SYNOPSIS OF LECTURES
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Lecture 2: Basic Facts II

1 More About Locally Nilpotent Derivations

k is any field of characteristic zero, and B is a commutative k-domain.

1.1 Kernel Algorithm

The following algorithm for computing the kernel of a locally nilpotent derivation is due to van den Essen. It is based on earlier ideas of Tan.

Assume B is finitely generated over k: $B = k[g_1, ..., g_n]$. Given $D \in \mathrm{LND}_k(B)$, set $A = \ker D$. Choose a local slice $r \in B$ of D, and set $f = Dr$. Let $\pi_r : B \to A_f$ be the corresponding Dixmier map. Recall that, for any $b \in B$,

$$\pi_r(b) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i b \frac{r^i}{f^i} \ .$$

Given g_i, choose m minimal so that $h_i := f^m \pi_r(g_i) \in A$. Now let $A_0 \subset A$ be the subring defined by $A_0 = k[h_1, ..., h_n, f]$. Inductively, define A_i to be the subring generated by

$$\{h \in B | fh \in A_{i-1}\} \ .$$

Theorem 1 *If A is finitely generated, then $A = A_N$ for some $N \geq 0$.*

Example: Weitzenböck Derivations.

1.2 Homogeneous Derivations

Gradings, degrees, homogeneous derivations

Theorem 2 *Suppose B is the graded ring $B = \oplus_{i \in \mathbb{N}} B_i$, where $\mathbb{N}$ denotes the non-negative integers, and let $D \in \mathrm{LND}_k(B)$ be given. Suppose that D admits a decomposition $D = \sum_{0 \leq i \leq n} D_i$, where each $D_i \in \mathrm{Der}_k(B)$ is homogeneous of degree i relative to this grading.*

(a) $D_n \in \mathrm{LND}_k(B)$

(b) *If* $f \in \ker D$ *and* $f = \sum_{0 \le i \le m} f_i$ *for* $f_i \in B_i$*, then* $f_m \in \ker D_n$*.*

2 The Exponential Map

Given $D \in \mathrm{LND}_k(B)$, the **exponential function** determined by D is $\exp D : B \to B$, where

$$\exp D(f) = \sum_{i \ge 0} \frac{1}{i!} D^i f \ .$$

Note that, since D is locally nilpotent, $\exp D$ is well-defined.

Theorem 3 *Suppose* $D \in \mathrm{LND}_k(B)$*.*

(a) $\exp(D) \in \mathrm{Aut}_k(B)$

(b) *If* $[D, E] = 0$ *for some* $E \in \mathrm{LND}_k(B)$*, then* $(D+E) \in \mathrm{LND}_k(B)$ *and*

$$\exp(D + E) = (\exp D) \circ (\exp E) \ .$$

3 $\mathbb{G}_a$-Actions

DEFINITION: $\mathbb{G}_a = (k, +)$, the additive group of the field k.

Assume that $X = \mathrm{Spec}(B)$ is an algebraic k-variety.

Given $D \in \mathrm{LND}_k(B)$, by Theorem 5 we obtain a group homomorphism

$$\eta : (\ker D, +) \to \mathrm{Aut}_k(B) \ , \eta(f) = \exp(fD) \ .$$

In addition, if $D \neq 0$, then η is injective. Restricting η to the constants k, we obtain the algebraic representation $\eta : \mathbb{G}_a \hookrightarrow \mathrm{Aut}_k(B)$. Geometrically, this means that D induces the faithful algebraic $\mathbb{G}_a$-action $\exp(tD)$ on X, $t \in k$.

Conversely, let $\rho : (\mathbb{G}_a \times X) \to X$ be an algebraic $\mathbb{G}_a$-action over k. (Equivalently, $\mathbb{G}_a$ acts on the coordinate ring $B = k[X]$.) Then ρ induces a derivation $\rho'(0)$, where differentiation takes places relative to $t \in \mathbb{G}_a$.

To be more precise, at the level of coordinate rings, $\rho^* : B \to B[t]$ is a k-algebra homomorphism (since ρ is a morphism of algebraic k-varieties).

Given $t \in k$ and $f \in B$, we also denote the action of t on f by $t \cdot f$. Define the function

$$\delta : B \to B \ , \ \delta = \epsilon \circ \frac{d}{dt} \circ \rho^* \ ,$$

where $\epsilon : B[t] \to B$ denotes evaluation at $t = 0$.

Proposition 1 *$\delta \in \mathrm{LND}_k(B)$.*

In summary, there is a bijective correspondence between $\mathrm{LND}_k(B)$ and the set of all algebraic $\mathbb{G}_a$-actions on $X = \mathrm{Spec}(B)$, where $D \in \mathrm{LND}_k(B)$ induces the action $\exp(tD)$, and where the action ρ induces the derivation $\delta = \rho'(0)$, as described above. In addition, the kernel of the derivation coincides with the invariant ring of the corresponding action,

$$\ker D = B^{\mathbb{G}_a} \ ,$$

since $Df = 0$ iff $\exp(tD)(f) = f$ for all $t \in k$.

4 Polynomial Rings

4.1 Variables, Automorphisms, and Gradings

Systems of variables, affine Cremona group, triangular and linear automorphisms, gradings

4.2 Derivations

partial derivatives, jacobian determinants, linear and triangular derivations, rank

4.3 Examples

4.4 Triangular Automorphisms

Proposition 2 *Every triangular automorphism of $k^{[n]}$ is of the form $L \circ (\exp T)$, where L is a diagonal matrix and T is a triangular derivation.*

5 Classical Results of Seidenberg and Vasconcelos

Theorem 4 *Let* $K = \mathrm{frac}(B)$ *and let* $B' \subset K$ *be the integral closure of* B *in* K*. Given* $D \in \mathrm{Der}_k(K)$*, if* $DB \subset B$*, then* $DB' \subset B'$*.*

Theorem 5 *Suppose* $R \subset B$ *is a subring over which* B *is integral. If* $D \in \mathrm{Der}_k(B)$ *restricts to a locally nilpotent derivation of* R*, then* $D \in \mathrm{LND}_k(B)$*.*

Lecture 3: Rentschler's Theorem

Rentschler's Theorem dates from 1968.

Theorem 6 *Let* k *be a field of characteristic zero. If* $D \in \mathrm{LND}_k(k[x,y])$ *is non-zero, then there exists* $f \in k[x]$ *and a tame automorphism* $\alpha \in GA_2(k)$ *such that* $\alpha D \alpha^{-1} = f(x)\partial_y$*.*

Geometrically, this says that every planar $\mathbb{G}_a$-action is conjugate, by a tame automorphism, to a triangular action $t(x,y) = (x, y + tf(x))$. Rentschler also showed that his theorem implies *Jung's Theorem*, which appeared in 1942, and which asserts that every plane automorphism is tame in the characteristic zero case. Rentschler's proof of Jung's Theorem is a compelling illustration of the importance of locally nilpotent derivations and $\mathbb{G}_a$-actions in the study of affine algebraic geometry.

Jung's Theorem was the predecessor of the well-known *Structure Theorem* for the group $GA_2(k)$. Recall that $GA_2(k)$ denotes the full group of algebraic automorphisms of the plane $\mathbb{A}^2_k$; $Af_2(k)$ is the affine linear subgroup, whose elements are products of linear maps $L \in GL_2(k)$ and translations $T = (x + a, y + b)$, $a, b \in k$; and $BA_2(k)$ is the subgroup of triangular automorphisms $\alpha = (ax + b, cx + f(x))$, $(a, c \in k^*, b \in k, f \in k[x])$. A *tame* automorphism is composed of linear and triangular factors.

Theorem 7 (Structure Theorem) *For any field* k*,* $GA_2(k)$ *has the amalgamated free product structure*

$$GA_2(k) = Af_2(k) *_B BA_2(k) \ ,$$

where $B = Af_2(k) \cap BA_2(k)$*.*

6 The Polynomial Ring in Two Variables over a Field

The main goal of this section is to give a self-contained elementary proof of Rentschler's Theorem. For the remainder of this section, let $A = k[x, y]$, the polynomial ring in two variables over a field k of characteristic zero. We argue only from first principles. In particular, recall the following from *Lecture 1* and *Lecture 2*: Suppose $D \in \text{LND}_k(A)$.

- **Principle 1.** $\ker D$ is factorially closed.
- **Principle 2.** If $Df \in fA$ for $f \in A$, then $Df = 0$. If $Df \in gA$ and $Dg \in fA$ for $f, g \in A$, then either $Df = 0$ or $Dg = 0$.
- **Principle 3.** Any pair $a, b \in \mathbb{Z}$ determines a $\mathbb{Z}$-grading on A via $\deg(x^i y^j) = ai + bj$. Let $f \in \ker D$ be given. If D_N is the highest homogeneous summand of D, and if f_d is the highest homogeneous summand of f, then $D_N \in \text{LND}_k(A)$ and $f_d \in \ker(D_N)$.
- **Principle 4.** For every $h \in A$, $Dh = h_x Dx + h_y Dy$.

6.1 Newton Polygon of f

Lemma 1 *Given non-zero $D \in \text{LND}_k(A)$, and given non-zero $f \in \ker D$, $\text{Newt}(f)$ is a triangle T with vertices of the form $(0,0)$, $(m,0)$, and $(0,n)$.*

6.2 Proof of Rentschler's Theorem

6.3 Newton Polygon of D

Lemma 2 *If $D \neq 0$, then $\text{Newt}(D)$ is equal to the triangle with vertices $(-1,-1), (\lambda,-1)$, and $(-1,\mu)$, where*

$$\lambda = \max\{i | D_{(i,-1)} \neq 0\} \cup \{-1\} , \quad \mu = \max\{j | D_{(-1,j)} \neq 0\} \cup \{-1\} .$$

6.4 Proof of Jung's Theorem

Every automorphism of A defines a pair of locally nilpotent derivations of A, namely, the partial derivatives relative to the given coordinate functions. Thus, in the characteristic zero case, Rentschler's Theorem is a powerful tool in the study of plane automorphisms.

Theorem 8 (Jung's Theorem) *The group $GA_2(k)$ of algebraic automorphisms of $k[x, y]$ is generated by its linear and triangular subgroups, $GL_2(k)$ and $BA_2(k)$.*

6.5 The Structure Theorem

The Structure Theorem follows from Jung's Theorem. Our proof follows Wright.

Lecture 5: Intro to Dimension Three

NOTATION: k is a field of characteristic zero, and $B = k[x, y, z]$ is the three dimensional polynomial ring over k.

7 Fundamental Theorems

Theorem 9 (Miyanishi's Theorem) *If k is a field of characteristic zero, the quotient of any non-trivial algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$. Equivalently, the kernel of a non-zero locally nilpotent k-derivation of $k^{[3]}$ is isomorphic to $k^{[2]}$.*

Definition. Given $f, g \in B$, $D_{(f,g)}$ will denote the derivation

$$D_{(f,g)}h = \frac{\partial(f, g, h)}{\partial(x, y, z)} \ .$$

Theorem 10 (Daigle) *Given $D \in \mathrm{LND}_k(B)$, $D \neq 0$, choose $f, g \in B$ such that $\ker D = k[f, g]$. Then $D = aD_{(f,g)}$ for some $a \in \ker D$.*

Theorem 11 (Zurkowski's Theorem) *If k is an algebraically closed field of characteristic zero, and if D is a non-zero homogeneous locally nilpotent k-derivation of B relative to some positive grading on B, then $\ker D = k[f, g]$ for homogeneous $f, g \in B$.*

Definition. If $D \in \mathrm{LND}_k(B)$ is homogeneous and $\ker D = k[f, g]$ for homogeneous polynomials f and g, then D is of *type* $(\deg f, \deg g)$.

If $D \in \mathrm{LND}_k(B)$, let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ denote the **quotient map**, induced by the inclusion $\ker D \hookrightarrow B$.

Theorem 12 (Bonnet's Theorem) *If k is a field of characteristic zero, and if $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient map associated to a non-trivial algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$, then π is surjective.*

Theorem 13 (Kaliman's Theorem) *Every free algebraic $\mathbb{C}^+$-action on $\mathbb{C}^3$ is a translation in a suitable polynomial coordinate system. Equivalently, if $B = \mathbb{C}^{[3]}$ and $D \in \mathrm{LND}_k(B)$, and if $(DB) = B$ (the ideal generated by the image), then $Ds = 1$ for some $s \in B$.*

Theorem 14 (Principal Ideal Theorem) *Let $D \in \mathrm{LND}_k(B)$ and $A = \ker D$. Then $A \cap DB$ is a principal ideal of A.*

Theorem 15 (Intersection of Kernels) *Given non-zero $D_1, D_2 \in \mathrm{LND}_k(B)$, exactly one of the following 3 statements is true.*

1. $(\ker D_1 \cap \ker D_2) = k$
2. *There exist $f, g, h \in B$ such that* $\ker D_1 = k[f, g]$, $\ker D_2 = k[f, h]$, *and* $(\ker D_1 \cap \ker D_2) = k[f]$.
3. $\ker D_1 = \ker D_2$

Corollary 1 *Suppose $D_1, D_2 \in \mathrm{LND}_k(B)$ are non-zero and have distinct kernels. If* $\ker D_1 = k[f]^{[1]}$ *and $D_2 f = 0$, then* $\ker D_2 = k[f]^{[1]}$.

8 Questions of Triangulability

Bass's example, triangulability criteria, Daigle's example

9 The $(2,5)$ Example

> Do there exist locally nilpotent derivations of the polynomial ring $k[x_1,...,x_n]$ having maximal rank n?

It was known at the time that, for $n=1$, the answer is positive, and for $n=2$ the answer is negative (by Rentschler's Theorem). But for $n \geq 3$ the answer was not known. In 1996, the author produced the following example of a locally nilpotent derivation on $B=k[x,y,z]$ having rank three. Define polynomials

$$F=(xz-y^2)\ ,\ G=(zF^2+2x^2yF+x^5)\ .$$

Define the k-derivation $\Delta : B \to B$ by $\Delta = D_{(F,G)}$.

Theorem 16 $\Delta \in \mathrm{LND}_k(B)$*,* $\ker \Delta = k[F,G]$*, and* $\mathrm{rank}\,(\Delta)=3$*.*

> The polynomial $R := R = (x^3+yF)$ is a local slice of Δ, with $\Delta R = -FG$. We also have:

Theorem 17 *Let Δ be the homogenous $(2,5)$ derivation of $B=k[x_1,x_2,x_3]$. Given $n \geq 4$, extend Δ to Δ° on $B[x_4,...,x_n]=k^{[n]}$ by setting $\Delta^\circ x_i = x_{i-1}^5$, $4 \leq i \leq n$. Then Δ° is homogeneous and locally nilpotent of rank n.*

10 Local Slice Constructions

In an effort to understand and generalize the $(2,5)$ example above, we define local slice constructions. This led to discovery of large new families of rank-3 elements of $\mathrm{LND}(B)$. Also working just after the appearance of the $(2,5)$ example, Daigle used a geometric approach, quite different from the method of local slice constructions, to find additional rank-3 examples. At the time, it appeared that the two methods produced the same examples.

10.1 Definition and Main Facts

10.2 Examples of Fibonacci Type

10.3 Type $(2, 4m+1)$

10.4 Triangular Derivations

11 Open Questions

1. For $B = k[x,y,z]$, can every irreducible $D \in \mathrm{LND}_k(B)$ be obtained from a partial derivative $\frac{\partial}{\partial x}$ by a sequence of local slice constructions (up to change of coordinates)?

2. Is every *tame* $\mathbb{G}_a$-acion on $\mathbb{A}^3$ conjugate to a triangular action?

Lectures 8 and 9: Questions of Finite Generation

Our primary interest is in the following question, which is one of the most basic questions one can ask about $\mathbb{G}_a$-actions.

> **(Main Problem)** Let k be any field of characteristic 0. If $\mathbb{G}_a$ acts algebraically on $\mathbb{A}^n$, is the ring of invariants finitely generated? Equivalently, if D is any locally nilpotent k-derivation of the polynomial ring $k^{[n]}$, is the kernel of D finitely generated as a k-algebra?

This represents a special case of Hilbert's Fourteenth Problem. It turns out that, if $n \leq 3$ the answer is *yes*, and if $n \geq 5$ the answer is generally *no*. Remarkably, only when $n = 4$ is the answer not known!

Historical Note. In its modern form, the famous Fourteenth Problem of Hilbert is as follows.

> For a field k, let $k^{[n]}$ denote the polynomial ring in n variables over k, and let $k^{(n)}$ denote its field of fractions. If K is a subfield of $k^{(n)}$ containing k, is $K \cap k^{[n]}$ finitely generated over k?

The main case of interest at the time of Hilbert was invariant rings for algebraic subgroups of $GL_n(\mathbb{C})$ groups acting on $\mathbb{C}^n$ as a vector space. But one can also consider the case of invariant rings of more general algebraic group actions on varieties:

> For a field k, suppose the linear algebraic k-group G acts algebraically on an affine k-variety V. Is the invariant ring $k[V]^G$ finitely generated?

It turns out that the answer depends on the type of group G which is acting, according to the classical decomposition of algebraic groups into unipotent and reductive components. For reductive groups, the Fourteenth Problem has a positive answer.

> **(Hilbert's Finiteness Theorem)** *If k is any field, and G is a finite or reductive algebraic k-group acting by algebraic automorphisms on an affine k-variety V, then the algebra of invariants $k[V]^G$ is finitely generated over k.*

In the late 1950s, Nagata published his celebrated counterexamples to Hilbert's Fourteenth Problem. One of these examples uses the unipotent group $\mathbb{G}_a^{13}$ acting linearly on $\mathbb{A}_k^{32}$, and Nagata proves that the invariant ring of this action is not finitely generated. In the language of derivations, this can be realized by 13 commuting linear (triangular) derivations D_i of the polynomial ring $k^{[32]}$ for which the subring $\cap_{i=1}^{13} \ker D_i$ is not finitely generated. Nagata's results are valid for any field k which *locally finite*, i.e., is not an algebraic extension of a finite field.

12 The Theorem of Weitzenböck

Invariant theory originally concerned itself with groups of vector space transformations, so the linear algebraic $\mathbb{G}_a$-actions were the first $\mathbb{G}_a$-actions to be studied, and they have a long and interesting history.

> **(Weitzenböck's Theorem)** *If k is a field of characteristic zero, and if $\mathbb{G}_a$ acts algebraically on $X = \mathbb{A}_k^n$ by linear transformations, then $k[X]^{\mathbb{G}_a}$ is finitely generated.*

Seshadri gave a proof for any characteristic 0 field in his 1962 paper.

13 Linear Counterexamples to the Fourteenth Problem

13.1 Nagata's Examples

Prior to the appearance of Nagata's examples, Rees constructed a counterexample to Zariski's generalization of the Fourteenth Problem, which asks:

> Let R be a normal affine ring over a field k. If L is a field with $k \leq L \leq \mathrm{frac}(R)$, is $(R \cap L)$ an affine ring?

In Rees' example, $\mathrm{frac}(R \cap L)$ contains the function field of a non-singular cubic projective plane curve, and cannot therefore be a counterexample to Hilbert's problem. But Rees' example was very important in its own right, and indicated that counterexamples to Hilbert's problem might be found in a similar fashion.

Shortly thereafter, Nagata discovered two counterexamples to Hilbert's problem.

13.2 Standard Unipotent and Torus Actions

> **(Mukai's Theorem)** *Let $\mathbb{C}^n$ act on $\mathbb{C}^{2n}$ by the standard action. If $G \subset \mathbb{C}^n$ is a general linear subspace of codimension r, and if*
>
> $$\tfrac{1}{r} + \tfrac{1}{n-r} \leq \tfrac{1}{2}$$
>
> *then the invariant ring is not finitely generated.*

It follows that S^G is not finitely generated if $\dim G = m \geq 3$ and $n \geq m^2/(m-2)$. Thus, there exists algebraic linear actions of $\mathbb{G}_a^3$ on $\mathbb{C}^{18}$ and of $\mathbb{G}_a^4$ on $\mathbb{C}^{16}$ whose rings of invariants are not finitely generated.

Then in 2002, Tanimoto used the example of Daigle and Freudenburg (below) to construct a linear counterexample in which $(\mathbb{G}_a^7 \rtimes \mathbb{G}_a)$ acts on $\mathbb{A}^{13}$.

To date, these are the smallest linear counterexamples to the Fourteenth Problem, both in terms of the dimension of the group which acts ($m = 3$), and the dimension of the space upon which it acts ($n = 13$).

14 Non-Finitely Generated Kernels I

This section explores the famous example of Paul Roberts and some of the rich theory which has flowed from it.

14.1 Roberts' Examples

15 Non-Finitely Generated Kernels II

15.1 The Basic Action in Dimension Four

Let $R = k[a,s,t,u]$ denote the polynomial ring in four variables, and let D be the basic linear derivation on R, i.e.,

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \text{or} \quad D = a\partial_s + s\partial_t + t\partial_u \ .$$

Theorem 18 *There exists a sequence of homogeneous polynomials $w_n \in R$ $(n \geq 0)$ such that $w_0 = 1$; $w_1 = s$; $Dw_n = t_n w_{n-1}$ for all $n \geq 1$; and $\deg w_n = \delta_n$ for all n.*

15.2 Dimension Five

Our next goal is to prove the following.

Theorem 19 *Let Δ be the triangular monomial derivation of $k[x,s,t,u,v] = k^{[5]}$ defined by*

$$\Delta = x^3\partial_s + s\partial_t + t\partial_u + x^2\partial_v \ .$$

Then the kernel of Δ is not finitely generated as a k-algebra.

Lemma 3 *Let $K = \oplus_{i\in\mathbb{N}} K_i$ be a graded k-domain such that $K_0 = k$, and let δ be a homogeneous locally nilpotent k-derivation of K. Given homogeneous $\alpha \in \ker\delta$ which is not in the image of δ, let $\tilde{\delta}$ be the extension of δ to $K[T]$ defined by $\tilde{\delta}T = \alpha$, where T is a variable over K. Suppose β_n is a sequence of non-zero elements of $\ker\tilde{\delta}$ having leading T-coefficients $b_n \in K$. If $\deg b_n$ is bounded, but $\deg_T \beta_n$ is not bounded, then $\ker\tilde{\delta}$ is not finitely generated over k.*

Lemma 4 *Let $A = \ker \Delta$, and let $\pi_v : S \to A_{\Delta v}$ be the Dixmier map for Δ induced by the local slice v. For each $m \geq 1$, $(-1)^{3m}(3m)!\pi_v(xw_{3m})$ is an element of A of the form*

$$xv^{3m} + \sum_{i=0}^{3m-1} \beta_i v^i \qquad (\beta_i \in k[x, s, t, u]) \ .$$

15.3 Dimension Seven

Given $n \geq 3$, let $\mathcal{D}_n$ be the Roberts derivation on $B = k[X, Y, Z, S, T, U, V]$ defined by

$$\mathcal{D}_n = (X^n)\partial_S + (Y^n)\partial_T + (Z^n)\partial_U + (XYZ)^{n-1}\partial_V \ .$$

This derivation has many symmetries.

Theorem 20 *Let $\mathcal{D}_3$ be the Roberts derivation on $B = k[X, Y, Z, S, T, U, V]$ defined by*

$$\mathcal{D}_3 = (X^3)\partial_S + (Y^3)\partial_T + (Z^3)\partial_U + (XYZ)^2\partial_V \ .$$

The kernel of $\mathcal{D}_3$ is not finitely generated.

Theorem 21 *Let Δ_n be the triangular monomial derivation of $k^{[5]} = k[x, s, t, u, v]$ defined by*

$$\Delta_n = x^{n+1}\partial_s + s\partial_t + t\partial_u + x^n\partial_v \ .$$

If $n \geq 2$, then the kernel of Δ_n is not finitely generated as a k-algebra.

Corollary 2 *Let $\mathcal{O} = k[a, b] = k^{[2]}$, and let d_n be the triangular $\mathcal{O}$-derivation of $\mathcal{O}[x, y, z] = k^{[5]}$ defined by*

$$d_n = a^n\partial_x + (ax + b)\partial_y + y\partial_z \ .$$

If $n \geq 2$, then the kernel of d_n is not finitely generated as an $\mathcal{O}$-algebra.

One may thus view the derivation d_n on $k^{[5]}$ as Roberts' derivation $\mathcal{D}_n$ on $k^{[7]}$ with all of the (obvious) symmetries removed.

16 Positive Results and Open Questions

16.1 Kernels in Dimension Four

Theorem 22 *Let k be an algebraically closed field of characteristic zero, and let R be a k-affine Dedekind domain or a localization of such a ring. The kernel of any triangular R-derivation of $R[x, y, z]$ is finitely generated as an R-algebra.*

Corollary 3 *If T is a triangular derivation of $k[w, x, y, z]$, then the kernel of T is finitely generated.*

Theorem 23 *For each integer $n \geq 3$, there exists a triangular derivation of $k[w, x, y, z]$ whose kernel, though finitely generated, cannot be generated by fewer than n elements.*

16.2 Elementary Derivations

Theorem 24 (van den Essen, Janssen) *Let D be an elementary derivation of $B = k[x_1, ..., x_n]$ for which $Dx_1 = \cdots = Dx_i = 0$ and $Dx_j \in k[x_1, ..., x_i]$ for $j > i$.*

(a) If either $i \leq 2$ or $(n - i) \leq 2$ then $\ker D$ *is finitely generated.*

(b) If 1 is in the ideal generated by the image of D, then $\ker D$ *is a polynomial ring.*

Theorem 25 (Khoury) *For $n \leq 6$, the kernel of every elementary monomial derivation of $k^{[6]}$ is generated by at most 6 elements.*

16.3 Open Questions

Question. If D is a locally nilpotent derivation of $k[x, y, z, w]$, is the kernel of D always finitely generated?

Question. (Nagata's Second Problem) Let K be a subfield of the field $k(x_1, ..., x_n)$ such that $\dim_k K = 3$. Is $K \cap k[x_1, ..., x_n]$ always finitely generated?

Question. Is the fixed ring of every algebraic *linear* action of $\mathbb{G}_a^2$ on $\mathbb{A}^n$ finitely generated? Equivalently, if M_1 and M_2 are commuting nilpotent matrices, is $\ker M_1 \cap \ker M_2$ finitely generated?

Question. If F is a field of positive characteristic, and $\mathbb{G}_a$ acts linearly algebraically on $\mathbb{A}^n_F$, is the ring of invariants finitely generated?

Lecture 11: Line Embeddings and Slices

17 Application to Line Embeddings

As above, set $B = k[x,y,z]$. Abhyankar introduced the algebraic embeddings $\theta_n : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ defined by $\theta_n(t) = (t + t^{n+2}, t^{n+1}, t^n)$. For $n \leq 4$, it was shown by Craighero that θ_n can be *rectified*, i.e., that there exist $u, v, w \in k[x,y,z]$ such that $B = k[u,v,w]$ and $\theta_n^* : B \to k[t]$ is given by

$$\theta_n^*(u) = \theta_n^*(v) = 0 \quad \text{and} \quad \theta_n^*(w) = t \ .$$

Another proof of rectifiability for $n = 3, 4$ is given by Bhatwadekar and Roy using:

Proposition 3 *Let $P \in k[x,y,z]$ have the form*

$$P = x + f(x,z)z + \lambda z^s y \ ,$$

where $f \in k[x,z]$, $s \in \mathbb{Z}$ is non-negative, and $\lambda \in k^*$. Then P is a variable of $k[x,y,z]$.

We give the explicit straightening of θ_3 via derivations.

18 Slices

- **Problem.** If $D \in \mathrm{LND}(k^{[n+1]})$ has a slice, is $\ker D \cong k^{[n]}$?

19 Torus Action Formula

Theorem 26 $\rho_t = \exp(-\lambda D)|_{\lambda=(1-t^n)s}$.

Example 1 *The simple action of $\mathbb{G}_m$ on $\mathbb{A}^n$ given by*

$$t(x_1, ..., x_n) = (x_1, ..., x_{n-1}, t^N x_n)$$

is of the form $\exp(-\lambda D)|_{\lambda=(1-t^N)x_n}$, where D is the partial derivative $D = \frac{\partial}{\partial x_n}$.

20 Asanuma's Torus Actions

The main result Asanuma proves is this.

Theorem 27 *Let k be an infinite field. If there exists a non-rectifiable closed embedding $\mathbb{A}^m \to \mathbb{A}^n$, then there exist non-linearizable faithful algebraic $(k^*)^r$-actions on $\mathbb{A}^{1+n+m}$ for each $r = 1, ..., 1+m$.*

To put his theorem to use, Asanuma uses an earlier result of Shastri.

Theorem 28 *Every open knot type admits a polynomial representation in $\mathbb{R}^3$.*

20.1 The Derivation Associated to an Embedding

Theorem 29 *Suppose $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ is a rectifiable embedding, and let D be the induced triangular derivation (**??**) of $k[u, v, \mathbf{x}]$. If $k[x, y, z] = k[F, G, H]$, where $\phi^*(F) = v$ and $\ker \phi^* = (G, H)$, then*

$$\ker D = k[u, F(u\mathbf{x} + \phi(v)), \tfrac{1}{u}G(u\mathbf{x} + \phi(v)), \tfrac{1}{u}H(u\mathbf{x} + \phi(v))] \ .$$

In particular, D is conjugate to a partial derivative.

20.2 Two-Dimensional Torus Action

Theorem 30

1. *$B^G = k[u\mathbf{x} + \phi(v)] \cong k^{[3]}$;*
2. *The fixed point set $\mathrm{Fix}(G)$ is the line defined by the ideal $I = (u, \mathbf{x})$;*
3. *The canonical embedding $\mathrm{Fix}(G) \hookrightarrow \mathrm{Spec}(B^G)$ is equivalent to the embedding $\phi : k^1 \hookrightarrow k^3$.*

20.3 The One-Dimensional Torus Action

Theorem 31

1. *$B^H = k[u\mathbf{x} + \phi(v), s] \cong k^{[4]}$;*
2. *The fixed point set $\mathrm{Fix}(H)$ is the surface defined by the ideal $J = (u, \mathbf{x} - \phi'(v)s)$, and $\mathrm{Fix}(H) \cong k^2$;*
3. *The canonical embedding $\mathrm{Fix}(H) \hookrightarrow \mathrm{Spec}(B^H)$ is equivalent to the embedding $(1 \times \phi) : k^2 \hookrightarrow k^4$.*

Corollary 4 *If the embedding $\phi : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ is not rectifiable, the induced algebraic actions of $\mathbb{G}_m$ and $\mathbb{G}_m^2$ on $\mathbb{A}^5$ are not linearizable.*

Kaliman's construction

21 Open Questions

Let $\phi : \mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ denote the trefoil knot embedding given above.

> **Question.** If D is the triangular derivation of $\mathbb{R}^{[5]}$ associated to ϕ, is $\ker D$ a polynomial ring?

If the answer here is negative, Asanuma's constructions provide a counterexample to the Real Cancellation Problem.

We next consider the complexification of ϕ.

> **Question.** Let $\Phi : \mathbb{C}^1 \to \mathbb{C}^3$ be the extension of ϕ. Is Φ algebraically rectifiable?

Theorem 32 (Derksen, van den Essen, van Rossum) *Suppose D is a locally nilpotent derivation of $B = k[u, x, y, z] \cong k^{[4]}$ (k a field of characteristic zero) such that $Du = 0$, and $Ds = 1$ for some $s \in B$. Then D is conjugate to the partial derivative ∂_z.*

In particular, the Cancellation Problem has an affirmative answer for every triangular derivation of $k^{[4]}$ having a slice.

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