on a fan Δ is a map $h\colon |\Delta| \to \mathbb{Q}^k$ that is linear on every cone $\sigma \in \Delta$. For a support map $h\colon |\Delta| \to \mathbb{Q}^k$ on a fan Δ in a lattice N, let γ be the cone in $\widehat{N} := N \times \mathbb{Z}^k$ generated by the graph Γ_h of h in $N_{\mathbb{Q}} \times \mathbb{Q}^k$. Define Λ_h to be the set of cones in $\widehat{N}$ consisting of all those faces of γ that are contained in a face δ of γ whose relative interior δ° meets Γ_h:

$$\Lambda_h := \bigcup_{\delta \prec \gamma;\, \delta^\circ \cap \Gamma_h \neq \emptyset} \mathfrak{F}(\delta)\,.$$

Note that Λ_h is a quasifan, that means it satisfies all the axioms of a fan except that its elements may not be strictly convex.

The support map $h\colon |\Delta| \to \mathbb{Q}^k$ is called *convex*, if the projection $P\colon N_{\mathbb{Q}} \times \mathbb{Q}^k \to N_{\mathbb{Q}}$ is injective on the support $|\Lambda_h|$. If the support map $h\colon |\Delta| \to \mathbb{Q}^k$ is convex, then the projected cones form a quasifan

$$\Sigma_h := \{P(\delta);\ \delta \in \Lambda_h\}$$

in the lattice N. We call Σ_h the *quasifan associated to h*. By construction, every maximal cone of Σ_h is generated by cones of Δ.

We say that a convex support map $h\colon |\Delta| \to \mathbb{Q}^k$ is *strictly convex* if its associated quasifan Σ_h equals Δ. Using this notion we can characterize divisoriality, generalizing the corresponding well-known statement on projectivity of toric varieties and existence of strictly convex support functions:

Proposition 5.12. *Let Δ be a fan in a lattice N. Then the toric variety X associated to Δ is divisorial if and only if Δ admits a strictly convex support map.*

Every convex support map $h\colon |\Delta| \to \mathbb{Q}^k$ on a fan Δ in a lattice N defines in a canonical way a toric morphism from the associated toric variety X to a divisorial toric variety.

Namely, let Σ_h be the associated quasifan, denote by L the maximal sublattice contained in all the cones of Σ_h, and let $F_h \colon N \to N/L$ be the projection. Then the cones $F_h(\sigma)$, $\sigma \in \Sigma_h$ form a fan Δ_h in N/L. Thus F_h gives rise to a toric morphism $f_h \colon X \to X_h$.

These toric morphisms are crucial for the construction of the toric quasiprojective reduction. Their basic properties are:

- There are only finitely many toric morphisms $f_i \colon X \to X_i$ of the above type, say $f_1, \ldots, f_r$.
- Each toric variety X_i is divisorial.
- Every toric morphism $f' \colon X \to X'$ to a divisorial toric variety X' factors through one of the f_i

The first observation is clear since the maximal cones of the fans Σ_h are generated by cones of Δ. For the second property, see [5, Proposition 2.7], and the third is established in [5, Proof of Theorem 3.2].

Now, the further construction of the toric divisorial reduction goes along the lines of the proof of the general existence Theorem 1.4: Consider the product morphism $f := f_1 \times \ldots \times f_r$, and let Y denote the normalization of the closure of the image $f(X)$ in $Y_1 \times \cdots \times Y_r$. Then we have:

Proposition 5.13. *The canonical toric morphism $X \to Y$ is the toric divisorial reduction of X.*

6. Categorical Quotients: Some counterexamples

Let us now turn to the classical notion of a categorical quotient for an algebraic group action. In contrast to Mumford [26, Definition 0.5], we introduce a separated and a nonseparated version. The precise definitions are:

Definition 6.1. Let G denote any algebraic group.

(i) Let $\mathfrak{C}$ be the category of algebraic prevarieties, and let X be a prevariety with a morphical G-action. A *categorical prequotient* for the G-action is a quotient for the class of G-invariant morphisms $X \to X'$ to prevarieties X'.
(ii) Let $\mathfrak{C}$ be the category of algebraic varieties, and let X be a variety with a morphical G-action. A *categorical quotient* for the G-action is a quotient for the class of G-invariant morphisms $X \to X'$ to varieties X'.

As the basic example 2.10 shows, these two objects may differ for a given group action on a variety. Further examples are provided by the following observation, which is an easy consequence of Proposition 3.2:

Remark 6.2. Every good prequotient for a reductive group action is also a categorical prequotient, and every good quotient is a categorical quotient.

In general, the problem of existence of categorical (pre-)quotients is difficult. The purpose of this section is to show that existence of a categorical quotient and of its nonseparated analogue are independent from each other. To this end we give the following examples obtained by means of Toric Invariant Theory:

- A onedimensional subtorus action on a fourdimensional quasiaffine toric variety admitting a categorical prequotient but no categorical quotient, see Proposition 6.3.

- A onedimensional subtorus action on a fourdimensional quasiprojective toric variety admitting neither a categorical quotient nor a categorical prequotient, see Proposition 6.4.
- A onedimensional subtorus action on a threedimensional quasiprojective toric variety admitting a categorical quotient but no categorical prequotient, see Proposition 6.5.

These examples are taken from [3], where the reader also finds the full proofs for the respective statements. For convenience, at the end of this section we present an elementary proof for the first example.

We work over the field $\mathbb{C}$ of complex numbers. The first example is an open toric subvariety $X \subset \mathbb{C}^4$ together with a onedimensional subtorus $H \subset (\mathbb{C}^*)^4$, namely

$$X := \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \times \mathbb{C}^2, \qquad H := \{(t,t,1,t^{-1});\ t \in \mathbb{C}^*\}.$$

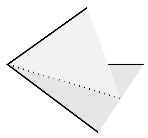
We first determine the toric prequotient and the toric quotient. For this we describe the situation in terms of fans: The toric variety X arises from the fan Δ in $\mathbb{Z}^4$ having the following two maximal cones:

$$\sigma_1 := \operatorname{cone}(e_1, e_2), \qquad \sigma_2 := \operatorname{cone}(e_3, e_4).$$

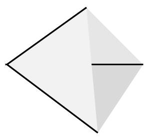
The subtorus H corresponds to the onedimensional sublattice L of $\mathbb{Z}^4$ generated by the vector $(1, 1, 0, -1)$. Thus, L is the kernel of the projection $P \colon \mathbb{Z}^4 \to \mathbb{Z}^3$ given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To obtain the toric quotient, we have to project the maximal cones σ_1 and σ_2 along L, and then to enter the loop of Algorithm 2.7. These two steps are shown in the following two pictures:



$P(\sigma_1)$, $P(\sigma_2)$



$P(\sigma_1) + P(\sigma_2)$

The first picture gives the system of fans $\widetilde{S}$ defining the toric prequotient space: $\widetilde{\Delta}_{ii}$ is the fan of faces of $P(\sigma_i)$, and $\widetilde{\Delta}_{12} = \widetilde{\Delta}_{21}$ is the trivial fan. By Proposition 3.13, the toric prequotient $p \colon X \to \widetilde{X}$ is even a good prequotient. This proves the first statement of the following:

Proposition 6.3. *In the above notations, we have for the action of H on X:*

(i) $p \colon X \to \widetilde{X}$ *is a categorical prequotient.*
(ii) *There is no categorical quotient.*
(iii) $\widetilde{X}$ *admits no separation.*

By the above picture, the toric quotient space is $\mathbb{C}^3$, and the toric quotient map is not surjective. This is crucial in the proof of ii), given at the end of this section. The third statement is a direct consequence of ii).

Our next example admits neither a categorical quotient nor a categorical prequotient. Let X denote the toric variety obtained by glueing the two affine charts

$X_1 = \mathbb{C}^4$ and $X_2 = \mathbb{C}^3 \times \mathbb{C}^*$ along the common subset $(\mathbb{C} \times \mathbb{C}^*)^2$, using the gluing map

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 z_2^2, z_2^{-1}, z_3, z_4).$$

On the toric variety X, we consider the action of the one dimensional subtorus H of $(\mathbb{C}^*)^4$ given by

$$H := \{(t^{-2}, 1, t, t);\ t \in \mathbb{C}^*\}.$$

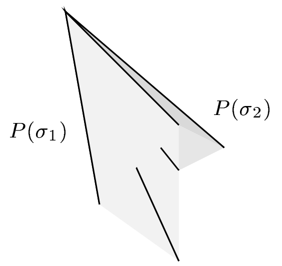
Let us again describe this setting in terms of convex geometry. The toric variety X arises from the fan Δ in $\mathbb{Z}^4$ having the maximal cones

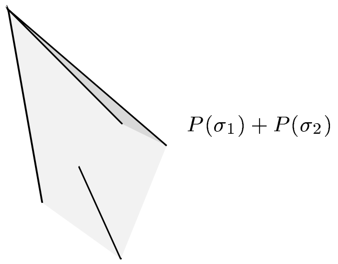
$$\sigma_1 := \mathrm{cone}(e_1, e_2, e_3, e_4), \qquad \sigma_2 := \mathrm{cone}(e_1, 2e_1 - e_2, e_3).$$

The subtorus H corresponds to the line generated by the vector $(-2, 0, 1, 1)$ and hence is the kernel of the projection $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Projecting the cones σ_1 and σ_2 along the line L, we obtain a constellation as in the left picture below. The algorithms for the construction of toric quotient and prequotient both lead to a single cone $\sigma := P(\sigma_1) + P(\sigma_2)$ in $\mathbb{Z}^3$ shown in the right picture:





So the induced toric morphism $p\colon X \to \widetilde{X}$ to the affine toric variety defined by σ is both the toric prequotient and the toric quotient for the action of H on X. Note that p is not surjective. This is crucial in the proof of the following:

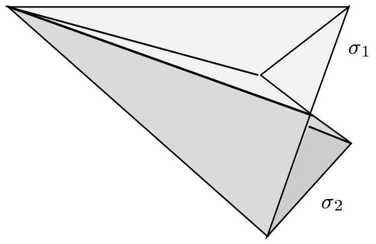
Proposition 6.4. *The action of H on X admits neither a categorical prequotient nor a categorical quotient.*

In the last example, we present a subtorus action admitting a categorical quotient but no categorical prequotient. We consider the toric variety X obtained from gluing two copies of $\mathbb{C}^3$ along the open subset $\mathbb{C} \times (\mathbb{C}^*)^2$ by the following map:

$$(z_1, z_2, z_3) \mapsto (z_1 z_2^2 z_3^2, z_2^{-1}, z_3^{-1}).$$

As the subtorus we take $H := \{(1, 1, t);\ t \in \mathbb{C}^*\}$. In terms of convex geometry, X is the toric variety arising from a fan Δ in $\mathbb{Z}^3$ with two maximal cones, namely

$$\sigma_1 := \operatorname{cone}(e_1, e_1 - e_2, e_1 + e_2 + e_3), \quad \sigma_2 := \operatorname{cone}(e_1, e_1 + e_2, e_1 - e_2 - e_3).$$



Let $P\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ denote the projection. Then P maps both cones, σ_1 and σ_2 onto the cone $\sigma := \operatorname{cone}(e_1 + e_2, e_1 - e_2)$ in $\mathbb{Z}^2$. The toric morphism $p\colon X \to \widetilde{X}$ to the affine toric variety $\widetilde{X}$ defined by σ turns out to be both the toric quotient and the toric prequotient.

Proposition 6.5. *The map $p\colon X \to \widetilde{X}$ is a categorical quotient for the action of H on X but this action admits no categorical prequotient.*

Finally, as announced we give the proof of non-existence of a categorical quotient in the first example. The proof is elementary; we can do without using the language of fans.

Proof of Proposition 6.3. Assume that there exists a categorical quotient $p\colon X \to Y$. Then it follows from the universal property of categorical quotients that p is surjective and that the canonical action of the torus $T := (\mathbb{C}^*)^4$ on X (denoted by $t\cdot x$) induces a (set theoretical) action of T on Y such that p is equivariant. Let

$$f\colon X \to \mathbb{C}^3, \qquad (x_1, x_2, x_3, x_4) \mapsto (x_1x_4, x_2x_4, x_3).$$

Then $f|_T\colon T \to (\mathbb{C}^*)^3$ is a surjective homomorphism of tori and induces a T-action on $\mathbb{C}^3$ making f equivariant. Moreover, f is constant on the orbits of the subtorus $H = \{(t, t, 1, t^{-1});\ t \in \mathbb{C}^*\} \subset T$ and an explicit calculation yields

$$f(X) = \mathbb{C}^3 \setminus (T\cdot(0,1,0) \cup T\cdot(1,0,0)).$$

In particular, $f(X)$ is not open in $\mathbb{C}^3$. Now, by the universal property of p, there is a morphism $\widetilde{f}\colon Y \to \mathbb{C}^3$ such that $f = \widetilde{f} \circ p$. We will show that $\widetilde{f}$ must be injective. Since Y is necessarily irreducible, this implies that $\widetilde{f}$ is an open map, hence $\widetilde{f}(Y) = f(X)$ is open in $\mathbb{C}^3$ and we arrive at a contradiction.

To obtain injectivity of $\widetilde{f}$, since p is surjective, it suffices to show that p is constant on the fibres of f. First consider the open T-stable subset

$$U_1 \;:=\; \{x \in X;\ x_4 \neq 0\} \;=\; X \setminus (T\cdot(1,1,1,0) \cup T\cdot(1,1,0,0)).$$

Then U_1 is also H-stable and the restriction $f|_{U_1}$ separates H-orbits. Hence p is constant on the fibres of $f|_{U_1}$. Next we consider the set

$$U \;:=\; U_1 \cup T\cdot(1,1,1,0) \;=\; X \setminus T\cdot(1,1,0,0).$$

Note that

$$f(T\cdot(1,1,1,0)) \;=\; T\cdot(0,0,1) \;=\; f(T\cdot(0,0,1,1)) \in f(U_1).$$

Hence, in order to prove that p is constant on the fibres of the restriction $f|_U$, it suffices by T-equivariance to show that p maps the set

$$f^{-1}(0,0,1) \cap T\cdot(1,1,1,0) \;=\; T_{(0,0,1)}\cdot(1,1,1,0)$$

to the point $p(0,0,1,1)$. So, let $t \in T_{(0,0,1)}$. Then t is of the form $(t_1, t_2, 1, t_4)$. Since p is T-equivariant and H-invariant, we see

$$\begin{aligned} p(t\cdot(1,1,1,0)) &= t\cdot p(1,1,1,0) \;=\; t\cdot \lim_{s\to 0} p(1,1,1,s) \\ &= t\cdot \lim_{s\to 0} p((s,s,1,s^{-1})\cdot(1,1,1,s)) \;=\; t\cdot \lim_{s\to 0} p(s,s,1,1) \\ &= t\cdot p(0,0,1,1) \;=\; p(0,0,1,t_4) \\ &= p((t_4^{-1}, t_4^{-1}, 1, t_4)\cdot(0,0,1,1)) \;=\; p(0,0,1,1). \end{aligned}$$

Thus p is even constant on the fibres of $f|_U$. Finally, we have to treat the set $T\cdot(1,1,0,0)$. Since

$$f(T\cdot(1,1,0,0)) = (0,0,0) \notin f(U),$$

we only need to prove that p is constant on $T\cdot(1,1,0,0)$. So far we know

$$\begin{aligned} \widetilde{f}^{-1}(\overline{T\cdot(0,0,1)}) &= p(f^{-1}(T\cdot(0,0,1))) \cup p(f^{-1}(0,0,0)) \\ &= p(T\cdot(1,1,1,0)) \cup p(T\cdot(1,1,0,0)), \end{aligned}$$

where the closure is taken with respect to the complex topology. In particular, the above set is analytic and contains $p(T\cdot(1,1,1,0))$ as an open subset. Since $T\cdot(1,1,0,0)$ lies in the closure of $T\cdot(1,1,1,0)$ and these sets are separated by f, we obtain in addition

$$p(T\cdot(1,1,0,0)) \;=\; \overline{p(T\cdot(1,1,1,0))} \setminus p(T\cdot(1,1,1,0)).$$

Now, $p(T\cdot(1,1,1,0)) = p(T\cdot(0,0,1,1))$ is of dimension one. Thus it follows that $p(T\cdot(1,1,0,0))$ is of dimension zero and hence it is a point. $\square$

7. Categorical quotients: Criteria for existence

After having studied counterexamples in the preceding section, we here give some positive results. We consider a subgroup action on a toric variety, and present existence criteria for a categorical quotient in terms of the toric quotient. For simplicity, we work over the field $\mathbb{C}$ of complex numbers.

The key to our results is the notion of a weakly proper morphism used in [3]. Let us recall the definition. By a local curve in a point x of a variety X we mean a holomorphic mapping germ $\gamma\colon \mathbb{C}_0 \to X_x$ arising from an algebraic curve, i.e., there is an algebraic curve X' in X through x with $\gamma(\mathbb{C}_0) \subset X'_x$.

Definition 7.1. We call a morphism of varieties $p\colon X \to Y$ *weakly proper* if for every local curve γ in $y \in Y$ there is a commutative diagram of holomorphic mapping germs

$$\begin{array}{ccc} \mathbb{C}_0 & \xrightarrow{\widetilde{\gamma}} & X_x \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C}_0 & \xrightarrow[\gamma]{} & Y_y \end{array}$$

with a local curve $\widetilde{\gamma}$ in $x \in X$, and a nonconstant holomorphic mapping germ $\alpha\colon \mathbb{C}_0 \to \mathbb{C}_0$.

A purely algebraic formulation similar to this definition was introduced by Kollàr [Ko], namely the so-called weak lifting property for discrete valuation rings. Note that weakly proper maps are necessarily surjective. In fact, an equivalent characterization of weakly properness is the following, see [7, Proposition 1.3]:

Remark 7.2. A surjective morphism $X \to Y$ of algebraic varieties is weakly proper if and only if it is universally submersive, i.e. if Y carries the quotient topology, and this property is preserved under base change, compare [26, Section 0].

The most important feature of weakly proper morphisms is that they are quotient morphisms in the following sense, compare [3, Proposition 1.1]:

Proposition 7.3. *Let X, X', Y be varieties, suppose that Y is normal and let $p\colon X \to Y$ be a weakly proper morphism. If a morphism $f\colon X \to X'$ is constant on the fibers of p, then $f = f' \circ p$ holds with a unique morphism $f'\colon Y \to X'$.*

Let us now come back to the toric setting. Let $H \subset T_X$ be a closed subgroup of the big torus of a toric variety X, and let $p\colon X \to Y$ denote the associated toric quotient. Aiming at a criterion for the existence of a categorical quotient derived from Proposition 7.3, we consider the following two questions:

- Is the toric quotient $p\colon X \to Y$ - at least in certain cases - weakly proper?
- Is every H-invariant morphism $f\colon X \to X'$ of varieties constant on the fibres of $p\colon X \to Y$?

To answer the first question, there is a simple general characterization of weakly properness in terms of fans. Let $p\colon X \to Y$ be a dominant toric morphism and let $P\colon \Sigma \to \Delta$ be the associated map of fans. Then we have, see [3, Proposition 1.2]:

Proposition 7.4. *The toric morphism $p\colon X \to Y$ is weakly proper if and only if $P(|\Delta|) = |\Sigma|$.*

The second of the above questions in general is not easy to decide. However, in the following situations one obtains positive answers to both questions and hence existence criteria for categorical quotients, compare [3, Corollary 3] and [7, Corollary 6.5]:

Theorem 7.5. *Let $H \subset T_X$ be a subgroup of the big torus of a toric variety X, and let $p\colon X \to Y$ be the associated toric quotient.*

(i) *If $\dim(T_X/H) \leq 2$, then $p\colon X \to Y$ is a categorical quotient for the action of H on X.*
(ii) *If X arises from a fan with convex support, then $p\colon X \to Y$ is a categorical quotient for the action of H on X.*

Moreover, assuming that the toric quotient is weakly proper, one can get a positive answer for the second question, provided the toric quotient space is of maximal possible dimension, see [7, Corollary 6.4]:

Theorem 7.6. *If the toric quotient $p\colon X \to Y$ for the action of a subtorus $H \subset T_X$ is weakly proper, and $\dim(Y) = \dim(X) - \dim(H)$ holds, then $p\colon X \to Y$ is a categorical quotient for the action of H on X.*

We conclude with two elementary observations on toric quotients. In the first one, we consider a special case of weakly proper toric quotients, namely those parametrizing closed orbits:

Proposition 7.7. *Let the toric quotient $p\colon X \to Y$ of a subtorus action $H \subset T_X$ arise from a map of fans $P\colon \Delta \to \Sigma$. Then the following statements are equivalent:*

(i) *The toric quotient $p\colon X \to Y$ parametrizes the closed H-orbits.*
(ii) *For each $\tau \in \Sigma$ there is a cone $\sigma \in \Delta$ with $P(\sigma) = \tau$ and $P^{-1}(\tau^\circ) \cap |\Delta| \subset \sigma$.*

If one of these conditions holds, then $p\colon X \to Y$ is a categorical quotient for the action of H on X.

Proof. First suppose that $p\colon X \to Y$ parametrizes the closed H-orbits. Let $\tau \in \Sigma$ and consider the distinguished point y_τ. Then the closed H-orbit in $p^{-1}(y_\tau)$ is of the form $H{\cdot}x_\sigma$ with some $\sigma \in \Delta$. Note that $P(\sigma^\circ) \subset \tau^\circ$ holds. Moreover, we obtain for the fibre:

$$p^{-1}(y_\tau) = \{x \in X;\ x_\sigma \in \overline{H{\cdot}x}\} \subset X_\sigma.$$

This implies $P^{-1}(\tau^\circ) \cap |\Delta| \subset \sigma$. We claim that moreover $P(\sigma) = \tau$ holds. Otherwise, there were a face $\varrho \prec \sigma$ such that $P(\varrho^\circ)$ is contained in $\tau^\circ \setminus P(\sigma^\circ)$. Using [32, Lemma 1.4], we see that x_σ is not contained in the closure of $H{\cdot}x_\varrho$. But this contradicts $x_\varrho \in p^{-1}(y_\tau)$.

Suppose now that Condition (ii) holds. This implies in particular, that during the computation of $P\colon \Delta \to \Sigma$, Algorithm 2.7 does not enter its loop. So the initialization of Algorithm 2.7 immediately produces a quasifan in N/L, where L denotes the primitive sublattice defined by H in the lattice N of Δ.

Given τ and σ as in (ii), we can conclude that the maximal linear subspace contained in $\sigma + L_{\mathbb{Q}}$ is $\ker(P)_{\mathbb{Q}}$. That means $\ker(P)_{\mathbb{Q}} = \mathrm{lin}(\sigma_L)$ in the notation of Section 3, and we obtain from Remark 3.4 that the restriction $X_\sigma \to X_\tau$ of p is the good quotient for the action of H on X_σ.

Now, $P^{-1}(\tau^\circ) \cap |\Delta| \subset \sigma$ implies that the fibre of the distinguished point y_τ is contained in X_σ. By the properties of the good quotient $X_\sigma \to X_\tau$, the orbit $H \cdot x_\sigma$ is the only closed H-orbit in $p^{-1}(y_t)$. The general fibres $p^{-1}(y)$ are treated by translating them into fibres $p^{-1}(y_\tau)$ via elements of T_X.

The supplement is an immediate consequence of Proposition 7.3. But it can also be verified directly as follows:

Let $f\colon X \to Z$ be an H-invariant morphism. Consider pairs $\tau \in \Sigma$ and $\sigma \in \Delta$ as in Condition (ii). As outlined above, restricting p yields good quotients $X_\sigma \to X_\tau$. Thus each restriction $f_\sigma := f|_\sigma$ factors as $f_\sigma = f'_\sigma \circ p$. In particular, each f_σ is invariant by $H' := \ker(p)$. Hence the f'_σ patch together to the desired factorizing morphism $f'\colon Y \to Z$. □

Remark 7.8. If a toric quotient $p\colon X \to Y$ for a subtorus $H \subset T_X$ separates H-orbits and X arises from a fan with convex support, then $p\colon X \to Y$ is a good quotient.

Our last observation is that the toric quotient for a subgroup action on a toric variety solves one of the initial tasks of geometric invariant theory in our context: it provides a geometric realization of the sheaf of invariant functions.

Proposition 7.9. *Let $p\colon X \to Y$ be the toric quotient for the action of a subgroup $H \subset T_X$ on a toric variety X. Then we have $\mathcal{O}_Y = p_*(\mathcal{O}_X)^H$.*

Proof. Using Remark 2.5, one easily reduces to the case that H is a torus. Moreover, since the sheaves in question are quasicoherent it suffices to check equality on the invariant affine charts of the toric variety Y.

Suppose that X arises from a fan Δ in a lattice N, and let $L \subset N$ be the primitive sublattice determined by $H \subset T_X$. We use the construction of the toric quotient given in 2.7. So, let $N_1 := N/L$, and let $P_1 \colon N \to N_1$ be the projection. The loop of 2.7 produces a quasifan Σ_1 in N_1.

In the strictification step one divides by the maximal sublattice $L_1 \subset N_1$ contained in all cones of Σ_1. Let $N_2 := N_1/L_1$, and let $P_2 \colon N_1 \to N_2$ be the projection. Then the cones $P_2(\tau_1)$, $\tau_1 \in \Sigma_1$, form a fan Σ in N_2, and with $P := P_2 \circ P_1$, the toric quotient $p \colon X \to Y$ arises from the map fans $P \colon \Delta \to \Sigma$.

We shall need the dual lattice homomorphisms of $P_1 \colon N \to N_1$ and $P \colon N \to N_2$. With $L_0 := P_1^{-1}(L_1)$, these are the inclusions

$$P_1^* \colon L^\perp \to M, \qquad P^* \colon L_0^\perp \to M.$$

Consider a maximal cone $\tau \in \Sigma$, and set $\tau_1 := P_2^{-1}(\tau)$. Then τ_1 is a maximal cone of Σ_1. Since τ_1 was obtained applying the loop of Algorithm 2.7, we know

$$\tau_1 = \sum_{\sigma \in \Delta,\ P_1(\sigma) \subset \tau_1} P_1(\sigma)\,.$$

Moreover, since $\tau_1 = \tau_1 + \ker(P_2)$ holds, we have $P_1^{-1}(\tau_1) = P^{-1}(\tau)$. Together with the above formula, this gives us for the dual cones:

$$P^{-1}(\tau)^\vee \cap L_0^\perp \;=\; P_1^{-1}(\tau_1)^\vee \cap L^\perp \;=\; \bigcap_{\sigma \in \Delta,\ P_1(\sigma) \subset \tau_1} (\sigma^\vee \cap L^\perp)\,. \tag{7.9.1}$$

Now let us compare the sections of the sheaves $\mathcal{O}_Y$ and $p_*(\mathcal{O}_X)^H$ over the affine chart Y_τ. On the one hand we have by definition

$$\mathcal{O}_Y(Y_\tau) \;=\; \mathbb{K}[\tau^\vee \cap (N/L_0)^*] \;=\; \mathbb{K}[P^{-1}(\tau)^\vee \cap L_0^\perp];$$

and on the other hand we have

$$\mathcal{O}_X(p^{-1}(Y_\tau))^H = \bigcap_{\substack{\sigma \in \Delta \\ P(\sigma) \subset \tau}} \mathcal{O}_X(X_\sigma)^H = \bigcap_{\substack{\sigma \in \Delta \\ P(\sigma) \subset \tau}} \mathbb{K}[\sigma^\vee \cap L^\perp] = \mathbb{K}\Bigg[\bigcap_{\substack{\sigma \in \Delta \\ P_1(\sigma) \subset \tau_1}} (\sigma^\vee \cap L^\perp)\Bigg].$$

Thus Equation 7.9.1 yields the desired equality. □

8. Appendix: Toric (pre-)varieties

In this appendix we recall the basic definitions and the combinatorial description of toric varieties and prevarieties. For the separated case, we refer to the standard textbooks of Fulton [17] and Oda [28]. The extension of the combinatorial description to the nonseparated setting can be found in [4].

Definition 8.1. (i) A *toric prevariety* is a normal prevariety X together with an algebraic torus $T_X \subset X$ embedded as an open subset and a regular action $T_X \times X \to X$ extending the multiplication $T_X \times T_X \to T_X$. A *toric variety* is a separated toric prevariety.

(ii) A morphism $f \colon X \to X'$ of toric prevarieties is a called a *toric morphism*, if it restricts to a group homomorphism $\varphi \colon T_X \to T_{X'}$ and $f(t \cdot x) = \varphi(t) \cdot f(x)$ holds for all $(t, x) \in T_X \times X$.

An important feature of the categories of toric varieties and prevarieties is the fact that they allow a complete description in terms of combinatorial data. In order to sketch the basic constructions, we first fix some notation from convex geometry:

By a lattice we mean a free finitely generated $\mathbb{Z}$-module. The associated rational vector space of a lattice N is $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N$. If $F: N \to N'$ is a homomorphism of lattices, then we denote the associated linear map $N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$ again by F. The dual lattice of a lattice N is $M := \operatorname{Hom}(N, \mathbb{Z})$. The canonical pairing is denoted by

$$M \times N \to \mathbb{Z}, \qquad (u, v) \mapsto \langle u, v \rangle := u(v).$$

When we speak of a cone in N, we always think of a convex polyhedral cone in $N_{\mathbb{Q}}$. If a cone τ is a face of a cone σ, we write $\tau \prec \sigma$. The relative interior of a cone σ is denoted by σ°. The dual cone of a cone σ in N is

$$\sigma^{\vee} := \{u \in M_{\mathbb{Q}};\ \forall_{v \in \sigma}\ \langle u, v \rangle \geq 0\}.$$

We now sketch how one associates to a given strictly convex cone σ in a lattice N an affine toric variety X_σ: Consider the semigroup $S_\sigma := M \cap \sigma^{\vee}$ and its semigroup algebra $\mathbb{K}[S_\sigma]$. Define

$$X_\sigma := \operatorname{Spec}(\mathbb{K}[S_\sigma]), \qquad T_\sigma := \operatorname{Spec}(\mathbb{K}[M]).$$

Then X_σ is an affine variety. Moreover the algebraic torus T_σ is embedded canonically as an open subvariety into X_σ, and its group structure extends to an action on X_σ such that for each $u \in S_\sigma$ the function $\chi^u \in \mathcal{O}(X_\sigma)$ satisfies

$$\chi^u(t \cdot x) = \chi^u(t)\chi^u(x) \quad \text{for all } (t, x) \in T_\sigma \times X_\sigma.$$

Toric morphisms arise as follows: Let σ, σ' be cones in respective lattices N, N', and let $F: N \to N'$ be a lattice homomorphism with $F(\sigma) \subset F(\sigma')$. Then F induces a pullback homomorphism $S_{\sigma'} \to S_\sigma$, hence a homomorphism $\mathbb{K}[S_{\sigma'}] \to \mathbb{K}[S_\sigma]$ which in turn defines a toric (!) morphism $X_\sigma \to X_{\sigma'}$.

Arbitrary toric varieties are obtained by gluing affine toric varieties along open subvarieties, and similarly one obtains toric morphisms by gluing. The corresponding notions in convex geometry are the following:

Definition 8.2. (i) A *fan* in a lattice N is a finite set Δ of strictly convex cones in N satisfying the following two conditions:
- any two cones of Δ intersect in a common face,
- if $\sigma \in \Delta$, then Δ also contains all faces of σ.

(ii) A *map of fans* Δ, Δ' in lattices N, N' respectively is given by a lattice homomorphism $F: N \to N'$ such that for each $\sigma \in \Delta$ there is some $\sigma' \in \Delta'$ with $F(\sigma) \subset \sigma'$.

We shall denote by $\Delta^{(k)}$ the subset of all k-dimensional cones of a fan Δ. So, $\Delta^{(1)}$ is the set of rays of Δ. The set of maximal cones is denoted by $\Delta^{\max}$. The support of a fan is the union of all its cones; it is denoted by $|\Delta|$. A map of fans will be written as $F: \Delta \to \Delta'$.

Let Δ be a fan in a lattice N. Then for any two cones $\sigma_1, \sigma_2 \in \Delta$ we have canonical toric morphisms $X_{\sigma_1 \cap \sigma_2} \to X_{\sigma_i}$. These morphisms turn out to be open embeddings, and we can use them to glue together the affine toric varieties X_σ, $\sigma \in \Delta$, to a toric variety X_Δ.

Moreover, if $F: \Delta \to \Delta'$ is a map of fans, then for every $\sigma \in \Delta$ we have a toric morphism $X_\sigma \to X_{\sigma'}$, where $\sigma' \in \Delta'$ is any cone containing the image $F(\sigma)$. These toric morphisms patch together to a well defined toric morphism $f: X_\Delta \to X_{\Delta'}$.

Note that $\mathfrak{TV}: \Delta \mapsto X_\Delta$, $F \mapsto f$ is a covariant functor from the category of fans to the category of toric varieties. The basic result in the theory of toric varieties is the following:

Theorem 8.3. *$\mathfrak{TV}\colon \Delta \mapsto X_\Delta$, $F \mapsto f$ is an equivalence of the category of fans with the category of toric varieties.*

We turn to the nonseparated setting. Roughly speaking, toric prevarieties are obtained by gluing toric varieties without taking care about separatedness. We sketch the combinatorial description given in [4].

Definition 8.4. A *system of fans* is a finite collection $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ of fans in a lattice N such that we always have $\Delta_{ij} = \Delta_{ji}$ and every set $\Delta_{ij} \cap \Delta_{jk}$ is a subfan of Δ_{ik}.

Let $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ be a system of fans in a lattice N. We define the associated toric prevariety $X_\mathcal{S}$ to be the gluing of the toric varieties $X_i := X_{\Delta_{ii}}$ along the open subvarieties $X_{ij} := X_{\Delta_{ij}}$. The properties of 8.4 guarantee that this is well defined.

Let us describe the orbits of the big torus $T_\mathcal{S} := \mathrm{Spec}(\mathbb{K}[M])$ of the toric prevariety $X_\mathcal{S}$. Setting $(\sigma, i) \sim (\sigma, j)$ if $\sigma \in \Delta_{ij}$, we obtain an equivalence relation on the set of labelled cones

$$\mathfrak{F}(\mathcal{S}) := \{(\sigma, i);\ i \in I,\ \sigma \in \Delta_{ii}\}.$$

We denote by $\Omega(\mathcal{S})$ the set of equivalence classes, and by $[\sigma, i]$ the class of an element $(\sigma, i) \in \mathfrak{F}(\mathcal{S})$. Each distinguished point $x_{(\sigma,i)} \in X_i$, see [17, page 29], defines a *distinguished point* $x_{[\sigma,i]} \in X_\mathcal{S}$ which depends only on the class $[\sigma, i]$ of (σ, i) in $\Omega(\mathcal{S})$.

Remark 8.5. The assignment $[\sigma, i] \mapsto T_\mathcal{S} \cdot x_{[\sigma,i]}$ defines a bijection from $\Omega(\mathcal{S})$ to the set of $T_\mathcal{S}$-orbits of the toric prevariety $X_\mathcal{S}$.

The face relation induces a partial ordering on the set $\Omega(S)$, namely $[\tau, j] \prec [\sigma, i]$ if τ is a face of σ and $[\tau, i] = [\tau, j]$. This partial ordering reflects the behaviour of orbit closures in $X_\mathcal{S}$:

Remark 8.6. A point $x_{[\sigma,i]}$ lies in the closure of the orbit $T_\mathcal{S} \cdot x_{[\tau,j]}$ if and only if $[\tau, j] \prec [\sigma, i]$. In particular, for the minimal invariant affine open neighbourhood $X_{[\sigma,i]}$ of $x_{[\sigma,i]} \in X_\mathcal{S}$ one has

$$X_{[\sigma,i]} = \bigcup_{[\tau,j]\prec[\sigma,i]} T_\mathcal{S} \cdot x_{[\tau,j]} = \bigcup_{[\tau,j]\prec[\sigma,i]} X_{[\tau,j]}.$$

We come to toric morphisms. Since in the nonseparated case there is no identity theorem, a toric morphism is no longer determined by its behaviour on the big torus. Thus a combinatorial description requires additional data: Let $\mathcal{S}$ and $\mathcal{S}'$ be systems of fans in lattices N and N' respectively.

Definition 8.7. A *map of systems of fans* from $\mathcal{S}$ to $\mathcal{S}'$ is a pair $(F, \mathfrak{f})$, where $F\colon N \to N'$ is a lattice homomorphism and $\mathfrak{f}\colon \Omega(\mathcal{S}) \to \Omega(\mathcal{S}')$ is an order preserving map with

$$\mathfrak{f}([\sigma, i]) = [\sigma', i'] \Rightarrow F_\mathbb{R}(\sigma^\circ) \subset (\sigma')^\circ.$$

Suppose that $(F, \mathfrak{f})$ is a map of the systems $\mathcal{S}$ and $\mathcal{S}'$ of fans. Then, for every $i \in I$ and $\sigma \in \Delta_{ii}$, the lattice homomorphism F defines a toric morphism

$$f_{[\sigma,i]}\colon X_{[\sigma,i]} \to X'_{\mathfrak{f}([\sigma,i])}.$$

By the conditions of Definition 8.7 these toric morphisms glue together to a toric morphism $f\colon X_\mathcal{S} \to X_{\mathcal{S}'}$. Here the map $\mathfrak{f}\colon \Omega(\mathcal{S}) \to \Omega(\mathcal{S}')$ controls the behaviour on the distinguished points:

Remark 8.8. For every $[\sigma, i] \in \Omega(\mathcal{S})$ we have $f(x_{[\sigma,i]}) = x'_{\mathfrak{f}([\sigma,i])}$.

By construction, $\mathfrak{TP}\colon \mathcal{S} \mapsto X_{\mathcal{S}}$, $(F, \mathfrak{f}) \mapsto f$ is a covariant functor from the category of systems of fans to the category of toric prevarieties. In fact we have, see [3, Theorem 3.6]:

Theorem 8.9. *$\mathfrak{TP}\colon \mathcal{S} \mapsto X_{\mathcal{S}}$, $(F, \mathfrak{f}) \mapsto f$ is an equivalence of the category of systems of fans with the category of toric prevarieties.*

References

[1] A. A'Campo-Neuen, J. Hausen: Quotients of toric varieties by the action of a subtorus. Tohoku Math. J. 51, 1–12 (1999)
[2] A. A'Campo-Neuen, J. Hausen: Quasi-projective reduction of toric varieties. Math. Z., Vol. 233, No. 1, 697–708 (2000)
[3] A. A'Campo-Neuen, J. Hausen: Examples and counterexamples for existence of categorical quotients. J. Algebra, 231, 67–85 (2000)
[4] A. A'Campo-Neuen, J. Hausen: Toric prevarieties and subtorus actions. To appear in Geom. Dedicata, math.AG/9912229
[5] A. A'Campo-Neuen, J. Hausen: Categorical quotients for divisorial toric varieties. Preprint, math.AG/0001124
[6] A. A'Campo-Neuen, J. Hausen: Orbit spaces of small tori. Preprint, Konstanzer Schriften in Mathematik und Informatik 151
[7] A. A'Campo-Neuen: Quotients of Toric Varieties with Curve Lifting Property. Preprint, math.AG/0003204
[8] A. A'Campo-Neuen, J. Hausen, S. Schröer: Homogeneous coordinates and quotient presentations of toric varieties. To appear in Math. Nachr., math.AG/0005083
[9] P. Berthelot, A. Grothendieck, L. Illusie et. al.: Theorie des Intersections et Théorème de Riemann-Roch, SGA 6, Springer LNM 225 (1971)
[10] A. Białynicki-Birula: Categorical quotients. J. Algebra 239, 35–55 (2001)
[11] A. Białynicki-Birula, J. Święcicka: A recipe for finding open subsets of vector spaces with a good quotient. Colloq. Math., Vol. 77, No. 1, 97-114 (1998)
[12] A. Borel: Linear Algebraic Groups, Second enlarged edition. Springer, New York, 1991.
[13] M. Borelli: Divisorial varieties. Pacific J. Math. 13, 375–388 (1963)
[14] D. Cox: The homogeneous coordinate ring of a toric variety. J. Alg. Geom. 4, 17–50 (1995)
[15] H. Derksen, H. Kraft: Constructive invariant theory. Alev, J. (ed.) et al., Algebre non commutative, groupes quantiques et invariants. Septieme contact Franco-Belge, Reims, France, June 26–30, 1995. Paris: Société Mathématique de France. Semin. Congr. 2, 221–244 (1995)
[16] M. Eikelberg: The Picard group of a compact toric variety, Result. Math. 22, 509–527 (1992)
[17] W. Fulton: Introduction to toric varieties. The William H. Roever Lectures in Geometry, Princeton University Press (1993)
[18] H. A. Hamm: Very good quotients of toric varieties. In: Bruce, J. W. (ed.) et al. Real and complex singularities. Proceedings of the 5th workshop; São Carlos, Brazil, July 27-31, 1998. Chapman/Hall/CRC Res. Notes Math. 412, 61-75 (2000)
[19] J. Hausen: Equivariant embeddings into smooth toric varieties. To appear in Can. J. Math., math.AG/0005086
[20] J. E. Humphreys: Linear Algebraic Groups, 3rd printing. Springer, Berlin, Heidelberg, New York (1987)
[21] T. Kajiwara: The functor of a toric variety with enough invariant effective Cartier divisors. Tôhoku Math. J. 50, 139–157 (1998)
[22] M. Kapranov, B. Sturmfels, A. V. Zelevinsky: Quotients of toric varieties, Math. Ann. 290, 643–655 (1991)
[23] P. Kleinschmidt: A classification of toric varieties with few generators. Aequationes Math. 35, No.2/3, 254–266 (1988)
[24] P. Kleinschmidt, B. Sturmfels: Smooth toric varieties with small Picard number are projective. Topology 30, No.2, 289–299 (1991)
[Ko] J. Kollàr: Quotients spaces modulo algebraic groups. Ann. of Math. **145**, 33–79 (1997)
[25] H. Kraft: Geometrische Methoden der Invariantentheorie. Vieweg, Braunschweig (1984)

[26] D. Mumford, J. Fogarty, F. Kirwan: Geometric Invariant Theory, 3rd enlarged edition, Springer, Berlin (1994)
[27] Newstead: Introduction to Moduli Problems and Orbit Spaces. Tata Institute of Fundamental Research, Bombay (1978)
[28] T. Oda: Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, Springer, Berlin, Heidelberg, New York (1988)
[29] C. S. Seshadri: Quotient spaces modulo reductive algebraic groups. Ann. of Math. 95, 511-556 (1972)
[30] H. Sumihiro: Equivariant completion. J. Math. Kyoto Univ. 14, 1–28 (1974)
[31] B. Sturmfels: Algorithms in Invariant Theory. Springer, Wien New York (1993)
[32] J. Święcicka: Quotients of toric varieties by actions of subtori. Colloq. Math., Vol. 82, No. 1, 105–116 (1999)
[33] D. L Wehlau: Constructive invariant theory for tori. Ann. Inst. Fourier 43, No.4, 1055–1066 (1993)
[34] J. Włodarczyk: Embeddings in toric varieties and prevarieties. J. Alg. Geometry 2, 705–726 (1993)

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