GOOD QUOTIENTS OF REDUCTIVE GROUP ACTIONS - COMBINATORICS IN PROJECTIVE SPACES

JOANNA ŚWIĘCICKA

Introduction

In various situations, for instance connected with construction of moduli spaces for specified category of objects, it is important to describe the space of orbits for an action of an algebraic group on a an algebraic variety. First we show what are the most common obstructions for existence of orbit spaces and give arguments why we can obtain a good substitutes of orbit spaces in the case of an action of a reductive group G. In the next section we give some necessary definitions and theorems and recall main properties of the theory of good quotients. We shall emphasize the role of the maximal torus T of G in the theory of good quotients. The third section is devoted to the combinatorics of actions of an algebraic torus $T = (\mathbb{C}^*)^k$ on $\mathbb{P}^n$. In the fourth chapter we recall Mumford theory ([17]). Next, in the fifth chapter, we give an easy procedure to construct all open, T-invariant subsets of $\mathbb{P}^n$ with good quotients by T. Then we extend the results of previous chapters to the case of actions of any reductive group. We present some generalizations and open problems for actions on other varieties.

The theory of quotients of algebraic varieties was developed by mathematicians of the nineteenth century in the context of the classical Invariant Theory, i.e. as a theory of invariants of actions of classical groups on rings of polynomials. With the development of algebraic geometry, linear algebraic groups, group schemes and their actions on algebraic varieties and scheme, became an object of the independent interest.

One of the possible applications of results from the theory of group actions is the classification theory. Usually the procedure is first to find a correspondence between the set of classifying object and points of an algebraic variety X (e.g. of $\mathbb{P}^n, Grass(n,m), Hilb(X)$,etc.) in such a way that isomorphic objects correspond to points in the same orbit of an action of an algebraic group on X. Then comes the problem of finding the orbit space (or a good substitute of it) of X (or of an open set $U \subset X$). This way of looking at the problem was influenced by the book of David Mumford Geometric Invariant Theory [17]. For a long time Mumford's Theory was the main tool in constructing quotient spaces for group actions.

The aim of this paper is to describe the results obtained by A.Białynicki-Birula and the author in [9] using the methods described in [23] in a way accessible for nonspecialists. These results are based on the following observations:

(i) The case of an action of an algebraic torus T on $\mathbb{P}^n$ all subsets with projective quotient space defined in [17] can be described by some combinatorial

1991 *Mathematics Subject Classification.* 14L30.

Key words and phrases. group actions, orbit spaces, quotients.

This work was completed with the support of KBN Grant 2 PO3A 03808.

finite data, namely by the components of the fixed points intersecting the closure of the orbit.

(ii) The case of an action of a reductive group G can be reduced to the case of an action of a maximal torus $T \subset G$.

1. Reductive group and quotients

In what follows we shall always assume that $\mathbb{K}$ is an algebraically closed field and X is an algebraic $\mathbb{K}$-variety with an algebraic action of an algebraic group G.

Linear algebraic group G is an affine variety, therefore orbits for the action of G on X, in general, are not closed. Hence the orbit space in the naive sense i.e. a variety Y and a morphism $q : X \to Y$ such that, for any $y \in Y$, $q^{-1}(y)$ is an orbit usually does not exists. The first obvious remark is that we can expect only, that, for any $y \in Y$, the fiber $q^{-1}(y)$ contains exactly one closed orbit and we get classification (parameterization) of closed orbits. Secondly, a reasonable assumption is, that a quotient of an affine variety V by the action of G is affine and, moreover, that the ring of regular functions on the quotient is exactly the ring $\mathbb{K}[V]^G$ of G-invariant regular functions on V i.e. regular functions constant on orbits of G. For an action of an arbitrary algebraic group on an affine variety, the ring of such functions may be not finitely generated. Therefore we have to restrict our considerations to actions of algebraic groups of special kind.

Definition 1.1. Let G be a linear algebraic group. *A radical* of G is the maximal solvable, connected, normal subgroup $rad(G)$. *A unipotent radical* $rad_u(G)$ is the unipotent part of $rad(G)$ or the maximal solvable, unipotent connected, normal subgroup of G. A linear algebraic group G is *reductive* if $rad_u(G)$ is trivial. The group G is *semi-simple* if $rad(G) = \{e\}$.
□

We shall see that, for reductive groups and their actions on affine varieties, a good substitute of orbit spaces can be constructed. First we need another definition:

Definition 1.2. An affine algebraic group G is called *geometrically reductive* if, for any linear representation $\rho : G \to Gl_{\mathbb{K}}(n)$ and a non-zero vector $v \in \mathbb{K}^n$, there exists a non-constant G-invariant polynomial function $f \in \mathbb{K}[x_1, \ldots, x_n]$ such that $f(v) \neq 0$.
□

We have the following theorem ([18])

Theorem 1.3. *Let G be a geometrically reductive group acting on an affine variety X. Then the subalgebra $\mathbb{K}[V]^G$ is finitely generated over $\mathbb{K}$.*
□

An algebraic linear group G is geometrically reductive if and only if it is reductive. In fact a famous theorem by Haboush in [14] asserts that any reductive group is geometrically reductive. The converse was proved in [16].

It follows that if V is an affine algebraic variety with an action of a reductive group G then we have a G-invariant morphism $q : V \to Spec(\mathbb{K}[V]^G)$. The following lemma (see e.g. Key Lemma in [12]) implies that points of $Spec(\mathbb{K}[V]^G)$ correspond to closed orbits of the action of G on V:

Lemma 1.4. *Let G be reductive and let V be an affine G-variety. Assume that Z_1, Z_2 are G-invariant, closed subsets of V and $Z_1 \cap Z_2 = \emptyset$. Then there exists a function $f \in \mathbb{K}[V]^G$ such that $f(Z_1) = 0$ and $f(Z_2) = 1$.*
□

It was proved in [20], that if G is not reductive than there exists an action on an affine variety V, such that $\mathbb{K}[V]^G$ is not finitely generated.

We end this Section with Theorem 1.5 showing, that in the case of non-reductive group (e.g. G_a) acting on an affine variety we can obtain a ring of regular functions on any quasi-affine variety. There are examples of quasiaffine varieties V such that $\mathbb{K}[V]$ is not finitely generated $\mathbb{K}$-algebra (e.g.[19]).

Theorem 1.5. (Theorem 1 [24]) *Let $\mathbb{K}$ be a field and R an integrally closed $\mathbb{K}$-algebra. Then the following properties are equivalent:*

(i) *There exists a quasi-affine irreducible, reduced $\mathbb{K}$-variety V such that $R \simeq \mathbb{K}[V]$.*
(ii) *There exists an affine irreducible, reduced $\mathbb{K}$-variety V and a regular action of $G_a = (\mathbb{K}, +)$ on V defined over $\mathbb{K}$ such that $R \simeq \mathbb{K}[V]^{G_a}$.*

□

In the next section we shall define a notion of a *good quotient* of a G-variety X by an action of a (reductive) group G as a G-invariant morphism $q : X \to Y$ which locally comes from the affine situation.

2. Good quotients - main definitions and theorems

From now on we shall assume that G is an algebraic reductive group. Moreover we shall assume that $\mathbb{K}$ has the characteristic equal 0, though several results are valid in any characteristic.

Definition 2.1. Let Y be an algebraic space (not necessarily an algebraic variety) with a trivial action of G. A G-morphism $q : X \to Y$ is said to be *a good quotient* if the following conditions are satisfied:
i) q is affine,
ii)$\mathcal{O}_Y = q_*(\mathcal{O}_X)^G$.
If $q : X \to Y$ is a good quotient then the space Y is called *a quotient space* of X by G and is denoted by $X /\!/ G$. A good quotient $q : X \to Y$ is *a geometric (good) quotient* if, for any $y \in Y$ the variety $q^{-1}(y)$ contains exactly one orbit. In this case we denote the quotient space by X/G.
□

If $q : X \to Y$ is a good quotient of X by an action of G then q has several nice properties. In particular:

2.2. Assume that $q : X \to Y$ is a good quotient of X by an action of an algebraic group G. Then

(i) for any algebraic variety Z and a morphism $Z \to Y$ the projection on the second factor $q' : X \times_Y Z \to Z$ is a good quotient of $X \times_Y Z$ by G (with respect to the induced action);
(ii) the quotient space Y is an algebraic variety if and only if X has a covering by G-invariant affine open sets.;

(iii) if V_1, V_2 are G-invariant closed subsets of X and $V_1 \cap V_2 = \emptyset$, then $q(V_1) \cap q(V_2) = \emptyset$. In particular, for any $x_1, x_2 \in X$
$$q(x_1) = q(x_2) \Leftrightarrow \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset.$$
□

Another important result is the following

Proposition 2.3. (Prop. 1.1 in [7]). *Let X be a variety with an action of a reductive group G. Let U_1, U_2 be two open G-invariant subsets. If there exist good quotients $U_i \to U_i /\!\!/ G$ for $i = 1, 2$ then there exists a good quotient of $U_1 \cap U_2$.*
□

Definition 2.4. Let U be an open G-invariant subset of X. Then U is *G-saturated in X* if, for any $x \in U$, the closures of the G-orbit $G \cdot x$ in U and in X coincide.
□

Remark 2.5. Assume that U is G-saturated in X and a good quotient $q : X \to X /\!\!/ G$ exists. Then $q|U : U \to U /\!\!/ G$ is a good quotient and the induced morphism $U /\!\!/ G \to X /\!\!/ G$ is an open embedding.
□

Definition 2.6. An open G-invariant subset U in X is called *G-maximal* if a good quotient $U \to U /\!\!/ G$ exists and if U is maximal in X with respect to saturated inclusion in the family of all open, G-invariant subsets of X that admit good quotients with respect to the action of G.
□

It follows from Definition 2.6 and the Remark 2.5 that any open G-invariant subset U of X with a good quotient is contained in a G-maximal set as a saturated subset. Therefore

Corollary 2.7. *Let $V \subset X$ be a G-invariant open subset such that a good quotient $V /\!\!/ G$ exists. Then there exists a G-maximal set U with a good quotient $q : U \to U /\!\!/ G$ and an open set $W \subset U /\!\!/ G$ such that $V \subset U$ and $V = q^{-1}(W)$.*
□

It follows from the results of [21] and 2.2 (iii) that if $G = T$ is an algebraic torus, X is a normal algebraic variety and a good quotient of X by T exists, then $X /\!\!/ T$ is also a normal algebraic variety.

In every complex normal G-variety there are finitely many G-maximal sets. In fact, Main Theorem in [1] states:

Theorem 2.8. *Let X be a complex normal variety with an action of a reductive group G. Then the number of G-maximal subsets of X is finite.*
□

It follows that in order to study open subsets in X with good quotients by the action of G, it is sufficient to describe this finite set of G-maximal subsets of X and then every open set with a good quotient is obtained as an inverse image of an open subset in one of the quotient spaces. Assume that $U \subset X$ is an open, G-invariant set with a good quotient $p : U \to U /\!\!/ G$. If the quotient space $U /\!\!/ G$ is complete (e.g. projective) then U is a G-maximal. But there are examples of G-maximal subsets of projective varieties with non-complete quotient space (see [5]).

There are several results that are useful when studying the existence of a good quotient of an open subset of an algebraic variety. The most important is the possibility to reduce the question of the existence of quotient by G to the case of actions of one dimensional tori (Theorem 5.1 in [6]). Corollary 5.2 in [6] gives strong motivation to study actions of algebraic tori of any dimension.

Theorem 2.9. *Let G be a reductive group acting on a normal algebraic variety X. The following conditions are equivalent:*

(i) *there exists a good quotient $X \to X/\!\!/G$,*
(ii) *there exists a good quotient $X \to X/\!\!/H$ for every reductive subgroup $H \subset G$,*
(iii) *there exists a good quotient $X \to X/\!\!/T$, where T is a fixed maximal torus of G,*
(iv) *there exists a good quotient $X \to X/\!\!/T_0$, for every one-dimensional torus T_0 contained in a fixed maximal torus T in G.*

□

This result reduces the question of existence of "global"quotient by G to the problem of existence of "global" quotient by smaller group. The next result reduce the same question to a "local" problem:

Theorem 2.10. (Theorem C in [10]) *Let a reductive group G act on an algebraic variety X. Assume that, for any two points $x, y \in X$, there exists a an open G-invariant neighborhood $V_{x,y}$ of x and y such that there exists a good quotient $V_{x,y} \to V_{x,y}/\!\!/G$. Then there exists a good quotient $X \to X/\!\!/G$. If all quotient spaces $V_{x,y}/\!\!/G$ are algebraic varieties, then $X/\!\!/G$ is also an algebraic variety.*
□

In Section 4 we shall shall use the following result in order to obtain a simple proof some results of [17]:

Theorem 2.11. (Theorem B in [10]) *Assume that there exists an affine G-equivariant morphism $\Phi : X \to Z$ of a normal algebraic variety X with an action of G into an algebraic space Z with the trivial action of G. Then there exists a good quotient $X \to X/\!\!/G$. Moreover, if Z is an algebraic variety (Z is quasi-projective), then $X/\!\!/G$ is an algebraic variety ($X/\!\!/G$ is quasi-projective, resp.).*
□

In next Sections we shall use the construction given in [23]. We recall here main ideas and results.

In the rest of this section we shall assume that $\mathcal{U} = \{U_i : i \in \mathcal{I}\}$ is a collection of open subsets in X with indices in a set $\mathcal{I}$. (Elements of $\mathcal{I}$ are sometimes called vertices).

Definition 2.12. *A cell* in $\mathcal{I}$ is any finite subset of $\mathcal{I}$. For any cell $I \subset \mathcal{I}$ we define an open set

$$U(I) = \bigcap_{i \in I} U_i, .$$

For any collection Π of cells let

$$U(\Pi) = \bigcup_{I \in \Pi} U(I).$$

□

We come back to the case of an action of G (reductive) on X and in the rest of the paper we shall assume that, for any $i \in \mathcal{I}$, the set U_i is G-invariant and that there exists a good quotient $q_i : U_i \to U_i /\!\!/ G$.

Corollary 2.13. *Assume that the group G acts on an algebraic normal variety X and $U_i, i \in \mathcal{I}$ is a set of G-invariant open sets with a good quotients $U_i /\!\!/ G$. For any cell $I \subset \mathcal{I}$ a good quotient $U(I) \to U(I) /\!\!/ G$ exists.*

Proof. This is an immediate implication of 2.3 and definition of $U(I)$.
□

Definition 2.14. Assume that an algebraic, reductive group G acts on X and open subsets U_i, $i \in \mathcal{I}$ are G-invariant. Let I be a cell in $\mathcal{I}$. A *boundary vertex* of a cell I is any $i \in \mathcal{I}$ such that $U(I) \subset U_i$ and $U(I)$ is not saturated in U_i. The *boundary* $\delta(I)$ of I is the set of all boundary vertices of I.
□

Definition 2.15. Assume that the group G acts on an algebraic normal variety X and $U_i, i \in \mathcal{I}$ is a set of G-invariant open sets with a good quotients $U_i /\!\!/ G$. Consider a collection $\Pi = \{I_1, \ldots, I_s\}$ of cells in $\mathcal{I}$. We shall call Π a *good collection* of cells if, for any cells $I_i, I_j \in \Pi$, the following two conditions are satisfied:
(i) $I_i \cap I_j \neq \emptyset$
(ii) $I_i \cap I_j \subset \delta(I_i) \Rightarrow I_i \cap I_j \in \Pi$
□

We are now ready to formulate the main result of [23], Theorem 2.6:

Theorem 2.16. *Assume that, for any $i \in \mathcal{I}$, there exists a good quotient*
$$q_i : U_i \to U_i /\!\!/ G.$$
Then, for any good collection Π of cells in $\mathcal{I}$ there exists a good quotient
$$q : U(\Pi) \to U(\Pi) /\!\!/ G.$$
□

In order to state the next result we shall need two more definitions;

Definition 2.17. We say that X is an A_2-variety if, for any $x, y \in X$, there exists an open affine set $V_{x,y}$ containing x and y.
□

Remark 2.18. We can define A_∞-varieties: A_∞-variety is a variety such that any finite set of points is contained in an affine neighborhood. According to results of [15] (see also [26]) a smooth variety Y is quasi-projective if and only if it is an A_∞-variety.

Definition 2.19. Let X be a variety with an action of G and assume that U is an open G-invariant subset with a good quotient by G and the quotient space is A_2 variety. U is called $(G,2)$-maximal, if U is maximal with respect to the saturated inclusion in the set of open subsets with A_2-varieties as quotients.
□

In the construction of T-maximal subsets of $\mathbb{P}^n$ we shall use the following theorem:

Theorem 2.20. (Theorem 5.7 [23]) *Let X be a projective smooth G-variety with $Pic(X) = \mathbb{Z}$ and let $\mathcal{I}$ be the set of all subsets with projective quotients. Assume that U is $(G, 2)$-maximal. Then there exists a good collection Π of cells in $\mathcal{I}$ such that*

$$U = U(\Pi).$$

□

3. Combinatorics of actions of algebraic tori on $\mathbb{P}^n$

Let an algebraic torus $T = (\mathbb{C}^*)^k$ act on $\mathbb{P}^n$. The group $\mathsf{X}(T)$ of characters of T can be identified to $\mathbb{Z}^k$. Then for any $t = (t_1, \ldots, t_k) \in T$ and $\chi = (m_1, \ldots, m_k) \in \mathsf{X}(T)$,

$$\chi(t) = t^\chi = \prod_{i=1}^{k} t_i^{m_i}.$$

It is well known that any action of T on $\mathbb{P}^n$ is given (in a suitable homogeneous coordinate system) by a finite set of characters. More precisely we can choose a homogeneous coordinate system and characters $\chi_0, \ldots, \chi_n \in \mathsf{X}(T) \simeq \mathbb{Z}^k$ such that, for any $t \in T$ and $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$,

$$t(x) = (t^{\chi_0} \cdot x_0, \ldots, t^{\chi_n} \cdot x_n).$$

Denote by $Ch(T)$ or Ch the set of these characters.

Remark 3.1. Notice that any set of $n+1$ points in $\mathbb{Z}^k$ corresponds and determines an action of an algebraic torus $T \simeq (\mathbb{C}^*)^k$ on $\mathbb{P}^n$.
□

We may consider characters of T as an integral lattice in the Euclidean space $\mathsf{X}_\mathbb{R} = \mathsf{X}(T) \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^k$. To any point $x \in \mathbb{P}^n$ we shall attribute a set of characters corresponding to non-zero coordinates of x and define a convex polytope in $\mathbb{R}^k$:

$$P(x) = conv\{\chi_i; x_i \neq 0\}.$$

We denote by $vert(P(x))$ the set of vertices of the convex polytope $P(x)$ and we write $P \prec Q$ if P is a face of a convex polytope Q. Notice that there is a one-to-one correspondence between different elements of the set Ch and irreducible components of the subvariety of fixed points in $\mathbb{P}^n$. Namely to any $\chi \in Ch$ there corresponds a set $X_\chi \subset (\mathbb{P}^n)^T$ defined by the condition:

$$(x_0, \ldots, x_n) \in X_\chi \Leftrightarrow (x_i \neq 0 \Rightarrow \chi_i = \chi)).$$

We shall use collections of convex polytopes to describe open, T-invariant subsets of projective space with good quotients. Below we give necessary definitions and lemmas. Proofs of these lemmas may be found in [9].

Let T be a k-dimensional torus acting on $\mathbb{P}^n$ with characters $\chi_0, \ldots, \chi_n$ as above.

Definition 3.2. A convex polytope with vertices in the set $Ch = \{\chi_0, \ldots, \chi_n\}$ is called a distinguished polytope or d-polytope. For any set $V \subset \mathbb{P}^n$ we define the combinatorial closure $C(V)$ of V:

$$x \in C(V) \Leftrightarrow \exists y \in V \text{ such that } P(x) = P(y).$$

A set V is combinatorially closed (combinatorially defined) if $V = C(V)$.
□

Obviously, all combinatorially defined sets are T-invariant.

Lemma 3.3. *For any d-polytope P there exists a point $x \in \mathbb{P}^n$ such that $P(x) = P$. For any $x \in \mathbb{P}^n$*

$$dim(\overline{T \cdot x}) = dim P(x).$$

If $y \in \overline{Tx}$, then $P(y)$ is a face of $P(x)$.
□

Lemma 3.4.

$$vert(P(x)) = \{\chi \in Ch : X_\chi \cap \overline{Tx} \neq \emptyset\}.$$

□

Lemma 3.5. *Let Δ be a collection of d-polytopes. The set*

$$\bigcup_{P \in \Delta} \{x \in \mathbb{P}^n : P = P(x)\}$$

is open if and only if for any $P \in \Delta$ and any d-polytope Q,

$$P \subset Q \Rightarrow Q \in \Delta$$

□

Definition 3.6. For any d-polytope P let $U(P) = \{x \in \mathbb{P}^n : P \subset P(x)\}$ For any collection Δ of d-polytopes we define a combinatorially closed set

$$U(\Delta) = \bigcup_{P \in \Delta} U(P).$$

□

Corollary 3.7. *The set $U(\Delta)$ is open. In particular, for any d-polytope P, the set $U(P) = \{x \in \mathbb{P}^n : P \subset P(x)\}$ is open.*
□

Lemma 3.8. *Assume, that Δ_1, Δ_2 are two collections of d-polytopes and $\Delta_1 \subset \Delta_2$. Then $U(\Delta_1) \subset U(\Delta_2)$ and $U(\Delta_1)$ is T-saturated in $U(\Delta_2)$ if and only if for any $P \in \Delta_2,\ Q \in \Delta_1$*

$$P \prec Q \Rightarrow P \in \Delta_1.$$

□

An intersection of d-polytopes usually is not a d-polytope. Therefore we shall define smaller convex polytopes: elementary polytopes (e-polytopes).
Let P° denote the relative interior of a d-polytope P.

Definition 3.9. Let $P_1, ..., P_r$ be a collection of d-polytopes and

$$E = \bigcap_{i=1}^{k} P_i$$

Then E is an e-polytope if for any d-polytope P

$$E \cap P^\circ \neq \emptyset \Rightarrow E \subset P.$$

For any e-polytope E let $U(E) = \{x \in \mathbb{P}^n : E \subset P(x)\}$ and for any collection Ω of e-polytopes we define

$$U(\Omega) = \bigcup_{E \in \Omega} U(E).$$

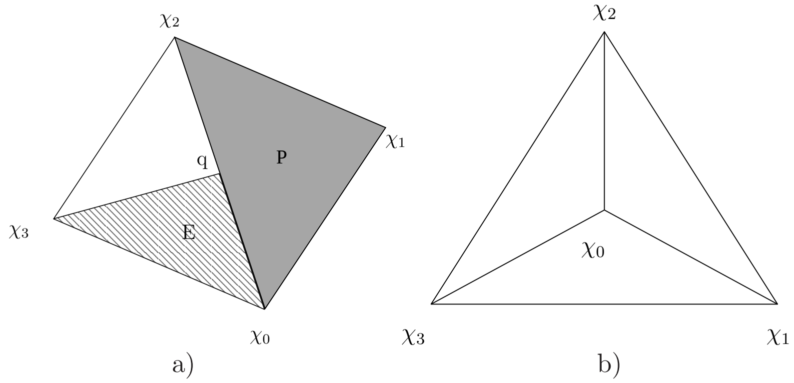


FIGURE 1. d-polytopes and e-polytopes.

Example 3.10. Let $(\mathbb{C}^*)^2$ act on $\mathbb{P}^3$ with characters $Ch = \{\chi_0, \chi_1, \chi_2 \chi_3\}$ as on Fig.1 a) or b).

Then in the case a) we have $P = conv(\chi_0, \chi_1, \chi_2)$ is a d-polytope but not an e-polytope and $E = conv(\chi_0, \chi_3, q)$ is an e-polytope but is not a d-polytope. The polytope consisting of the point q is also an e-polytope (but not a d-polytope).In the case b) the only d-polytope that is not an e-polytope is "the general polytope" $P = conv(\chi_0, \dots, \chi_3)$.
□

Notice that any $\chi \in Ch$ considered as a convex polytope (of dimension 0) is a d-polytope and an e-polytope. Obviously, if Ω is a collection of e-polytopes then $U(\Omega)$ is combinatorially closed and hence T-invariant. Moreover, it follows immediately from 3.5 that

3.11. For any collection Ω of e-polytopes $U(\Omega)$ is open.
□

Obviously, the following results follow from deinitions

3.12. Let P be a d-polytope with vertices $\{\chi_1 \dots, \chi_l\}$ and Ω is a collection of all e-polytopes contained in P.Then

$$U(P) = \bigcap_{E \in \Omega} U(E) = \bigcap_{i=1}^{l} U(\chi_i)$$

□

3.13. For any d-polytopes P, P_1 and any e-polytopes E, E_1 we have

(a) $P_1 \subset P \Leftrightarrow U(P) \subset U(P_1)$,
(b) $E \subset P \Leftrightarrow U(P) \subset U(E)$,
(c) $E \subset E_1 \Leftrightarrow E \prec E_1$,
(d) assume that $P_1 \subset P$. Then $U(P)$ is saturated in $U(P_1)$ unless $P_1 \prec P$,
(e) assume that $E \subset P$. Then $U(P)$ is saturated in $U(E)$ unless E is contained in a face of P,
(f) assume that $E_1 \subset E$ and $E_1 \neq E$. Then $U(E)$ is not saturated in $U(E_1)$.

□

Corollary 3.14. *Let $E_1, \dots, E_s$ be any collection of e-polytopes, let Q be the convex hull of $E_1, \dots, E_s$ and Π be a collection of d-polytopes containing Q. Then*

$$\bigcap_{i=1}^{s} U(E_i) = \{x \in \mathsf{X}_{\mathbb{R}}(T) : Q \subset P(x)\} = \bigcup_{P \in \Pi} U(P).$$

Moreover if $P_1, \dots, P_m$ is any collection of d-polytopes such that $\bigcap_{i=1}^{m} P_i = Q$ (e.g P_i are minimal with respect to inclusion d-polytopes in Π) then

$$\bigcap_{i=1}^{s} U(E_i) = U(P_1) \cup \cdots \cup U(P_m).$$

□

Example 3.15. Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{P}^3$ with characters $Ch = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ as on Picture 1 a). Let $\Pi = \{E_1, E_2\}$ be a collection of two e-polytopes: $E_1 = conv(\chi_0, \chi_1, A)$ $E_2 = conv(\chi_1, \chi_2, A)$. Then

$$U(E_1) = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 | x_0 \cdot x_1 \neq 0 \, and \, x_2 \neq 0\} \cup \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 | x_0 \cdot x_1 \neq 0 \, and \, x_3 \neq 0\}$$
$$= U(P_1) \cup U(P_2)$$

where $P_1 = conv(\chi_0, \chi_1 \chi_2)$ and $P_2 = conv(\chi_0, \chi_1, \chi_3)$.
□

Definition of e-polytopes implies that any e-polytope E is uniquely determined by a choice of a point in the relative interior of E.

3.16. Let E be an e-polytope and $p \in \mathsf{X}_{\mathbb{R}}(T)$. If $p \in E^{\circ}$ then, for any d-polytope P,

$$E \subset P \Leftrightarrow p \in P$$

hence

$$U(E) = \{x \in \mathbb{P}^n : p \in P(x)\}.$$

□

4. Sets of semistable points in $\mathbb{P}^n$.

In this section we shall recall briefly the theory of good quotients of sets of semistable points. This theory was created and developed by D.Mumford in his book Geometric Invariant Theory [17].

Let X be an algebraic variety with an action of a reductive group G. Assume that $\mathcal{L} \xrightarrow{\pi} X$ is a line bundle on X.

Definition 4.1. A *G-linearization* of a line bundle $\mathcal{L}$ on X is an action of G on $\mathcal{L}$ compatible with the action of G on X, i.e. a morphism $\overline{\sigma}$ such that the following diagram is commutative

$$\begin{tikzcd} G \times \mathcal{L} \arrow[r, "\overline{\sigma}"] \arrow[d, "1 \times \pi"'] & \mathcal{L} \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

where σ corresponds to the action of G on X. Each $g \in G$ induces a linear map $\mathcal{L}_x \to \mathcal{L}_{\sigma(g,x)}$ of fibers over $x \in X$ and $\sigma(g, x)$ respectively. In particular, for any G-invariant open set $U \subset X$, we have the induced action of G on $\Gamma(\mathcal{L}, U)$. A line bundle with a chosen linearization is called a *G-linearized line bundle*.

□

Usually linearized line bundles admit various linearizations. But we have the following

Proposition 4.2. *(Prop 1.4* [17]*) Let a connected algebraic group G act on an algebraic variety X. Assume that the group of characters of G is trivial. Then every line bundle admits at most one linearization.*

Remark 4.3. The proof of the above Proposition relies on the following fact: if $\overline{\sigma}$ and $\overline{\sigma_1}$ are two linearizations of the line bundle $\mathcal{L}$ (with respect too an action σ) then there exists a character $\alpha : G \to G_m$ of G, such that $\overline{\sigma_1}$ and $\overline{\sigma}$ differ by α, i.e. for any $g \in G$ and $x \in X$

$$(*) \qquad \overline{\sigma_1}|\mathcal{L}_x = \alpha(g)^{-1} \cdot \overline{\sigma}|\mathcal{L}_x.$$

On the other hand, if α is a character of G and $\overline{\sigma}$ is a linearization of $\mathcal{L}$, then $(*)$ defines a new linearization.

□

If G is semi-simple then the character group of G is trivial and hence, according to the above proposition, for any line bundle $\mathcal{L}$ on a G-variety X, there exists at most one G-linearization.

Let $\alpha : G \to G_m$ be a character of G. We fix one linearization $\overline{\sigma}$ of $\mathcal{L}$ and we denote by $\mathcal{L}_\alpha$ the linearized line bundle defined by $\overline{\sigma_1}$ in $(*)$. According to this notation, the line bundle $\mathcal{L}$ linearized by $\overline{\sigma}$ is denoted by $\mathcal{L}_0$.

Proposition 4.4. *(*Cor. 1.6 *in* [17]*) Assume that X is a normal algebraic variety with an action of a connected algebraic group G and $\mathcal{L}$ is a line bundle on X. Then there exists $n \in \mathbb{Z}$ such that $\mathcal{L}^n$ admits a G-linearization.*

□

Two propositions mentioned below follow easily from the characterization of ample line bundles in [13].

Proposition 4.5. *Assume that $X \to X/\!\!/G$ is a good quotient. Then, for any G-linearized line bundle $\mathcal{L}$ on X we have the induced line bundle $(p_*(\mathcal{L}))^G$ on the quotient $X/\!\!/G$. Moreover, if $(p_*(\mathcal{L}))^G$ is ample, then $\mathcal{L}$ is ample.*

□

Proposition 4.6. *Assume that $X \to X/\!\!/G$ is a good quotient and $\mathcal{E}$ is a linear bundle on $X/\!\!/G$. Then $p^*(\mathcal{E})$ is a G-linearized line bundle on X and $p^*(\mathcal{E})$ is ample when $\mathcal{E}$ is ample.*

□

For any G-linearized line bundle $\mathcal{L}$ on X we shall define open, G-invariant subsets of semistable and of stable points. For any $s \in \Gamma(X, \mathcal{L})$, let *sup*$(s)$ denote the set $\{x \in X : s(x) \neq 0\}$. For any $s \in \Gamma(X, \mathcal{L})$, the set *sup*$(s)$ is open. We say that s is a section with affine support, if *sup*(s) is affine.

Definition 4.7. Let $\mathcal{L}$ be a G-linearized line bundle over a G-variety X. A point $x \in X$ is said to be *semistable* if there exists $n \in \mathbb{N}$ and a section $s \in (\Gamma(X, (\mathcal{L})^n)^G$ with affine support, such that $x \in sup(s)$. The set of semistable points in X with respect to a G-linearized line bundle $\mathcal{L}$ is denoted by $X^{ss}_G(\mathcal{L})$. A point $x \in X$ is said to be *stable (properly stable)* if it is semistable and the orbit $G \cdot x$ is closed and of maximal dimension in $X^{ss}_G(\mathcal{L})$ (resp., the orbit is closed, of and the isotropy group is finite). The set of stable points is denoted by $X^s_G(\mathcal{L})$. We shall often omit the subscript G when there is no ambiguity and sometimes we shall write X^{ss} or X^s when the bundle $\mathcal{L}$ is fixed.

□

It follows directly from the definition, that the set $X^s(\mathcal{L})$ is open, G-invariant and saturated in $X^{ss}(\mathcal{L})$.

We have the following Lemma

Lemma 4.8. *Let X be an algebraic variety with an action of G, Y a quasi-projective variety and let $\pi : X \to Y$ be an affine morphism. Then there exists a linearized line bundle $\mathcal{L}$ over X such that $X = X^{ss}(\mathcal{L})$.*

Proof.

Let $\mathcal{L}_1$ be a very ample line bundle on Y. Then there exist sections $s_1, \dots, s_m$ such that $sup(s_i)$ form a covering of Y by affine sets. Sections $s_1, \dots, s_m$ induce G-invariant sections $t_1, \dots, t_m$ of the line bundle $\pi^*(\mathcal{L}_1)$. The sets $sup(t_i)$ form an affine covering of X. Hence $X = X^{ss}(\pi^*(\mathcal{L}))$.

□

Moreover the following is obvious:

4.9. Assume that $\mathcal{L}$ is a linearized line bundle on X. For any $n \in \mathbb{N}$ consider the line bundle $\mathcal{L}^n$ with the induced linearization. Then

$$X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}^n) \text{ and } X^s(\mathcal{L}) = X^s(\mathcal{L}^n).$$

□

It is easy to see that, for any $\alpha \in \mathsf{X}(T)$ and any $m \in \mathbb{N}$

$$X^{ss}(\mathcal{L}_\alpha) = X^{ss}(\mathcal{L}^m_{m\cdot\alpha}) \text{ and } X^s(\mathcal{L}_\alpha) = X^s(\mathcal{L}^m_{m\cdot\alpha}).$$

We have the following

Lemma 4.10. *Let $\mathcal{L}_0$ be a linearized line bundle on X and $\alpha \in \mathbb{Z}^k$. Then $x \in X^{ss}(\mathcal{L}_\alpha)$ if and only if there exists $n \in \mathbb{N}$ and a semi-invariant global section of $\mathcal{L}_0^n$ with the weight $n\cdot\alpha$ such that $x \in sup(s)$ and $sup(s)$ is affine. Similarly $x \in X^s(\mathcal{L}_\alpha)$ if and only if $x \in X^{ss}(\mathcal{L}_\alpha)$ and the orbit $G \cdot x$ is closed and of maximal dimension in $X^{ss}(\mathcal{L}_\alpha)$.*

The following theorem may be considered as the main result of [17]

Theorem 4.11. (comp 1.10 in [17]) *Let X be an algebraic normal variety with an action of G and $\mathcal{L}$ be a G-linearized line-bundle on X. There exists a good quotient*

$$q : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L}) /\!\!/ G.$$

The quotient spaces $X^{ss}(\mathcal{L}) /\!\!/ G$ and $X^{ss}(\mathcal{L})$ are quasi-projective. The set $X^s(\mathcal{L})$ is a saturated subset of $X^{ss}(\mathcal{L})$ and the morphism $q|X^s$ is a geometric good quotient.

□

Proof. We can choose a finite set $s_0, \dots, s_k \in (\Gamma(X, \mathcal{L}^n))^G$ (for some $n \in \mathbb{N}$) such that the supports $sup(s_i)$ for $i = 0, ..., k$ form a covering of $X^{ss}(\mathcal{L})$. Consider a $\mathbb{K}$-linear subspace V of $\Gamma(X^{ss}, \mathcal{L})$ generated by $s_i, i = 0, \dots, k$ restricted to $X^{ss}(\mathcal{L})$ and the induced morphism $\phi : X^{ss}(\mathcal{L}) \to \mathbb{P}(V)$. Obviously ϕ is an affine morphism and since s_i were G-invariant, ϕ is constant on G-orbits. We use now Theorem 2.11

and infer that there exists a good quotient $q : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})/\!\!/G$ and the quotient space is quasi-projective. According to 2.2 (iii) $q(x_1) = q(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2}$ therefore every fiber of $q|X^s \to X^s/\!\!/G$ is an orbit.
□

In [9] we have the following Lemma:

Lemma 4.12. *Let G be a reductive group acting on a projective and smooth variety X with $Pic(X) = \mathbb{Z}$. Let U be a G-invariant open subset of X such that a good quotient $U \to U/\!\!/G$ exists and $U/\!\!/G$ is quasi-projective. Then there exists a G-linearized line bundle $\mathcal{L}$ on X such that U is contained in $X^{ss}(\mathcal{L})$ as a G-saturated subset.*
□

We shall assume now that $G = T$ is an algebraic torus.

Let $\mathcal{L}$ be a T-linearized line bundle over X. Consider any linearization $\overline{\sigma}$ of $\mathcal{L}^m$ and the linearization $\overline{\sigma_0}$ induced on $\mathcal{L}^m$ by the chosen linearization of $\mathcal{L}$. Then, according to 4.3, there exists $\alpha \in \mathbb{Z}^k \simeq \mathsf{X}(T)$, a character of T such that the morphisms induced on the fibers of $\mathcal{L}^m$ by $\overline{\sigma}$ and $\overline{\sigma_1}$ differ by α. If we have a linearized line bundle $\mathcal{L}_0$ then, for any $m \in \mathbb{N}$, we denote by $\mathcal{L}_0^m$ the line bundle $\mathcal{L}^m$ with the linearization induced by $\mathcal{L}_0$. By abuse of notation we shall define, for any $\alpha \in \mathsf{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ the set of semistable (stable) points with respect to a "rational linearization" $\mathcal{L}_\alpha$ of $\mathcal{L}$.

Definition 4.13. Assume T acts on X and we have chosen a T-linearized line bundle $\mathcal{L}_0$. For any $\alpha \in \mathsf{X}_{\mathbb{Q}}(T)$ we define $X^{ss}(\mathcal{L}_\alpha)$ (resp. $X^s(\mathcal{L}_\alpha)$) as the set of semistable (resp. stable) points corresponding to the linearized line bundle $\mathcal{L}^m_{m\cdot\alpha}$, where $m \in \mathbb{N}$ is any integer, such that $m \cdot \alpha \in \mathsf{X}(T)$. The set $X^{ss}(\mathcal{L}_\alpha))$ (resp. $X^s(\mathcal{L}_\alpha)$) is called the set of α-semistable (resp. α-stable) points
□

We shall denote the space $\mathsf{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\mathsf{X}_{\mathbb{Q}}(T)$. It follows from Lemma 4.10 that

Corollary 4.14. *Let $\mathcal{L}_0$ be a linearized line bundle on X and $\alpha \in \mathsf{X}_{\mathbb{Q}}(T)$. Then $x \in X$ is α-semistable if and only if there exist an integer $m \in \mathbb{N}$, $m \cdot \alpha \in \mathsf{X}(T)$ and a semi-invariant section s of $\mathcal{L}_0$ with the weight $m \cdot \alpha$ such that $x \in sup(s)$ and $sup(s)$ is affine. A point $x \in X^{ss}(\mathcal{L}_\alpha)$ is α-stable if the orbit $T \cdot x$ is of maximal dimension and closed in $X^{ss}(\mathcal{L}_\alpha)$*
□

5. T-maximal subsets of $\mathbb{P}^n$

In this Section we shall assume that $X = \mathbb{P}^n$ and $G = T \simeq (\mathbb{C}^*)^k$ is an algebraic torus. We shall use now the results of the last two Sections to describe open, T-invariant subsets of $\mathbb{P}^n$ with good quotients. First notice that linearization of the line bundle $\mathcal{O}(1)$ is equivalent to the choice of the a lifting of the action of T to the action on $\mathbb{A}^{n+1}$. We can find now (as in Section 3) a coordinate homogeneous system in $\mathbb{A}^{n+1}$ and a set of characters $\chi_0, \dots, \chi_n \in \mathsf{X}(T) \simeq \mathbb{Z}^k$ in such a way that the action is given by

$$(*) \qquad t(x) = (t^{\chi_0} \cdot x_0, \dots, t^{\chi_n} \cdot x_n)$$

for $t \in T$ and $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$. We may assume that The set of characters is defined uniquely by the linearization of $\mathcal{O}(1)$: for any element $\alpha \in \mathbb{Z}^k$ the formula

$$(**) \qquad t(x) = ((\chi_0 - \alpha)(t) \cdot x_0, \ldots, (\chi_n - \alpha)(t) \cdot x_n)$$

corresponds to the same action but to a different linearization of the line bundle $\mathcal{O}(1)$. For the rest of this Section we fix the linearization of $\mathcal{O}(1)$, hence we fix the set $Ch = \{\chi_0, \ldots, \chi_n\}$ of characters of T. In the previous Section we have defined, for any $\alpha \in \mathsf{X}_{\mathbb{Q}}(T)$, sets of α-semistable and α-stable points in $\mathbb{P}^n$ corresponding to suitable the linearization of $\mathcal{O}(m)$. We shall denote these sets by $(\mathbb{P}^n)^{ss}(\alpha)$ and $(\mathbb{P}^n)^s(\alpha)$, respectively (then $(\mathbb{P}^n)^{ss}(0)$ (resp. $((\mathbb{P}^n)^s(0))$ denotes sets of semistable (resp. stable) points with respect to $\mathcal{O}(1)_0$)

We shall use definitions and notation of Section 3. Let Λ be a collection of all elementary polytopes (e-polytopes). We shall see that there is a one-to-one correspondence between all possible sets of semistable points corresponding to various linearizations of line bundles $\mathcal{O}(m)$ and elements of Λ.

Recall, that, for any $x \in \mathbb{P}^n$, we have a convex polytope $P(x)$ in the space $\mathsf{X}_{\mathbb{R}}(T)$. The following proposition follows easily from the fact that global sections of $\mathcal{O}(m)$ are exactly homogeneous polynomials of degree m of $n+1$ variables $(x_0, \ldots, x_n)$.

Proposition 5.1. *Let $\mathcal{O}(1)_0$ be a linearization corresponding to the choice of homogeneous coordinates $(x_0, \ldots, x_n)$ in $\mathbb{P}^n$ and the set of characters $Ch = \{\chi_0, \ldots, \chi_n\}$. Let $\alpha_1, \alpha_2 \in \mathsf{X}_{\mathbb{Q}}(T)$ and assume that E_0, E_1, E_2 are e-polytopes such that $0 \in E_0^\circ$ and $\alpha_i \in E_i^\circ$. Then*

(i) $(\mathbb{P}^n)^{ss}(0) = \{x \in \mathbb{P}^n : 0 \in P(x)\} = U(E_0)$ *and*
$(\mathbb{P}^n)^s(0) = \{x \in \mathbb{P}^n : 0 \in P(x)^\circ\}$*;*

(ii) $(\mathbb{P}^n)^{ss}(\alpha_1) = \{x \in \mathbb{P}^n : \alpha_1 \in P(x)\} = U(E_1)$ *and*
$(\mathbb{P}^n)^s(\alpha_1) = \{x \in \mathbb{P}^n : \alpha_1 \in P(x)^\circ\}$*;*

(iii) $(Pn)^{ss}(\alpha_1) = (\mathbb{P}^n)^s(\alpha_2) \iff E_1 = E_2$*;*

(iv) $(\mathbb{P}^n)^{ss}(\alpha_1) \subset (\mathbb{P}^n)^{ss}(\alpha_2) \iff E_2 \prec E_1$*.*

(v) $(\mathbb{P}^n)^{ss}(\alpha_1) = (\mathbb{P}^n)^s(\alpha_1) \iff E$ *is of maximal dimension (in the case of effective action, this amounts to* $dim(E) = k$*).*

□

Corollary 5.2. *Let T act on $\mathbb{P}^n$ as before.*

(1) *For any e-polytope E there exists an integer m and a linearization of the line bundle $\mathcal{O}(m)$ on $\mathbb{P}^n$ such that $U(E) = (\mathbb{P}^n)^{ss}(\mathcal{O}(m))$.*
(2) *For any integer m and a linearization of the line bundle $\mathcal{O}(m)$ there exists a unique e-polytope E such that $(\mathbb{P}^n)^{ss}(\mathcal{O}(m)) = U(E)$.*

□

Hence, for any e-polytope E, $U(E)$ is an open T-invariant set with a good, projective quotient.

Corollary 5.2 gives a combinatorial description of all sets of semistable points in $\mathbb{P}^n$ corresponding to all linearizations of line bundles. Since the group $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$, it follows from Lemma 7.14 in [9] that every open T-invariant subset with quasi-projective quotient is contained as a saturated subset in the set of semistable points with respect to a linearization of $\mathcal{O}(m)$, hence elements of Λ are in one to one correspondence with open subsets with projective quotients by T.

We shall use now the construction of [23], recalled in Section 3, and show an easy way to find all T-maximal subsets in $\mathbb{P}^n$.

First we shall use the fact that $\mathbb{P}^n$ is a toric variety with respect to the action of a maximal torus of $PGL(n)$ containing T.

Proposition 5.3. (Corollary 2.4 [22])*Let X be a toric variety with respect to an action of the torus S and let T be a subtorus of S. Assume that $U \subset X$ is a T-maximal subset of X. Then U is a toric variety with respect to the action of S.*

It follows, that any T-maximal subset of $\mathbb{P}^n$ is a toric variety and therefore the quotient space $U/\!\!/T$ is a toric variety as well. It follows (e.g from [25]) that $U/\!\!/T$ is an A_2-variety. Therefore we can use Theorem 2.20 and obtain any T-maximal set as an open set corresponding to a good collection of cells in the set indexed by elementary polytopes (see also Example 6.1 in [23]).

Denote by $\mathcal{I}$ the set of open T-invariant subsets with a projective quotient spaces or, equivalently, the set of all e-polytopes in $\mathsf{X}_Q(T)$.

We shall describe some good collections of cells in $\mathcal{I}$ and corresponding open sets (see 2.15 and 2.16). For any d-polytope P denote by $I(P)$ a cell in $\mathcal{I}$ corresponding to e-polytopes contained in P. According to 3.14 we may find a collection of d-polytopes $P_1, \dots, P_m$ such that

$$U(I) = \bigcup_{i=1}^{m} U(P_i).$$

In [9] we have defined some nice collections of d-polytopes:

Definition 5.4. A set of d-polytopes $\Delta = \{P_1, \dots, P_m\}$ has the property (η) if, for any $i, j = 1, \dots, m$

(1) $P_i \cap P_j \neq \emptyset$,
(2) $P_i \cap P_j \subset F \prec P_j \Rightarrow F \in \Delta$.

□

It is obvious that the following Lemma holds:

Lemma 5.5. *Let T acts on $\mathbb{P}^n$ as above.*

(i) *For any good collection Π of cells in $\mathcal{I}$ there exists a set of d-polytopes Λ with the property (η) such that*

$$\bigcup_{I \in \Pi} \bigcap_{E \in I} U(E) = \bigcup_{P \in \Lambda} U(P).$$

(ii) *Let Λ be a collection of d-polytopes and Π the corresponding collection of e-polytopes. Then Π is a good collection if and only if Λ has the property (η)*

□

As an immediate corollary from 2.20, 2.16 and 5.5 we get the following two Theorems:

Theorem 5.6. *Assume that Λ is a collection of d-polytopes satisfying (η).*

(1) *There exists a good quotient of the set*

$$U(\Lambda) = \bigcup_{P \in \Lambda} U(P).$$

(2) *The quotient space* $U(\Lambda)/\!\!/T$ *is quasi-projective if and only if*

$$\bigcap_{P\in\Lambda} P \neq \emptyset.$$

(3) $U(\Lambda)$ *is* T*-maximal if and only if for any d-polytope* Q *there exists* $P \in \Lambda$ *such that* $P \cap Q = \emptyset$.

PROOF. To prove (1) we use 5.5(ii) to infer that $U(\Lambda) = U(\Pi)$ where Π is a good collection of cells in the set of e-polytopes. This corresponds to a good collection of sets with a good quotients by T and hence, according to 2.16 the good quotient $U(\Pi)/\!\!/T$ exists.

Assume now that $\bigcap_{P\in\Lambda} P \neq \emptyset$ and let E be an e-polytope contained in $\bigcap_{P\in\Lambda} P$. Then $U(\Lambda)$ is contained in the set $U(E)$ of semistable points corresponding to E. It follows from Definition 5.4 and Lemma 3.13 that $U(\Lambda)$ is saturated in $U(E)$. Conversely, if $U(\Lambda)/\!\!/T$ is quasi-projective, then it is contained as a saturated subset in the set $U(E)$ of semistable points with respect to a linearization of $\mathcal{O}(k)$ for some $k \in \mathbb{N}$ (see e.g. 6.5), hence $\bigcap_{P\in\Lambda} P \neq \emptyset$.

The last statement follows from immediately from the definition (5.4), (1) above and Lemma 3.13

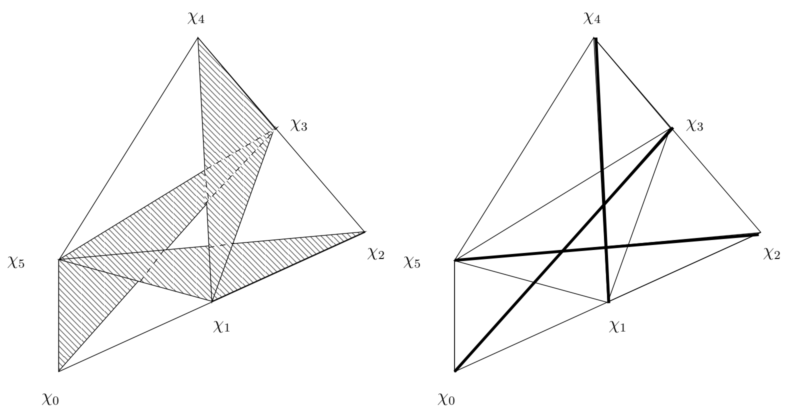
□

Theorem 5.7. *Let* U *be a* T*-maximal set in* $\mathbb{P}^n$. *Then there exists a collection* Π *of d-polytopes with the property* (η) *such that* $U = U(\Pi)$

PROOF. According to 5.3, the quotient space $U/\!\!/T$ is a toric variety, hence is $(T,2)$-maximal and we may use 2.20. Therefore, there exists a good collection Λ of e-polytopes such that $U = U(\Lambda)$. Then, by 5.5, there exists a collection Π of d-polytopes with the property (η) such that $U = U(\Pi)$.

□

As a consequence of 5.7 we get an easy tool to produce combinatorial pictures generating good collections of e-polytopes and hence open subsets of $\mathbb{P}^n$ with good quotients. In fact, it is enough to draw any set of mutually intersecting d-polytopes in $conv(Ch) \subset \mathsf{X}_{\mathbb{R}}(T)$.



a) Collection $\Pi = \{P_1, P_2, P_3\}$ b) Collection $\Pi_1 = \{D_1, D_2, D_3\}$

FIGURE 2

Example 5.8. Let $T \simeq (\mathbb{C}*)^2$ act on P^5 with characters $Ch = \{\chi_0, \dots, \chi_5\}$ as on Figure 2. Consider a collection $\Pi = \{P_1, P_2, P_3,\}$ of d-polytopes, where $P_1 = conv(\chi_0, \chi_3\chi_5)$, $P_2 = conv(\chi_1, \chi_2, \chi_5)$ and $P_3 = conv(\chi_1, \chi_3, \chi_4)$. It is easy to see that Π satisfies (η) and hence a good quotient $U(\Pi)/\!\!/T$ exists. Moreover, since $P_1 \cap P_2 \cap P_3 = \emptyset$, the quotient space $U(\Pi)/\!\!/T$ is not quasi-projective. According to 3.3 all points in $U(\Pi)$ have orbits of maximal dimension and hence a good quotient of $U(\Pi)$ is a geometric quotient.

Consider now the same action of T on $\mathbb{P}^5$ and a collection $\Pi_1 = \{D_1, D_2, D_3\}$ of d-polytopes, where $D_1 = conv(\chi_0, \chi_3)$, $D_2 = conv(\chi_1, \chi_4)$ and $D_3 = conv(\chi_2, \chi_5)$. Then a good quotient $U(\Pi_1)/\!\!/T$ exists but it is not a geometric quotient. $U(\Pi) \subset U(\Pi_1)$ but $U(\Pi)$ is not saturated in $U(\Pi_1)$.
□

So far we have described open subsets of $\mathbb{P}^n$ with good quotients by an action of an algebraic torus T. It is important to know when the quotient space of a T-maximal set in any T-variety X is complete. This problem for actions of tori of dimension one is solved in [2] . In [17]) the problem of finding open sets with projective quotients for actions of reductive groups was considered and the method of constructing such sets was provided. Other methods to study similar problems can be found in [3] and [8]. For actions of tori of any dimension on $\mathbb{P}^n$ the problem of constructing T-maximal sets with complete quotients is solved in [9]. To state this result we need another definition:.

Definition 5.9. Suppose we are given a collection of d-polytopes Π such that if $P \in \Pi$ and F is a face of P, then $F \in \Pi$. We say that Π is a subdivision of a d-polytope Q if $Q = \bigcup_{P\in\Pi} P$ and, for any $P_1, P_2 \in \Pi$, $P_1 \cap P_2$ is empty or is a face of P_1.
□

Remark 5.10. Consider a smooth curve C, a point $c_0 \in C$ and a morphism $f : C - \{c_0\} \to \mathbb{P}^n$. Assume that, for every $x \in f(C - \{c_0\})$, $P(x) = P$ for a given d-polytope P. Let $\Gamma_f \subset (C - \{c_0\}) \times X \subset C \times X$ be the graph of f. Then let $G \cdot \Gamma_f = \{(c, tf(c)) | c \in C - \{c_0\}, t \in T\}$ and $Z_f \subset X$ be the such that $\overline{G \cdot \Gamma_f} \cap (\{c_0\} \times X) = \{c_0\} \times Z_f$, i.e. Z_f be the projection of $\overline{G \cdot \Gamma_f} \cap (\{c_0\} \times X)$on the second factor. Then the $\Pi = \{P(x) | x \in Z_f\}$ i.e. the collection of d-polytopes corresponding to points in Z_f is a subdivision of P (see e.g. [11]).

Proposition 5.11. *(*Proposition 7.11 in [9]*). Let Π be a collection of d-polytopes satisfying (η) and $U = U(\Pi)$ (hence a good quotient $U/\!\!/T$ exists). Then $U/\!\!/T$ is complete if and only if for any subdivision $\{Q_1, \dots, Q_p\}$ of the "general" d-polytope $conv(\chi_0, \dots, \chi_n)$ there exists $j = 1, \dots, p$ and $P \in \Pi$ such that $P \subset Q_j$.*

Proof. The quotient space $U/\!\!/T$ is complete if and only if for any smooth curve C, a point $c \in C$ and a morphism $f : C - \{c\} \to U/\!\!/T$ there exists a morphism $g : C \to U/\!\!/T$ such that $f = g|C - \{c\}$. The proof of the proposition follows from Remark 5.10.

Example 5.12. Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{P}^5$ with characters $\chi_0, \dots, \chi_5$ as on Fig.3. Consider a collection $\Lambda = \{P_1, P_2, P_3, P_4\}$ of d-polytopes, where P_1, P_2, P_3 are as in 5.8 and $P_4 = conv(\chi_1, \chi_3, \chi_5)$. It is obvious that Δ satisfies (η), and T-orbit of any point $x \in U(\Delta)$ has maximal dimension, hence a geometric good quotient $U(\Delta)/T$ exists.

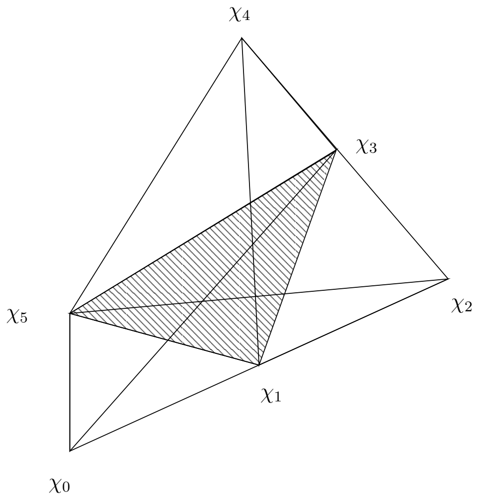


FIGURE 3. $P_4 = conv(\chi_1, \chi_3\chi_5)$

It is easy to check that for any subdivision $\Delta = \{Q_1, \ldots, Q_l\}$ of the general d-polytope $conv(\chi_0, \chi_2, \chi_4)$ there exists $i = 1, \ldots, l$ and $j = 1, \ldots, 4$ such that $P_j \subset Q_i$ and hence, according to 5.11, the quotient space $U(\Delta)/T$ is complete. The intersection of d-polytopes of Δ is empty therefore the quotient space is not projective.
□

6. G-MAXIMAL SUBSETS IN $\mathbb{P}^n$

In this Section we shall assume that a reductive group G acts on $\mathbb{P}^n$. We shall use the results of the previous Section to produce sets with good quotients by the action of G. It follows from 2.9 that every G-maximal subset of X is contained in a T-maximal set. In general it is not clear if the set of G-orbits contained in a T-maximal set is open and has a good quotient by G. Fortunately, this is the case in the situation considered here, as is stated in the following theorem:

Theorem 6.1. (*Theorem C,* [9]*) Let a reductive group G act on a projective space $\mathbb{P}^n$. Let T be a fixed maximal torus of G.*

(I). Let V be a T-maximal subset of $\mathbb{P}^n$ and $U = \bigcap_{g\in G} g \cdot V$. Then U is open, G-invariant and there exists a good quotient $U /\!\!/ G$.

(II). For any G-maximal set U, there exists a T-maximal set V containing U as a T-saturated subset and, for any such V, we have $U = \bigcap_{g\in G} g \cdot V$.

The proof of this theorem is a consequence of a sequence of lemmas. We fix the notation: let $B \subset G$ be a Borel subgroup containing T, $N(T)$ - the normalizer of T in G and let $W = N(T)/T$ be the Weyl group of G. Let V be a T-maximal set in $\mathbb{P}^n$.

Lemma 6.2. *There exists a closed B-invariant subset $F \subset \mathbb{P}^n$ such that*

$$\mathbb{P}^n - \bigcap_{w\in W} w \cdot V = N(T) \cdot F = W \cdot F$$

Lemma 6.3. *$\bigcap_{g\in G} g\cdot V$ is open. Moreover*

$$B(\mathbb{P}^n - V) = \mathbb{P}^n - \bigcap_{g\in G} g\cdot V.$$

Lemma 6.4. *$\bigcap_{g\in G} g\cdot V$ is saturated, with respect to any one-dimensional torus $T_0\subset T$, in $\bigcap_{w\in W} w\cdot V$.*

The proof of the theorem follows from these lemmas by Reduction Theorem 2.9.
□

Theorem 6.1 is not satisfactory if we look for G-maximal sets. For a given T-maximal set $U\subset\mathbb{P}^n$ it is not known if the open set $\bigcap_{g\in G} g\cdot U$ is G-maximal. In fact it is not clear if it is non-empty or when the quotient space is complete.

The problem of describing open sets with projective quotients is completely solved - at least for actions of semi-simple groups on $\mathbb{P}^n$- by Mumford's Theory. Let $\mathcal{L}$ be a linearized line bundle on $\mathbb{P}^n$, T be a maximal torus in G and let $U = (\mathbb{P}^n)^{ss}_T(\mathcal{L})$. In [17] Mumford proved that

$$\bigcap_{g\in G} g\cdot U = (\mathbb{P}^n)^{ss}_G(\mathcal{L}).$$

and that the quotient of $\bigcap_{g\in G} g\cdot U$ by G is projective, hence $\bigcap_{g\in G} g\cdot U$ is G-maximal (when it is non-empty). The converse to this result in the case of an action of a semi-simple group G follows from the fact that the $Pic(\mathbb{P}^n)=\mathbb{Z}$ and any line bundle admits at most one G-linearization.

In fact we have the following

Theorem 6.5. ([9], Theorem 8.6*) Let a semi-simple group G act on $\mathbb{P}^n$. Let U be an open G-invariant subset of $\mathbb{P}^n$ such that a good quotient $q: U\to U/\!\!/G$ exists. Then either $U/\!\!/G$ is quasi-projective, or is an algebraic space, but not an algebraic variety. Moreover, if $U/\!\!/G$ is quasi-projective then U is contained as a saturated subset in the unique G-maximal set coming from the Mumford construction.*
□

We have still a conjecture which is not fully solved even for actions on projective spaces:

Conjecture 6.6. ([8]) Let G be a semi-simple group and $N(T)$ be the normalizer of a maximal torus $T\subset G$. Assume that G acts on a projective, smooth variety X and $U\subset X$ is an $N(T)$-invariant, open subset such that a good quotient $U\to U/\!\!/T$ exists. Then $\bigcap_{g\in G} g\cdot U/\!\!/G$ exists in the category of algebraic spaces. Moreover, if $U/\!\!/T$ is complete then $\bigcap_{g\in G} g\cdot U/\!\!/G$ is a complete algebraic space.
□

According to Theorem 6.1, the first part of this conjecture is true in the case $X=\mathbb{P}^n$ but the second part is still open. If $G = Sl_2$ and we consider only geometric quotients (see Definition 2.1) the conjecture is true in general.

Theorem 6.7. *(*Theorem 2 in [4]*) Suppose that X is a projective smooth variety with an action of $G = Sl_2$, $T\subset G$ is a maximal torus and $N(T)$ the normalizer of T in G. Let $U\subset X$ be an $N(T)$-invariant open subset of X such that the geometric quotient $U\to U/T$ exists and is complete. Then the geometric quotient*

$$\bigcap_{g\in G} g\cdot U \to \bigcap_{g\in G} g\cdot U/G$$

exists and the quotient space is a complete algebraic space.

□

The proof of this theorem for $X = \mathbb{P}^n$ follows immediately from the fact that there exists exactly one open $N(T)$ invariant subset of $\mathbb{P}^n$ with complete quotient and hence in this case we are in the situation considered by Mumford. But we have still open questions:

6.8. Let a reductive (or semi-simple) group G act on $\mathbb{P}^n$, U be a T-maximal set of $\mathbb{P}^n$ and $V = \bigcap_{g\in G} g\cdot U$.

(i) When V is G-maximal?
(ii) When $V/\!\!/G$ is complete?

□

References

1. A. Białynicki-Birula, *Finiteness of the number of maximal open sets with a good quotient*, Transformation groups **3** (1998), no. 4, 301–319.
2. A. Białynicki-Birula and J. Święcicka, *Complete quotients by algebraic torus actions*, Lectures Notes in Mathematics **Group Actions and Vector Fields** (1982), no. 956, 10–20.
3. ______, *Generalized moment functions and orbit spaces*, Amer. J. Math. **109** (1987), 229–238.
4. ______, *On complete orbit spaces of $sl(2)$ actions*, Coll. Math. **55** (1988), 229–243.
5. ______, *On exotic orbit spaces of tori acting on projective varieties*, Can. Math Soc., Conference proceedings **10** (1989), 25–30.
6. ______, *A reduction theorem for existence of good quotients*, Amer. J. Math. **113** (1990), 189–201.
7. ______, *A combinatorial approach to geometric invariant theory*, Proc. Sophus Lie Memorial Conf., pp. 115–127, Scand. Univ. Press, Oslo, 1992.
8. ______, *On projectivity of good quotients*, Journal of Algebra **177** (1995), 961–966.
9. ______, *Open subsets of projective spaces with a good quotient by an action of a reductive group*, Trans. Groups **1** (1996), 153–185.
10. ______, *Three theorems on existence of good quotients*, Math. Annalen **307** (1997), 143–149.
11. A. Białynicki-Birula and A. Sommese, *A conjecture about quotients by tori*, Adv. Stud. Pure Math. (1986), no. 8, 59–68.
12. I. Dolgachev, *Introduction to eometric invariant theory*, Lecture Notes Series, no. 25, Res. Inst. of Math., Seoul Nat. Univ., Korea, 1994.
13. A. Grothendieck, *Eléments de géométrie algébrique*, Publ. Math. IHES.
14. W. Haboush, *Reductive groups are geometrically reductive*, Ann. of math. **102** (1975), no. 2, 67–83.
15. S. L. Kleiman, *Toward a numerical theory of ampleness*, Annals of Math. **84** (1966), 293–344.
16. T. Miyata and M. Nagata, *Note on semi-reductive groups*, J. Math. Kyoto Univ. **3** (1963/64), 379–382.
17. D. Mumford, *Geometric invariant theory*, Ergeb.Math., vol. 34, Springer Verlag, 1982.
18. M. Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.
19. A. Neeman, *Steins, affines and Hilbert's fourteenth problem*, Annals of Math. **127** (1988), 229–244.
20. V.L. Popov, *Hilberts theorem on invariants*, Soviet. Math. Dokl. **20** (1979), 1318–1322.
21. H. Sumihiro, *Equivariant completions*, Journal of Math. Kyoto University **14**, 1–28.
22. J. Święcicka, *Quotients of toric varieties by actions of subtori*, Coll. Math **82** (1999), no. 1, 105–116.
23. ______, *A combinatorial construction of sets with good quotients by an action of a reductive group*, Coll. Math. **87** (2001), no. 1, 85–102.
24. J. Winkelmann, *Invariant rings and quasiaffine quotients*, arXiv:math.AG/0007076 v1 12 Jul 2000.
25. J. Włodarczyk, *Embeddings in toric varieties and prevarieties*, Journal of Alg. Geom. (1993), no. 2, 705–726.

26. ———, *Maximal quasiprojective sets and the Kleiman-Chevalley quasiprojectivity criterion*, J. Sci. Univ. Tokyo (1999), no. 6.

Uniwersytet Warszawski, Instytut Matematyki, ul.Banacha 2, 02-097 Warszawa, Poland
E-mail address: jswiec@mimuw.edu.pl