VECTOR BUNDLES ON ALGEBRAIC CURVES

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The object of these notes is to describe the classification of vector bundles on algebraic curves and the structure of the corresponding moduli spaces. For convenience we work throughout over the complex numbers and let C denote an irreducible non-singular projective algebraic curve. We also work almost entirely in terms of varieties rather than schemes, although a scheme-theoretic approach has technical advantages. Most of the basic theory that we shall need is covered in Chapter I of Hartshorne's book; the main technique required from the later chapters of the book is sheaf cohomology. Chapter IV contains a lot of material on curves; we shall need only a few basic facts.

We restrict attention to the moduli spaces of vector bundles, ignoring the many interesting and important extensions to parabolic bundles, pairs of various types, coherent systems and above all G-bundles.

The notes have been slightly adapted from those issued with the original lectures, but are still not in their final form. In particular I have included only a very small number of references. Comments (by email to newstead@liv.ac.uk) are welcome.

1. Lecture 1 – Quotients

1.1. **Moduli and quotients.** Moduli problems in algebraic geometry are concerned with the classification of certain objects (e.g. algebraic curves, vector bundles on a fixed algebraic variety, sets of points in $\mathbb{P}^n$) under an equivalence relation (e.g. isomorphism of curves, isomorphism of bundles, projective equivalence). The general procedure for tackling such a problem is

- fix some **discrete invariants** (e.g. genus, Hilbert polynomial, number of points);
- find a **family** of objects, parametrised by a variety S, which includes all the desired objects as members, together with an action of an algebraic group G on S such that orbits correspond to equivalence classes of objects;
- construct a quotient of S by G.

The purpose of this lecture is to consider the third part of this programme.

1.2. **Types of quotient.** Let G be an algebraic group acting rationally on a variety X (that is, through a morphism $G \times X \to X$). We would like to construct a **quotient** for this action; what could this mean?

(i) **Orbit space**. It may happen that the set of orbits X/G has a natural structure of variety. In fact this rarely happens even if one allows more general structures such as schemes or "algebraic spaces". Actually there is always a dense Zariski-open G-invariant subset U of X for which U/G exists [Rosenlicht 1963], but we do not have a very good geometrical grasp of U/G.

(ii) **Categorical quotient**. We say that $\phi : X \to Y$ (or just Y) is a *categorical quotient* of X by G if every morphism $X \to Z$ which is constant on orbits factors uniquely through ϕ. This has good functorial properties, but not necessarily good geometric ones.

We shall be looking for a categorical quotient with good geometrical properties.

1.3. **Affine quotients.** For any variety X, we let $A(X)$ denote the algebra of morphisms $X \to \mathbb{C}$. The action of G on X determines an action of G on $A(X)$. If $X \to Z$ is constant on orbits, we have a homomorphism $A(Z) \to A(X)^G$. If Y is a categorical quotient, the homomorphism $A(Y) \to A(X)^G$ is an isomorphism.

Now suppose that X is an affine variety. In this case $A(X)$ is just the affine coordinate ring of X, and, by Hilbert's Nullstellensatz, we can recover X from $A(X)$. Moreover, a $\mathbb{C}$-algebra has the form $A(Z)$ for some affine variety Z if and only if it is finitely generated and has no nilpotent elements. The second property is necessarily true of $A(X)^G$, so the question to be answered is

Question. Is $A(X)^G$ finitely generated?

This is a version of Hilbert's 14th problem and the answer in general is NO [Nagata 1958]. However the answer is always YES if G is **reductive**; this includes many important cases, for example $GL(n)$, $SL(n)$ and $PGL(n)$.

Theorem 1.1. *Let G be a reductive group acting on an affine variety X. Then there exists an affine variety Y and a morphism $\phi : X \to Y$ such that*

(i) *ϕ is constant on orbits;*
(ii) *ϕ is surjective;*
(iii) *if U is a Zariski-open subset of Y, the induced homomorphism*

$$\phi^* : A(U) \to A(\phi^{-1}(U))^G$$

is an isomorphism;
(iv) *if W is a Zariski-closed G-invariant subset of X, then $\phi(W)$ is closed in Y;*
(v) *if W_1, W_2 are Zariski-closed G-invariant subsets of X and $W_1 \cap W_2 = \emptyset$, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.*

Corollary 1.2. *For any Zariski-open subset U of Y, U is a categorical quotient of $\phi^{-1}(U)$ by G.*

Corollary 1.3. *If $\phi(x_1) = \phi(x_2)$, then $\overline{O(x_1)} \cap \overline{O(x_2)} \neq \emptyset$.*

Corollary 1.4. *If G acts on $\phi^{-1}(U)$ with closed orbits, then $U = \phi^{-1}(U)/G$ (i.e. U is an orbit space).*

Note that Corollary 1.3 states that ϕ separates orbits to the greatest extent possible, consistent with Y being a variety. This is perhaps more significant than the simple fact that $A(X)^G$ is finitely generated.

1.4. **Projective quotients.** The case in which we are most interested is when X is **projective**; this is because we would like our quotients to be **complete** (**compact**) or at least to have natural compactifications. To apply the previous section, we need to consider **affine open subsets** of X which are invariant under G. We suppose $X \subset \mathbb{P}^n$ and that G acts **linearly**, that is through a representation of G in $GL(n+1)$. Then G acts on the polynomial ring $\mathbb{C}[X_0, \dots, X_n]$. If f is any G-invariant homogeneous polynomial for this action, then $X_f = \{x \in X : f(x) \neq 0\}$ is a G-invariant affine open subset of X.

Definition. A point $x \in X$ is

- *semistable* for the action of G if there exists f such that $x \in X_f$;
- *stable* for the action of G if $\dim O(x) = \dim G$ and there exists f as above such that G acts on X_f with closed orbits. (In GIT, this is called "properly stable".)

Lemma 1.5. *The subsets X^{ss}, X^s of X of semistable (resp. stable) points of X are G-invariant Zariski-open subsets on X.*

Exercise. Prove Lemma 1.5.

Definition. $\phi : X \to Y$ is

- a *good quotient* of X by G if ϕ is an affine morphism (that is, the inverse image of every affine Zariski-open set in Y is affine) and (i)–(v) of Theorem 1.1 hold;
- a *geometric quotient* if it is a good quotient and also an orbit space.

It is conventional to write $Y = X//G$ when Y is a good quotient and $Y = X/G$ when Y is a geometric quotient. The concept of "good quotient" given here is due to Seshadri; the idea originates in GIT, but the definition given there is not identical with ours. Our "geometric quotient" differs from that in GIT by the insistence that ϕ be an affine morphism.

Theorem 1.6. *Let G be a reductive group acting linearly on a projective variety X. Then*

(i) *there exists a good quotient $\phi : X^{ss} \to Y$ and $Y = X^{ss}//G$ is projective;*
(ii) *there exists a Zariski-open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and $Y^s = X^s/G$ is a geometric quotient of X^s;*
(iii) *for $x_1, x_2 \in X^{ss}$, $\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{O(x_1)} \cap \overline{O(x_2)} \cap X^{ss} \neq \emptyset$;*
(iv) *for $x \in X^{ss}$, x is stable if and only if $\dim O(x) = \dim G$ and $O(x)$ is closed in X^{ss}.*

Remark 1.7. In some ways the most significant point of this theorem is that Y is **projective**. This means both that it is complete and that it can be embedded in projective space.

Remark 1.8. For G not necessarily reductive, Rosenlicht's result gives a dense Zariski-open subset U of X such that U/G exists with all the properties of a geometric quotient except that the morphism ϕ need not be affine.

1.5. **References.** The main references for this lecture are GIT and my Tata notes.

2. Lecture 2 – Construction of moduli spaces

2.1. **The concept of a moduli space.** Let C be an irreducible non-singular projective algebraic curve of genus g. We consider algebraic vector bundles E over C of rank n and degree d. (Note here that $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$ and this group has a canonical positive generator; hence the Chern class $c_1(E)$ can be regarded as an integer – this is by definition the *degree* of E. Alternatively one can consider a rational section s of the line bundle $\det E$; then $\deg E = N - P$, where N and P are the numbers of zeroes and poles of s, counted with multiplicities – this number is indepedent of the choice of s.)

Definition.

(i) A *family* of vector bundles over C *parametrised by* a variety S is a vector bundle V over $C \times S$; for any $s \in S$, we write $V_s = V|C \times \{s\}$ and refer to this as the *member* of the family corresponding to s.
(ii) For any family V parametrised by S and any morphism $f : S' \to S$, we have an *induced family* $(id_C \times f)^*V$ parametrised by S'.
(iii) Two families V_1, V_2 parametrised by S are *equivalent* ($V_1 \sim V_2$) if $V_1 \cong V_2 \otimes p_S^* L$ for some line bundle L over S.

We can now define a contravariant functor

$$\mathbb{F} : \{\text{varieties}\} \to \{\text{sets}\}$$

$$\mathbb{F}(S) = \{\text{equivalence classes of families parametrised by } S\}.$$

We would like $\mathbb{F}$ to be **representable**, i.e. for there to exist a variety M and a **universal** vector bundle U over $C \times M$ such that, for any family V parametrised by S, there exists a unique morphism $\phi : S \to M$ such that

$$V \sim (id_C \times \phi)^* U.$$

If this were to happen we would call M a **fine moduli space** for vector bundles of rank n and degree d over C.

In principle we should allow **schemes** as parametrising spaces; in this problem, however, it makes no difference. Similarly using **holomorphic** bundles and looking for an analytic moduli space makes no difference; this is a fundamental fact first noted in a formal way by Serre.

2.2. **Some problems. 1.** Automorphisms create problems for the existence of moduli spaces. All vector bundles admit scalar multiples of the identity as automorphisms; this is the reason for the definition of equivalence of families given above. However some bundles have $\dim \mathrm{Aut} E \geq 2$ and this prevents the existence of moduli spaces. To avoid this we restrict attention to **simple** bundles, i.e. those for which $\mathrm{Aut} E = \mathbb{C}^*$.

2. Using simple bundles overcomes all local problems, but global problems (does M exist as a variety (or scheme) or only in some weaker sense, can it be embedded in a projective space etc.?) remain. It was to overcome these problems that Mumford introduced the concept of a **stable** bundle.

Definition. A vector bundle E over C is *stable* if, for every proper subbundle F of E,

$$\frac{\deg F}{\mathrm{rk} F} < \frac{\deg E}{\mathrm{rk} E}.$$

3. To construct a moduli space $M(n,d)$ for stable bundles of rank n and degree d over C, we proceed as follows

- Find a family $V \to C \times R$ including **all** stable bundles of the given rank and degree, together with an action of a reductive group G (actually $PGL(p)$) on R such that
$$V_{r_1} \sim V_{r_2} \Leftrightarrow r_1, r_2 \text{ lie in the same orbit.}$$
- Embed R in $\mathbb{P}^N$ so that G acts linearly.
- Take quotients $R^{ss}//G$ and R^s/G as in Lecture 1.

If we could show that R^s corresponds precisely to the stable bundles in the family V, we could then define $M(n,d) := R^s/G$. But where is the universal bundle U? We would like to define $U := V/G$, but it turns out that $G = PGL(p)$ does not act on V; $GL(p)$ (or $SL(p)$) does act on V, but does not act freely on R^s, so construction of U by **descent** doesn't work. However, if we settle for $\mathbb{F}$ being **co-representable**, everything is OK and $M(n,d)$ is defined as a **coarse moduli space**. There is no universal bundle but $M(n,d)$ does have a weaker universal property which is sufficient for many purposes.

4. If we want a projective quotient, we must consider $R^{ss}//G$. What does this correspond to? The obvious guess is *semistable* bundles (where we replace the $<$ in the definition of stable bundle by $\leq$). This does indeed work but of course the points of the quotient no longer correspond precisely to orbits (see below).

Remark 2.1. One can show that, in any family of bundles, the subsets of stable and semistable members of the family correspond to Zariski-open subsets of the parametrising variety. This is much harder than Lemma 1.5 and indeed was first proved by constructing moduli spaces as described below and using an a posteriori argument.

2.3. **Sheaves and cohomology.** The correspondence between vector bundles and sheaves is fundamental to the study of vector bundles. Let $\mathcal{O}$ denote the sheaf of local rings over C. By considering local sections of a vector bundle E (that is, sections defined over Zariski-open subsets of C), we obtain the sheaf of sections of E. This sheaf is a module over $\mathcal{O}$ and as such is **locally free**. Conversely any locally free sheaf determines a vector bundle, unique up to isomorphism. It is conventional to identify the concepts of vector bundle and locally free sheaf, and we shall do this.

It is sometimes necessary to use the bigger category of **coherent** sheaves over C; unlike locally free sheaves, these form an abelian category. For our case, where C is a non-singular curve, a coherent sheaf is locally free if and only if it is torsion-free; thus every coherent sheaf is the direct sum of a locally free sheaf and a torsion sheaf - and the latter is necessarily supported on a finite set of points of C. Note that the concepts of rank and degree can be extended to coherent sheaves; in particular a sheaf of rank 0 is precisely a torsion sheaf.

The introduction of sheaves allows us to use sheaf cohomology. In our case we have just H^0 and H^1, where H^0 is just the space of global sections. Since C is projective, H^0 and H^1 are finite-dimensional as vector spaces, and we write $h^i = \dim H^i$. We have the following fundamental results, valid for coherent sheaves over C, which as usual is an irreducible non-singular projective curve of genus g. We denote by K the **canonical** line bundle on C (that is, the dual of the tangent bundle).

Riemann-Roch Theorem. For any coherent sheaf E of rank n and degree d over C,

$$h^0(E) - h^1(E) = d + n(1-g).$$

Serre Duality Theorem. Let E^* denote the dual sheaf $Hom(E, \mathcal{O})$. There is a natural duality of vector spaces between $H^1(E)$ and $H^0(E^* \otimes K)$. In particular

$$h^1(E) = h^0(E^* \otimes K).$$

Exercise. Using these theorems, show that

$$\deg K = 2g-2, \quad h^0(K) = g.$$

2.4. **Stable bundles as quotient sheaves.**

Lemma 2.2. (i) *Every line bundle is stable.*
(ii) *If E is stable (semistable) and L is a line bundle, then $E \otimes L$ is stable (semistable).*
(iii) *If E is stable, then E is simple.*

Exercise. Prove Lemma 2.2.

Definition. A bundle E is *generated by its sections* if the evaluation map $H^0(E) \to E_x$ is surjective for all $x \in C$, or equivalently if there is an exact sequence

$$0 \to F \to \mathcal{O} \otimes W \to E \to 0,$$

where W is a vector space.

Lemma 2.3. *Let E be a semistable bundle of rank n and degree d with $d > n(2g-1)$. Then*

- *E is generated by its sections;*
- *$h^1(E) = 0$.*

Exercise. Prove Lemma 2.3.

Now fix a line bundle $\mathcal{O}(1)$ of degree 1 and fix n, d. Choose r so that $d + nr > n(2g-1)$. Assuming E is semistable and writing $E(r) = E \otimes \mathcal{O}(1)^r$, we have by Lemma 2.3 an exact sequence

$$0 \to G \to \mathcal{O} \otimes W_r \to E(r) \to 0,$$

where the induced linear map $W_r \to H^0(E(r))$ is an isomorphism and $h^1(E(r)) = 0$. So by Riemann-Roch

$$p := \dim W_r = h^0(E(r)) = d + n(r+1-g).$$

Note that E is a quotient of $F_r = \mathcal{O}(-r) \otimes W_r$.

The set of all sheaf quotients of F_r of rank n and degree d gives Grothendieck's Quot scheme Q_r, which is a projective scheme. Moreover there exists a **universal quotient** and in particular a sheaf V over $C \times Q_r$, flat over Q_r, such that $V_q := V|C \times \{q\}$ is precisely the quotient sheaf corresponding to the point $q \in Q_r$. (Flatness here is a technical condition which ensures that pull-backs behave well.) Note that, if we identify $GL(p)$ with the automorphism group of W_r, then $GL(p)$ acts on Q_r; moreover scalar matrices act trivially, so the action goes down to an action of $PGL(p)$ on Q_r.

We call a point $q \in Q_r$ **good** if the induced map

$$W_r \to H^0(E(r))$$

is an isomorphism.

Lemma 2.4. *Let q_1, q_2 be good points of Q_r. Then $V_{q_1} \cong V_{q_2}$ if and only if q_1, q_2 lie in the same orbit for the action of $PGL(p)$ on Q_r. Moreover the stabiliser of V_{q_1} is* $\mathrm{Aut}V_{q_1}/\{\lambda(id)\}$.

Exercise. Prove Lemma 2.4.

Now define

$$R := \{q \in Q_r : V_q \text{ is locally free and } q \text{ is good}\}.$$

It is easy to see that R is a Zariski-open $PGL(p)$-invariant subset of Q_r.

This completes the first part of the construction of $M(n, d)$.

2.5. **Computation of stability.** The next step is to embed R in $\mathbb{P}^N$ and compute the stability condition. The first construction for this was due to Mumford. Later on Gieseker gave a somewhat different construction which works also in higher dimensions. However, the most natural one is due to Simpson using Grothendieck's construction for a projective embedding of Q_r. In fact, Simpson's method extends to torsion-free sheaves on singular curves and to sheaves of pure dimension on higher-dimensional varieties.

We shall not give full details here but note that one advantage of this method is that Q_r is embedded as a projective scheme whereas R is only quasi-projective. The action of $PGL(p)$ extends to an action on $\mathbb{P}^N$ and, while this action is not linear, the induced action of $SL(p)$ is. We can thus apply the results of Lecture 1 to construct a quotient for the action.

It remains to show that the stability condition for the action of $SL(p)$ corresponds to bundle stability when restricted to R. Actually this is true only if we take r sufficiently large.

Proposition 2.5. *There exists an integer r_0 such that, for all $r \geq r_0$,*

(i) *if V_q is stable (semistable), then q is stable (semistable);*
(ii) *if q is semistable, then $q \in R$;*
(iii) *if $q \in R$ is stable (semistable), then V_q is stable (semistable).*

2.6. Conclusions.

Theorem 2.6. *Fix $r \geq r_0$. Then $Q_r^s/SL(p)$ is a coarse moduli space for stable bundles of rank n and degree d over C. It is quasi-projective and is an open subset of the projective variety $Q_r^{ss}//SL(p)$.*

According to the general theory and Proposition 2.5, the points of $Q_r^{ss}//SL(p)$ correspond to equivalence classes of semistable bundles. This equivalence can be described as follows. Any semistable sheaf E possesses a filtration (the *Jordan-Hölder* filtration)

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_m = E$$

by subbundles E_i, where E_i/E_{i-1} is a stable bundle of rank n_i and degree d_i with $d_i/n_i = d/n$ for $1 \leq i \leq m$. This filtration is not unique, but the associated graded object

$$\mathrm{gr}(E) = \bigoplus_{i=1}^{m} E_i/E_{i-1}$$

is determined by E.

Definition. Two semistable bundles E_1, E_2 are *S-equivalent* if $\mathrm{gr}(E_1) = \mathrm{gr}(E_2)$.

Given this, we now define

$$M(n,d) := Q_r^s/SL(W_r), \quad \widetilde{M}(n,d) := Q_r^{ss}//SL(W_r).$$

The points of $\widetilde{M}(n,d)$ then correspond to S-equivalence classes of semistable bundles.

This completes the construction. From the infinitesimal theory of Quot schemes, we deduce

- $M(n,d)$ is non-singular;
- if $M(n,d)$ is non-empty, then $\dim M(n,d) = n^2(g-1)+1$.

Moreover

- $\widetilde{M}(n,d)$ is a normal projective variety;
- if n and d are coprime, then $M(n,d) = \widetilde{M}(n,d)$; hence $M(n,d)$ is a non-singular projective variety.

None of this proves the most basic property of all, namely non-emptiness. In fact,

- if $g \geq 2$, $M(n,d)$ is non-empty for all n, d;
- if $g = 1$, $M(n,d)$ is non-empty if and only if $(n,d) = 1$ [Atiyah 1957];
- if $g = 0$, $M(n,d)$ is non-empty if and only if $n = 1$; in fact in this case every vector bundle is a direct sum of line bundles [Grothendieck 1957 in this form].

Finally, we can ask whether $M(n,d)$ is a fine moduli space, that is whether there exists a universal bundle.

Proposition 2.7. *$M(n,d)$ is a fine moduli space if and only if $(n,d) = 1$.*

The problem with constructing a universal bundle and hence proving this proposition is that the action of $PGL(p)$ on R does not lift to the universal quotient V. However the action of $GL(p)$ does lift in such a way that the scalar matrix λI acts by multiplication by λ. It turns out that, when $(n, d) = 1$, one can use V to construct a line bundle L on R on which the action of $GL(p)$ lifts in such a way that λI acts by λ^{-1}. We can now replace V by $V \otimes p_R^* L$, which as a family is equivalent to V and on which $PGL(p)$ acts. A descent argument now gives a universal bundle on $C \times M(n, d)$. This argument fails if $(n, d) \neq 1$, and Ramanan [1973] showed that in this case the universal bundle does not exist.

2.7. **References.** My Tata notes, Le Potier.

3. Lecture 3 – Topology of Moduli Spaces

In this lecture we consider the topology of $M(n,d)$. Here we use "topology" in the sense of the usual complex topology, not the Zariski topology. Recall that C is an irreducible non-singular projective algebraic curve over $\mathbb{C}$, or equivalently a compact Riemann surface, of genus g.

3.1. **Description of $M(n,d)$ using representations.** This goes back to Weil [1938] and was fully developed by Narasimhan and Seshadri [1965].

Recall that

$$\pi_1 := \pi_1(C) = \langle a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle.$$

A representation $\rho : \pi_1 \to GL(n,\mathbb{C})$ determines a vector bundle E_ρ with a holomorphic structure; the construction also works for families.

Theorem 3.1. [Weil] *A holomorphic vector bundle over C has the form E_ρ if and only if every indecomposable summand of E has degree 0.*

However it is difficult to determine when $E_{\rho_1} \cong E_{\rho_2}$; Weil remarked that this should be easier for **unitary** representations and Narasimhan and Seshadri proved

Theorem 3.2.
(i) *If $\rho_1, \rho_2 : \pi_1 \to U(n)$, then $E_{\rho_1} \cong E_{\rho_2}$ if and only if ρ_1, ρ_2 are equivalent under the conjugation action of $PU(n)$.*
(ii) *A holomorphic vector bundle over C has the form E_ρ with ρ unitary if and only if it is a direct sum of stable bundles of degree 0; in particular E_ρ is always semistable.*

Using the generators for π_1, we can identify the unitary representations of ρ with the points of a closed subspace $R_0 \subset U(n)^{2g}$. Theorem 3.2 then gives us a map $R_0 \to \widetilde{M}(n,0)$, which is surjective and is an orbit space for the action of $PU(n)$. Moreover we have a restriction $R_0^s \to M(n,0)$, which is a principal $PU(n)$-bundle. Note that R_0 does not possess a holomorphic structure, but Narasimhan and Seshadri proved that $\widetilde{M}(n,0)$ has a natural structure of analytic space and has a certain universal property for holomorphic families of semistable bundles. In fact this structure is an algebraic structure and is the same as the one described in Lecture 2.

In the case $n = 1$, we have $R_0 = U(1)^{2g} = (S^1)^{2g}$; hence

Corollary 3.3. $M(1,0) \cong (S^1)^{2g}$.

In fact, $M(1,0)$ is just the **Jacobian** of C and this result is classical.

Question. What happens if $d \neq 0$?

Narasimhan and Seshadri show that we can take suitable representations of an extended group. We obtain a new subspace $R_d \subset U(n)^{2g}$ given by

$$R_d := \{(A_1, A_2, \ldots, A_g, B_g) \in U(n)^{2g} : \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = e^{-\frac{2\pi i d}{n}} I\}.$$

Theorem 3.2 applies with (ii) replaced by

: (ii)′ *A holomorphic bundle has the form E_ρ with $\rho \in R_d$ if and only if it is a direct sum of stable bundles E_i with*
$$\frac{\deg E_i}{\mathrm{rk} E_i} = \frac{d}{n}.$$

From this we obtain $R_d \to \widetilde{M}(n,d)$ and $R_d^s \to M(n,d)$. Of course, if $(n,d) = 1$, then $R_d = R_d^s$. In principle this gives a complete topological description and one may note in particular

Theorem 3.4. *The topological structure of $M(n,d)$, $\widetilde{M}(n,d)$ depends only on g, n and d and not on the holomorphic structure of C.*

Remark 3.5. If L is a line bundle of degree d over C, we can define $M_L(n,d)$, $\widetilde{M}_L(n,d)$ by admitting only bundles of determinant L. We can give topological descriptions of these spaces exactly as above by replacing $U(n)$ by $SU(n)$. Note also that the determinant defines a morphism
$$\det : M(n,d) \to M(1,d)$$
whose fibre over L is precisely $M_L(n,d)$; with respect to the usual topology, this is a locally trivial fibration.

3.2. **Digression: Morse theory.** A natural question to ask is whether there exists a natural **perfect Bott-Morse function** on $M(n,d)$. If so, we could give a precise topological description of $M(n,d)$, at least in the case $(n,d) = 1$. [Here a *Bott-Morse* function is a differentiable function whose critical set is a finite disjoint union of submanifolds whose normal bundles have non-degenerate Hessians. Such a function is *perfect* if all the differentials in the associated spectral sequence are zero – this implies that all the homology coming from the critical submanifolds survives in the homology of the complete manifold without any cancellation.]

For $M_L(2,1)$, this question has been answered. We can consider the function $\mathrm{Tr}A_g$ (see previous section); since $A_g \in SU(2)$, this function takes real values between -2 and 2, and is constant on orbits for the conjugation action of $PU(2)$, hence defines a bounded real-valued function on $M_L(n,d)$. The critical submanifolds are easy to calculate and it is not hard to show the function is Bott-Morse; so, once we know the homology of $M_L(2,1)$, it is simple to determine whether it is perfect. This indeed turns out to be the case (as observed a few years ago by Jeffrey and Weitsman). However, it is only very recently that an a priori proof has been given.

Theorem 3.6. [Thaddeus 2000] $\mathrm{Tr}A_g$ *defines a perfect Bott-Morse function on* $M_L(2,1)$.

3.3. **Homology and cohomology.** Note first that we have a map
$$M_L(n,d) \times M(1,0) \to M(n,d)$$
$$(E, L') \mapsto E \otimes L'.$$

This is an unramified covering with group $(\mathbb{Z}/n)^{2g}$, and this group acts trivially on the rational cohomology. It follows that

$$H^*(M(n,d),\mathbb{Q}) \cong H^*(M_L(n,d),\mathbb{Q}) \otimes H^*(M(1,0),\mathbb{Q}),$$

this isomorphism being an isomorphism of graded algebras. Now, by Corollary 3.3, $M(1,0) \cong (S^1)^{2g}$, so its cohomology is known. It is therefore reasonable to restrict attention to the fixed determinant case.

The first result here was a formula for the Betti numbers

$$b_i = \dim H^i(M_L(2,1),\mathbb{Q}).$$

A few years later, Harder observed that this formula could be expressed easily in terms of the **Poincaré polynomial**

$$P_t(M_L(2,1)) := \sum b_i t^i$$

and we give this version here.

Theorem 3.7. [Newstead 1967, Harder 1970] *For $g \geq 1$,*

$$P_t(M_L(2,1)) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Exercise. Prove that the RHS of this formula is a polynomial of degree $6g-6$ in t.

Using Theorem 3.7, one can obtain cohomology generators.

Theorem 3.8. [Newstead 1972] $H^*(M_L(2,1),\mathbb{Q})$ *is generated by classes*

$$\alpha \in H^2, \quad \psi_1,\ldots,\psi_{2g} \in H^3, \quad \beta \in H^4.$$

These classes are naturally defined using the Chern classes of the universal bundle (this exists by Proposition 2.7). Moreover, one can obtain an additive basis for each H^j in terms of these generators, but a detailed description of the relations was not clear at this stage.

The next major result was due to Harder and Narasimhan [1975]; they obtained an inductive formula for $P_t(M_L(n,d))$ for $(n,d)=1$ by number-theoretic methods. This was made more explicit by Desale (Bhosle) and Ramanan [1975].

Next came the fundamental paper of Atiyah and Bott [1962]. Included in this paper, which has had an enormous influence on subsequent developments, was a proof that $H^*(M_L(n,d),\mathbb{Z})$ is torsion-free. They also gave generators for this ring.

In the meantime Mumford had proposed a method of obtaining relations among the cohomology generators. For $M_L(2,1)$ these relations were shown to be sufficient first by Kirwan [1992] and then by many other people. Indeed a much simpler description of the relations has been obtained.

In general, for $(n,d)=1$, Zagier has produced a (complicated) explicit formula for $P_t(n,d)$. For $(n,d)\neq 1$, things are more complicated even in the simplest case $(n,d)=(2,0)$ because the moduli spaces are either non-compact (if one uses $M_L(n,d)$) or singular (if one uses $\widetilde{M}_L(n,d)$). Instead of using ordinary cohomology, it is now better to use something which takes account of the singularities, for example intersection cohomology or the cohomology of a partial desingularisation. There are partial answers due to Kirwan and others, but as yet no really satisfactory description.

3.4. **Cohomology of $M_L(2,1)$.** We write for convenience $H_g^* = H^*(M_L(2,1),\mathbb{Q})$. We have already noted in Theorem 3.8 that H_g^* is generated by classes

$$\alpha \in H^2, \quad \psi_1,\dots,\psi_{2g} \in H^3, \quad \beta \in H^4.$$

In fact these classes are defined in a natural way. By Proposition 2.7, there exists a universal bundle U on $C \times M_L(2,1)$ and this bundle is determined up to tensoring by a line bundle lifted from $M_L(2,1)$. In particular the endomorphism bundle End$U$ is determined. Using the formulae for the Betti numbers, we can write $c_2(\mathrm{End}U)$ in terms of its Künneth components:

$$c_2(\mathrm{End}U) = 2f\alpha - \beta - 4\sum_{i=1}^{2g} e_i\psi_i,$$

where f is the positive generator of $H^2(C,\mathbb{Z})$ and $e_1,\dots,e_{2g}$ is a symplectic basis of $H^1(C,\mathbb{Z})$ such that $e_ie_{i+g} = -f$ for $1 \le i \le g$ and all other e_ie_j with $i<j$ are 0. We write also

$$\gamma := 2\sum_{i=1}^{g} \psi_i\psi_{i+g} = \left(\sum_{i=1}^{2g} e_i\psi_i\right)^2 [X].$$

Note that γ is independent of the choice of basis $e_1,\dots,e_{2g}$. The classes α, ψ_i, β are all integer classes, but they do not generate the integral cohomology. It is true however that α generates $H^2(C,\mathbb{Z})$ and the ψ_i generate $H^3(C,\mathbb{Z})$.

Let $H_{I,g}^*$ denote the subalgebra of H_g^* generated by α, β and γ. This is in fact a naturally defined subalgebra; there is a natural action of $Sp(2g,\mathbb{Z})$ on H_g^* and $H_{I,g}^*$ is precisely the invariant subalgebra. Moreover $\psi_1,\dots,\psi_{2g}$ is a symplectic basis of $H^3(C,\mathbb{Z})$ for this action and γ is the unique (up to scalar multiples) invariant 2-form.

Recently a very nice form for the relations in $H_{I,g}^*$ has been obtained independently by several authors or groups of authors: Zagier, King and Newstead, Siebert and Tian, Baranovskii, Herrera and Salamon. Let I_g denote the ideal of relations, so that

$$H_{I,g}^* = \mathbb{Q}[\alpha,\beta,\gamma]/I_g.$$

Theorem 3.9.

$$I_g = \langle \zeta_g, \zeta_{g+1}, \zeta_{g+2}\rangle,$$

where $\zeta_h \in \mathbb{Q}[\alpha,\beta,\gamma]$ are determined for $h \ge 0$ by $\zeta_0 = 1$ and

$$\zeta_{h+1} = \alpha\zeta_h + h^2\beta\zeta_{h-1} + 2h(h-1)\gamma\zeta_{h-2}.$$

Example.

- $\zeta_1 = \alpha$;
- $\zeta_2 = \alpha^2 + \beta$;
- $\zeta_3 = \alpha^3 + 5\alpha\beta + 4\gamma$;
- $\zeta_4 = \alpha^4 + 14\alpha^2\beta + 16\alpha\gamma + 9\beta^2$.

Hence

- $H^*_{I,1} = \mathbb{Q}$;
- in $H^*_{I,2}$, the relations reduce to $\beta = -\alpha^2$, $\gamma = \alpha^3$, $\alpha^4 = 0$. So

$$H^*_{I,2} = \mathbb{Q}[\alpha]/\langle \alpha^4 \rangle.$$

Exercise. Calculate $H^*_{I,3}$.

As a step on the way to proving this, one can prove another interesting result. For this, we note that γ determines a *primitive decomposition* of the exterior algebra $\Lambda^* H^3_g$ – if we write

$$\Lambda^k_0 := \ker\left(\gamma^{g-k+1} : \Lambda^k H^3_g \to \Lambda^{2g-k+2} H^3_g\right),$$

then

$$\Lambda^* H^3_g = \bigoplus_{k=0}^{g} \Lambda^k_0 \otimes \mathbb{Q}[\gamma]/\langle \gamma^{g-k+1} \rangle.$$

Theorem 3.10. *As a* $\mathbb{Q}[\alpha, \beta, \gamma]$*-module,*

$$H^*_g = \bigoplus_{k=0}^{g} \Lambda^k_0 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}.$$

Example.

- $H^*_1 = \mathbb{Q}$;
- Additively $H^*_2 = \mathbb{Q} + \alpha\mathbb{Q} + (\psi_1, \psi_2, \psi_3, \psi_4)\mathbb{Q} + \alpha^2\mathbb{Q} + \alpha^3\mathbb{Q}$, the relations (apart from the obvious ones) being given by

$$\psi_1\psi_3 = \psi_2\psi_4 = \frac{1}{4}\alpha^3, \quad \psi_1\psi_2 = \psi_1\psi_4 = \psi_2\psi_3 = \psi_3\psi_4 = 0.$$

Theorem 3.10 was conjectured by Mumford. From this theorem and Theorem 3.7, one can deduce that the Poincaré polynomial of $H^*_{I,g}$ is

$$\frac{(1-t^{2g})(1-t^{2g+2})(1-t^{2g+4})}{(1-t^2)(1-t^4)(1-t^6)}.$$

This strongly suggests a result of the type of Theorem 3.9. Of course, Theorems 3.9 and 3.10 together give the complete structure of $H^*(M_L(2,1), \mathbb{Q})$.

3.5. **Intersection numbers.** The determination of intersection numbers on moduli spaces has become of great importance in questions of geometry, topology and physics; they are related, for example, to the invariants constructed by Donaldson, Gromov and Seiberg and Witten. Knowledge of the relations in rational cohomology is not enough to determine these numbers, although the converse is true at least in the coprime case. For $M_L(2,1)$, an explicit formula for the intersection numbers has been given by Thaddeus [1992]. More generally, for $(n, d) = 1$, formulae in terms of asymptotic expansions of integrals have been obtained by Witten using physical methods and proved mathematically by Jeffrey and Kirwan [1998]. Witten has also given formulae for $M(2,0)$. A paper by Jeffrey, Kiem, Kirwan and Woolf, currently in preparation, will rederive Witten's calculations for $M(2,0)$ and give formulae for intersection pairings on $\widetilde{M}(n,d)$ in intersection cohomology and on the partial resolution of its singularities in the general case.

3.6. **References.** The material in this lecture is scattered in many papers, but there is a very nice account in

- M. Thaddeus, An introduction to the topology of the moduli space of stable bundles on a Riemann surface, Geometry and Physics (Aarhus, 1995), Lecture Notes in Pure and Applied Mathematics, Vol. 184, Marcel Dekker, 1997.

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4. Lecture 4 – Geometry of moduli spaces I

4.1. **Irreducibility and rationality.** For simplicity, we restrict from now on to the case $g \geq 2$, which implies in particular that all the moduli spaces we consider are non-empty.

Proposition 4.1. *(i) $\widetilde{M}(n,d)$ is irreducible;*
(ii) $\widetilde{M}_L(n,d)$ is unirational.

Proof. By tensoring by a line bundle, we can suppose $d > n(2g-1)$. Then, by Lemma 2.3, every semistable bundle E of rank n and degree d is generated by its sections. A dimensional argument due to Serre now shows that we have an exact sequence

$$0 \to \mathcal{O}^{\oplus(n-1)} \to E \to L \to 0,$$

where $L = \det E$. For fixed L, the exact sequences of this type are classified by

$$V_L := \operatorname{Ext}(L, \mathcal{O}^{\oplus(n-1)}) \cong H^1(L^* \otimes \mathcal{O}^{\oplus(n-1)}).$$

The semistable E correspond to a Zariski-open subset W_L of the vector space V_L, and the universal property of the moduli space gives us a surjective morphism $W_L \to \widetilde{M}_L(n,d)$ as required for (ii).

For (i), we need to note that, as L varies in $M(1,d)$, the vector spaces V_L have constant dimension and therefore fit together to form a vector bundle V over the irreducible variety $M(1,d)$. (As a sheaf $V = R^1 p_{M(1,d)*}(P^* \otimes \mathcal{O}^{\oplus(n-1)})$, where P is a universal bundle on $C \times M(1,d)$.) The Zariski-open subset of V corresponding to semistable E is irreducible, and we have a surjective morphism $W \to \widetilde{M}(n,d)$. This proves (i).

Exercise. Prove that, if E is a semistable bundle over C which is generated by its sections, then there exists an exact sequence $0 \to \mathcal{O}^{\oplus(n-1)} \to E \to L \to 0$, where $L = \det E$.

Corollary 4.2. *If $(n,d) = 1$, $M_L(n,d)$ is simply-connected.*

Proof. This is an immediate consequence of (ii) above.

The question of whether $M_L(n,d)$ is actually **rational** is more complicated. The exact sequence used in the proof above gives the result for $d \equiv -1 \bmod n$ [taking $d = n(g-1) + (n-1)$]. Since the 1970s, there have been many extensions to this result, but no complete answer even when $(n,d) = 1$ until the following result was proved recently.

Theorem 4.3. [King/Schofield 1999] *$M(n,d)$ is birational to $M(h,0) \times \mathbb{P}^{(n^2-h^2)(g-1)}$, where $h = (n,d)$.*

Idea of proof. The theorem is proved by constructing a dominant rational map $\mu : M(n,d) \to M(h,0)$ whose generic fibre is rational. The construction is carried out in steps, a key point being to show that each step preserves a certain *Brauer class*; this allows an induction giving the result.

Corollary 4.4. *If $(n,d) = 1$, $M_L(n,d)$ is rational.*

Proof. The dominant rational map constructed in the proof of the theorem restricts to a dominant rational map of fixed determinant moduli spaces (though with different determinants). Now note that $M_L(1,0)$ is a point.

4.2. **Projective embeddings.** The construction of $\widetilde{M}(n,d)$ in Lecture 2 gives a projective embedding of the moduli space, or equivalently a **very ample** line bundle, but it is difficult to identify this line bundle.

To construct explicit projective embeddings, we proceed differently. It is somewhat simpler to use $\widetilde{M}_L(n,d)$ and we shall do this. The first result was

Theorem 4.5. [Newstead 1968, Narasimhan/Ramanan 1969] *If $g = 2$, then*

- *$M_L(2,1)$ is an intersection of 2 quadrics in $\mathbb{P}^5$;*
- *$\widetilde{M}_L(2,0) \cong \mathbb{P}^3$.*

For **hyperelliptic curves** of genus $g \geq 3$, a similar description for $M_L(2,1)$ was given by Desale and Ramanan [1976], but for $\widetilde{M}_L(2,0)$ one obtains only a projective embedding of $\widetilde{M}_L(2,0)/i$, where i denotes the action of the hyperelliptic involution on the moduli space. For $n \geq 3$ on hyperelliptic curves, a similar construction gives moduli for bundles with orthogonal or spin structure rather than the basic moduli spaces [Ramanan 1981, Bhosle 1984]; the results in rank 2 are in some sense a fortuitous accident.

Before proceeding further, we recall that the **Picard group** (the group of line bundles) of $\widetilde{M}_L(n,d)$ has been determined.

Theorem 4.6. [Ramanan 1973 for $(n,d) = 1$, Drezet/Narasimhan 1989 in general]

$$\mathrm{Pic}\widetilde{M}_L(n,d) \cong \mathbb{Z}.$$

There is a construction for a generator of $\mathrm{Pic}\widetilde{M}_L(n,d)$ as a "determinant bundle". However, when $(n,d) = 1$, we don't need this. It follows from the rationality of $M_L(n,d)$ and a standard exact sequence that the homomorphism

$$c_1 : \mathrm{Pic}M_L(n,d) \to H^2(M_L(n,d), \mathbb{Z}),$$

given by the first Chern class, is an isomorphism. Moreover $H^2(M_L(n,d), \mathbb{Z}) \cong \mathbb{Z}$ and has a natural positive generator α (see section 3.4 for the case $n = 2$, but true in general).

Now let $\mathcal{L}$ denote the unique positive generator of $\mathrm{Pic}\widetilde{M}_L(n,d)$. We have

Theorem 4.7. (i) [Beauville 1988, Laszlo 1994, Brivio/Verra 1996, van Geemen/Izadi 2001] *For $\widetilde{M}_L(2,0)$, $\mathcal{L}$ is very ample except for $g \geq 3$, C hyperelliptic. In the exceptional case, $\mathcal{L}$ determines a morphism to projective space of degree* 2.

(ii) [Beauville 1991, Brivio/Verra 1999] *For $M_L(2,1)$, $\mathcal{L}$ is very ample in all cases.*

For $n \geq 3$, very little is known.

4.3. **Verlinde formulae.** It is of interest from various points of view to obtain formulae for $h^0(\mathcal{L}^k)$. In our context these numbers are related to the projective embeddings of the previous section. The numbers also arise in physics and, indeed the first viable formulae were obtained by physical methods. These were the original Verlinde formulae.

As an illustration, let us consider the case $(n,d)=1$ and write $M=M_L(n,d)$ for convenience. The first observation is that, because $c_1(K_M)=-2\alpha$, it follows that $h^i(\mathcal{L}^k)=0$ for $k\geq 0$, $i>0$. Hence, by Hirzebruch-Riemann-Roch,

$$h^0(\mathcal{L}^k)=\chi(\mathcal{L}^k)=(\mathrm{ch}\mathcal{L}^k)(\mathrm{td}M)[M]=e^{k\alpha}(\mathrm{td}M)[M].$$

Here ch denotes total Chern character and td$M$ denotes the Todd class of M, which can be computed in terms of the Chern classes of M, or alternatively in terms of the first Chern class and the Pontrjagin classes.

In the case $n=2$, $d=1$, it is known [Newstead 1972] that the total Pontrjagin class $p(M)$ is given by

$$p(M)=(1+\beta)^{2g-2}$$

(for the meaning of β, see section 3.4). We can then express td$M$ formally as

$$\mathrm{td}M=e^{\alpha}\left(\frac{\sqrt{\beta}/2}{sinh(\sqrt{\beta}/2)}\right)^{2g-2}.$$

So

$$h^0(\mathcal{L}^k)=e^{(k+1)\alpha}\left(\frac{\sqrt{\beta}/2}{sinh(\sqrt{\beta}/2)}\right)^{2g-2}[M].$$

Since the intersection numbers of M are known, this can be calculated and we can obtain an expression

$$h^0(\mathcal{L}^k)=(k+1)^{g-1}\sum_{j=1}^{2k+1}\frac{(-1)^{j+1}}{\left(\sin\frac{j\pi}{2k+2}\right)^{2g-2}}.$$

Exercise. Prove that the RHS of this expression is an integer.

4.4. **Torelli theorems.** For any C, the Jacobian $J=M(1,0)$ is an abelian variety. It is not true that J determines C, although in a sense this is true "generically". However, J has additional structure - there is a divisor Θ on J, determined up to translation, with $h^0(\Theta)=1$. Such a divisor is called a *principal polarisation.* The classical theorem of Torelli states that the pair (J,Θ) determines C.

Now suppose $n\geq 2$, $(n,d)=1$. There is nothing remarkable in conjecturing that $M(n,d)$ (perhaps with some added structure) should determine C, since we have a surjective morphism $M(n,d)\to J$. It is however somewhat surprising that $M_L(n,d)$ determines C.

Theorem 4.8. [Mumford/Newstead 1968 for $n=2$, Tyurin 1970 in general] *Suppose $n\geq 2$, $(n,d)=1$. Then $M_L(n,d)$ determines C.*

Idea of proof. We use the **intermediate Jacobian** J_2 associated with $H^3(M_L(n,d))$. There are actually two definitions of this (due to Weil and Griffiths), but these coincide because $H^{0,3} = 0$ since $M_L(n,d)$ is unirational. J_2 possesses a principal polarisation θ determined by $M_L(n,d)$. One shows that $(J_2, \theta) \cong (J, \Theta)$ by using the 13 component of the second Chern class of the universal bundle on $C \times M_L(n,d)$.

4.5. **Reference.** The key reference for rationality is

- A. D. King and A. Schofield, Rationality of moduli of vector bundles on curves, Indag, Math., N. S., 10 (1999), 519-535.

5. Lecture 5 – Geometry II Stratifications

In this lecture, we consider two ways in which the moduli spaces can be stratified.

5.1. **Maximal subbundles.** First suppose $n = 2$. Any vector bundle E then determines a **ruled surface** $\mathbb{P}(E)$ – these have been studied since the 19th century. A line subbundle of E determines a section (or *directrix curve*) σ of $\mathbb{P}(E)$. One can show that the self-intersection number of σ is given by

$$\sigma.\sigma = \deg(E/L) - \deg L.$$

Corrado Segre proved in the 1880s that every ruled surface has a directrix curve σ such that $\sigma.\sigma \leq g$. This theorem was forgotten and was reproved by Nagata [1970] in terms of bundles.

Theorem 5.1. *Every vector bundle E of rank 2 over C possesses a line subbundle L such that*

$$\deg(E/L) - \deg L \leq g.$$

Definition.

$$s(E) := \min_{L \subset E}(\deg(E/L) - \deg L)$$

$$U(s) := \{E \in M(2,d) : s(E) = s\}.$$

The line bundles $L \subset E$ such that $\deg(E/L) - \deg L = s(E)$ are called *maximal subbundles* of E. The subsets $U(s)$ are locally closed in $M(2,d)$ in the Zariski topology and define a stratification of $M(2,d)$ – the *Segre stratification.* Note that

$$E \text{ stable } \Leftrightarrow s(E) > 0$$

$$s(E) \equiv d \bmod 2.$$

Theorem 5.2. [Lange/Narasimhan 1983] *Let $0 < s \leq g$ and $s \equiv d \bmod 2$. Then $U(s)$ is non-empty and irreducible of dimension*

- $s + 3g - 2$ *if* $0 < s < g$
- $4g - 3$ *if* $s = g$.

In particular $U(g-1)$ or $U(g)$ is the unique open stratum.

Remark 5.3. Note that points of $U(s)$ come from exact sequences

$$0 \to L \to E \to L' \to 0$$

with $\deg L' - \deg L = s$. By Riemann-Roch, $h^1(L'^* \otimes L) = s + g - 1$ when $s > 0$. It follows that, if we identify scalar multiples, the non-trivial extensions of the above form can be parametrised by a variety of dimension

$$g + g + h^1(L'^* \otimes L) - 1 = s + 3g - 2.$$

This proves that $\dim U(s) \leq s + 3g - 2$. Once we have Theorem 5.2, it shows also that the general bundle in $E \in U(s)$ has finitely many maximal subbundles. By more refined arguments, one can show

- for $s < g - 1$, the general bundle $E \in U(s)$ has a unique maximal subbundle;

- for $s = g - 1$, the general E has 2^g maximal subbundles – this can be proved using the Porteous formula and intersection theory [Lange 1985].

Now consider general n.

Definition. Let $0 < r < n$. We define

$$s_r(E) := \min_{E' \subset E, \mathrm{rk} E' = r} (r \deg(E/E') - (n-r) \deg E')$$

$$U_r(s) := \{E \in M(n,d) : s_r(E) = s\}.$$

Note that E is stable (semistable) if and only if $s_r(E) > 0$ ($s_r(E) \geq 0$) for all r, $1 \leq r \leq n-1$. We have the following generalisation of the theorem of Segre and Nagata.

Theorem 5.4. [Mukai/Sakai 1985] $s_r(E) \leq r(n-r)g$.

For fixed r, the sets $U_r(s)$ define a stratification of $M(n,d)$.By a dimensional count, one can show that, for general $E \in M(n,d)$, $s_r(E) \geq r(n-r)(g-1)$. Moreover

$$s_r(E) \equiv rd \bmod n.$$

Unless $r = 1$ or $r = n-1$, this is not sufficient to determine the maximal value of $s_r(E)$. However Hirschowitz [1986], following Lange [1983], proved that

$$s_r(E) \leq r(n-r)(g-1) + \varepsilon \text{ with } 0 \leq \varepsilon < n.$$

Theorem 5.5. [Lange/Brambila-Paz 1998, Russo/Teixidor 1999] *Let $0 < s \leq r(n-r)(g-1) + n - 1$ and $s \equiv rd \bmod n$. Then $U_r(s)$ is non-empty and irreducible of dimension*

- $s + (n^2 + r^2 - nr)(g-1) + 1$ *if* $s \leq r(n-r)(g-1)$
- $n^2(g-1) + 1$ *if* $s > r(n-r)(g-1)$.

Remark 5.6. As in rank 2, one can show

- for $s < r(n-r)(g-1)$, the general bundle $E \in U_r(s)$ has a unique maximal subbundle E' – moreover E' and E/E' are both stable;
- for $s = r(n-r)(g-1)$, the general E has finitely many maximal subbundles and, if E' is any one of these, both E' and E/E' are stable – again this number can be computed using the Porteous formula and intersection theory (see Oxbury [2000], Okonek/Teleman [2001] for $r = 1$, Lange/Newstead [2002] for $(r, \deg E') = 1$ and Holla [2002] in general). For $r = 1$, the number is n^g and, in general, these numbers are a form of Gromov-Witten invariant.

5.2. **Brill-Noether theory.** Another stratification of $M(n,d)$ is given by defining, for $k \geq 0$,

$$B(n,d,k) := \{E \in M(n,d) : h^0(E) \geq k\}.$$

Obviously $B(n,d,0) = M(n,d)$. Moreover the $B(n,d,k)$ are Zariski-closed in $M(n,d)$ with $B(n,d,k+1) \subset B(n,d,k)$. (In the literature, $B(n,d,k)$ is usually denoted by $W^{k-1}_{n,d}$, but the notation used here is preferable.)

For $n = 1$, the study of these subvarieties was started by Brill and Noether in the 1870s. Modern versions had to wait until the 1970s and 1980s [Kempf, Kleiman/Laksov, Fulton/Lazarsfeld, Griffiths/Harris, Gieseker, Eisenbud/Harris]. The main results can be expressed in terms of the *Brill-Noether number*

$$\beta(1, d, k) = g - k(k - d + g - 1).$$

Every irreducible component of $B(1, d, k)$ has dimension $\geq \beta(1, d, k)$ and we have

(i) if $\beta(1, d, k) \geq 0$, then $B(1, d, k) \neq \emptyset$;
(ii) if $\beta(1, d, k) > 0$, then $B(1, d, k)$ is connected;
(iii) for C generic, $B(1, d, k)$ has dimension $\beta(1, d, k)$ whenever it is nonempty and $\neq M(1, d)$;
(iv) for C generic, $B(1, d, k)$ is irreducible whenever $\beta(1, d, k) > 0$;
(v) for C generic and $B(1, d, k) \neq M(1, d)$, $\mathrm{Sing}B(1, d, k) = B(1, d, k + 1)$.

The infinitesimal study of $B(1, d, k)$ is governed by the multiplication or *Petri map*

$$H^0(L) \otimes H^0(L^* \otimes K) \to H^0(K).$$

If $\deg L = d$ and $h^0(L) = k$, the cokernel of this map is isomorphic to the dual of the Zariski tangent space to $B(1, d, k)$ at L.

Proposition 5.7. *$B(1, d, k)$ is non-singular of dimension $\beta(1, d, k)$ at L if and only if the Petri map is injective.*

Exercise. Prove Proposition 5.7.

For general n, work started in the mid 1980s [Hirschowitz 1986, Sundaram 1987, Laumon 1991] followed by more extensive results in the 1990s and beyond. Here the Brill-Noether number is given by

$$\beta(n, d, k) = n^2(g - 1) + 1 - k(k - d + n(g - 1))$$

and the Petri map by

$$H^0(E) \otimes H^0(E^* \otimes K) \to H^0(E \otimes E^* \otimes K).$$

It remains true that every component of $B(n, d, k)$ has dimension $\geq \beta(n, d, k)$ and the analogue of Proposition 5.7 holds. However the analogues of statements (i)-(v) can all be false. There are now many results but no clear overall picture.

5.3. **Reference.** There is no generally accessible introduction to maximal subbundles; one needs to study the original papers.

For Brill-Noether theory, the best general reference (though not now completely up to date) is

- V. Mercat, Le problème de Brill-Noether: présentation, http://www.bnt.math.jussieu.fr/presentation.html

6. Lecture 6 – Deformations

In a sense the terms **deformations** and **moduli** mean the same thing. In practice, given some object X_0 in algebraic geometry, the **theory of moduli** concerns the global theory, that is giving a nice structure to he set of all possible deformations of X_0, while **deformation theory** concerns the local theory in the neighbourhood of X_0. In previous lectures, we have been concerned almost entirely with the global theory with only brief references to local properties (for example, the non-singularity of $M(n, d)$). In this last lecture, we shall consider the local theory.

6.1. **Basic ideas.** Deformation theory began with a fundamental series of papers by Kodaira and Spencer in the 1950s in a complex analytic context. In the 1960s it was developed by various authors, principally Kuranishi, M. Artin and Schlessinger, with major contributions from Grothendieck's theory of schemes.

First let X_0 be a non-singular projective variety. By a *deformation* of X_0 *parametrised by* (S, s_0), where s_0 is a point of the variety S, we mean a proper morphism $f : X \to S$, smooth in the neighbourhood of s_0, such that $f^{-1}(s_0) = X_0$. *Isomorphism* of deformations is defined in a local sense, meaning isomorphism in the neighbourhood of s_0.

Definition. The *infinitesimal deformation space* of X_0 is the sheaf cohomology group $H^1(TX_0)$, where TX_0 denotes the tangent bundle of X_0.

Remark 6.1. If we allow the parametrising variety to become a scheme, this has a simple interpretation; $H^1(TX_0)$ can be identified with the set of isomorphism classes of deformations of X_0 parametrised by the scheme $\mathrm{Spec}\mathbb{C}[\epsilon]$ where $\epsilon^2 = 0$.

Now suppose we have any deformation of X_0 parametrised by (S, s_0). It is then easy to see that we have a canonical linear map

$$TS_{s_0} \to H^1(TX_0)$$

from the Zariski tangent space of S at s_0 to the infinitesimal deformation space of X_0. This map is called the *infinitesimal deformation map.* If S is non-singular and $f : X \to S$ is everywhere smooth, there is a global version of this map, namely a map of sheaves

$$TS \to R^1 f_*(TX_f),$$

where TX_f denotes the tangent bundle along the fibres of f.

A fundamental theorem of Kuranishi asserts that X_0 possesses a **local moduli space** (M, m_0). This parametrises a deformation of X_0 with a suitable local universal property, for which in particular the infinitesimal deformation map is an isomorphism.

Remark 6.2.
(i) The space $H^2(TX_0)$ is an "obstruction space", providing possible obstructions to the non-singularity of M at m_0. In particular, if $H^2(TX_0) = 0$, then M is non-singular at m_0.
(ii) The elements of $H^0(TX_0)$ are the infinitesimal automorphisms of X_0. If $H^0(TX_0) = 0$, the automorphism group of X_0 is discrete (and frequently finite), but there can exist non-trivial automorphisms. It follows that the local moduli space need not be injectively parametrised even when $H^0(TX_0) = 0$.

Of course, if C is a non-singular projective curve, then $H^2(TC) = 0$. Moreover $TC = K^{-1}$ has degree $2-2g$. For $g \geq 2$, it follows that $H^0(TC) = 0$ and we can calculate $h^1(TC)$ using Riemann-Roch. Hence

Theorem 6.3. *For $g \geq 2$, the local moduli space of C is non-singular at C and has dimension $h^1(TC) = 3g-3$.*

In other words, a curve of genus g possesses $3g-3$ moduli, a fact first asserted by Riemann.

6.2. **Deformations of vector bundles.** There is a similar (in some ways simpler) theory for deformations of vector bundles. Indeed, if E is a coherent sheaf, we can define its infinitesimal deformation space as $\mathrm{Ext}^1(E,E)$; when E is a vector bundle (locally free sheaf), this space is isomorphic to

$$H^1(EndE),$$

where $EndE$ denotes the bundle of endomorphisms of E. The obstruction space is $H^2(EndE)$, which is 0 when the base space is a curve. So in this case the local moduli space is always non-singular. In fact, if E is stable, the global moduli space $M(n,d)$ is a local moduli space at E and we have

Theorem 6.4. *The tangent space $TM(n,d)_E$ is canonically isomorphic to $H^1(EndE)$. In particular, $M(n,d)$ is non-singular of dimension $n^2(g-1)+1$.*

There is again a global version of this. For $(n,d) = 1$, we have a universal bundle U on $C \times M(n,d)$ and hence a bundle $EndU$. In fact, even though U does not exist when $(n,d) \neq 1$, $EndU$ does exist, and we have

Corollary 6.5. *The tangent bundle $TM(n,d)$ is canonically isomorphic to*

$$R^1 p_{M*}(EndU),$$

where $p_M : C \times M(n,d) \to M(n,d)$ is the natural projection.

Note here that $R^1 p_{M*}(EndU)$ is defined as a sheaf, but in this case it is locally free and can therefore be regarded as a bundle.

6.3. **Deformations of moduli spaces.** In a very interesting paper, Narasimhan and Ramanan [1975] showed that, when $(n,d) = 1$ and L is a line bundle of degree d, deformations of $M_L(n,d)$ exactly copy those of the curve C. We suppose here that $g \geq 2$ and let U denote the universal bundle on $C \times M_L(n,d)$. For any $x \in C$, we write $U_x = U|\{x\} \times M_L(n,d)$. We can consider U as a family of bundles on $M_L(n,d)$ parametrised by C. We can now state the key theorem of Narasimhan and Ramanan in the following slightly modified form (in particular we have globalised the statement (i)).

Theorem 6.6. (i) *The infinitesimal deformation map*

$$TC \to R^1 p_{C*}(EndU)$$

is an injective map of vector bundles.

(ii) *For any $x \in C$,*

(a) $h^0(EndU_x) = 1$;
(b) $h^1(EndU_x) = 1$, *except possibly when* $g = 2$, $n = 2$;
(c) $h^2(EndU_x) = 0$, *except possibly when* $g = 2$, $n = 2, 3$ *or* $g = 3$, $n = 2$.

From this, together with a vanishing theorem of Akizuki and Nakano and the Leray spectral sequence, Narasimhan and Ramanan deduce

Theorem 6.7. (i) *The group of automorphisms of* $M_L(n, d)$ *is finite.*
(ii) $h^i(TM_L(n, d)) = 0$ *for* $i \neq 1$.
(iii) $h^1(TM_L(n, d)) = 3g - 3$.

Actually there is an exceptional case $g = 2$, $n = 3$ in (iii), but this was proved a little later by the same authors using a different method.

Remark 6.8. If we now compare this result with Theorem 6.3, we see that the infinitesimal deformation spaces of C and $M_L(n, d)$ both have dimension $3g - 3$. In fact the proof of Theorem 6.7 gives an isomorphism between these spaces and one can show that the local moduli spaces for these varieties are isomorphic. This gives a local version of the Torelli theorem of Lecture 4.

6.4. **Picard bundles.** In Theorem 6.7, we looked at deformations of the tangent bundle of $M_L(n, d)$. Here we shall discuss deformations of certain other bundles on the moduli space. For line bundles, this is due independently to Kempf and Mukai in the late 1970s; the vector bundle case, which is still under investigation, is due to Biswas, Brambila-Paz and Newstead.

Let us suppose that $(n, d) = 1$ and that U is a universal bundle on $C \times M_L(n, d)$. Let $E \in M(n', d')$ and suppose that

$$nd' + n'd > nn'(2g - 2).$$

Under these conditions, it follows that $h^1(E \otimes F) = 0$ for all $F \in M_L(n, d)$. Hence the direct image $p_{M*}(U \otimes p_C^* E)$ is a locally free sheaf on $M_L(n, d)$. We write

$$W_L(E) = p_{M*}(U \otimes p_C^* E)$$

and refer to $W_L(E)$ as a *Picard bundle.* Our object is to study the deformations of $W_L(E)$.

Theorem 6.9. *Let* $(n, d) = 1$, $g \geq 3$ *and* $nd' + n'd > nn'(2g - 2)$. *Let* $E, E_1, E_2 \in M(n', d')$. *Then*

(i) $\mathrm{Hom}(W_L(E_1), W_L(E_2)) = \begin{cases} 0 & \textit{if } E_1 \ncong E_2 \\ \mathbb{C} & \textit{if } E_1 \cong E_2 \end{cases}$
except possibly when $g = 3$, $n = 2$;
(ii) $h^1(EndW_L(E)) = n'^2(g - 1) + 1$
except possibly when $g = 3$, $n = 2, 3$ *or* $g = 4$, $n = 2$;
(iii) $h^2(EndW_L(E)) = 0$
except possibly when $g = 3$, $n = 2, 3, 4$, $g = 4$, $n = 2$ *or* $g = 5$, $n = 2$.

Note that (ii) tells us that the infinitesimal deformation space of $W_L(E)$ has the same dimension as the moduli space to which E belongs. This suggests that, at least if

$(n', d') = 1$, $M(n', d')$ should act as a moduli space for the bundles $W_L(E)$ on $M_L(n, d)$. This can be proved if all $W_L(E)$ are stable (which is true for example if $n' = 1$), but in general we know only that $W_L(E)$ is simple (by (i)). This is enough to assert that $M(n', d')$ is a local moduli space for every $W_L(E)$.

In the case $n = n' = 1$, much more is known (due to Kempf and Mukai), and in general there are many possible extensions of Theorem 6.9 to consider.

6.5. **Hecke transformations.** We will finish by hinting at one of the ingredients for the proofs of Theorems 6.6 and 6.9. It should be noted that other approaches are possible and may give stronger results, but the method indicated here (due to Narasimhan and Ramanan) has been used in other contexts and seems to be of some interest.

The idea is as follows. Let $(n, d) = 1$, $x \in C$ and consider U_x as above. We consider the projective bundle $\mathbb{P}(U_x)$ over $M_L(n, d)$. A point of $\mathbb{P}(U_x)$ consists of a stable bundle E and a line in the fibre E_x, or equivalently an exact sequence

$$0 \to E \to F \to \mathbb{C}_x \to 0,$$

where F is a vector bundle and $\mathbb{C}_x$ is the torsion sheaf supported at x with fibre $\mathbb{C}$. The bundles F form a family of bundles, parametrised by $\mathbb{P}(U_x)$, of determinant $L(x)$; so we have a rational map

$$\gamma : \mathbb{P}(U_x) - - \to M_{L(x)}(n, d + 1).$$

This map is not everywhere defined since F need not be stable. However, if we restrict to a certain open subset Z of $M_{L(x)}(n, d + 1)$, then $\gamma^{-1}(Z)$ is again an open subset of $\mathbb{P}(U_x)$ and $\gamma : \gamma^{-1}(Z) \to Z$ is a fibration with fibre a projective space. This fibration and the projection of $\mathbb{P}(U_x)$ onto $M_L(n, d)$ both have good cohomological properties, and this enables us to transport cohomology groups across this correspondence and hence sometimes to calculate them. However, this only works if the complement of $\gamma^{-1}(Z)$ in $\mathbb{P}(U_x)$ has sufficiently high codimension. This codimension depends on g and n and is the reason for the exceptional cases in the theorems. In particular, this method seems to be incapable of proving the following conjecture, with which we finish.

Conjecture 6.10. *With the notation of Theorem 6.9,*

$$h^i(End W_L(E)) = 0 \textit{ for all } i \geq 2.$$

6.6. **References.** A good reference for deformations is

- C. S. Seshadri, Theory of moduli, Proc. of Symposia in Pure Mathematics, Vol. XXIX, pp.263-304, AMS, 1975.

The key reference for section 6.3 is

- M. S. Narasimhan and S. Ramanan, Deformations of the moduli space of vector bundles over an algebraic curve, Ann. of Math. 101 (1975), 391-417.