A GEOMETRIC CONSTRUCTION OF TANGO BUNDLE ON $\mathbb{P}^5$

DANIELE FAENZI

Abstract. The Tango bundle T over $\mathbb{P}^5$ is proved to be the pull-back of the twisted Cayley bundle $C(1)$ via a map $f\colon \mathbb{P}^5 \to Q_5$ existing only in characteristic 2. The Frobenius morphism φ factorizes via such f.
Using f the cohomology of T is computed in terms of $S \otimes C$, $\varphi^*(C)$, $\mathrm{Sym}^2(C)$ and C, while these are computed by applying Borel-Bott-Weil theorem.
By machine-aided computation the mimimal resolutions of C and T are given; incidentally the matrix presenting the spinor bundle S over Q_5 is shown.

1. Introduction

The well-known Hartshorne conjecture states, in particular, that there are no indecomposable rank-2 vector bundles on $\mathbb{P}^n$, when n is greater than 5. However, one of the few rank-2 bundles on $\mathbb{P}^5$ up to twist and pull-back by finite morphisms is the Tango bundle T first given in [Tango 1]. Later Horrocks in [Horrocks] and Decker Manolache and Schreyer in [Decker-Manolache-Schreyer] descovered that it can be obtained starting from Horrocks rank-3 bundle: anyway it only exists in characteristic 2.

Here we prove that T is the pull-back of the twisted Cayley bundle $C(1)$ (defined over any field) via a map $f\colon \mathbb{P}^5 \to Q_5$ existing only in characteristic 2. This allows to compute its cohomology by Borel-Bott-Weil theorem (see 2.2). We observe non-standard cohomology of $\mathrm{Sym}^2 C$ in characteristic 2, although $C \otimes S$ behaves standardly (i.e. as if $\mathrm{char}(k) = 0$).

Different ways to prove that Tango's equations actually give $C(1)$ are shown in section (3). Lastly in section (4) we give the minimal resolution of T and some more computational remarks. We make extensive use of `Macaulay2` computer algebra system: computation-related material can be found on the url: `http://www.math.unifi.it/~faenzi/tango/`

I would like to thank Wolfram Decker for the observations on the Horrocks bundle and his help on machine computation and Edoardo Ballico for his careful remarks about Ekedahl's work and representations in positive characteristic. Also I am indebted with Giorgio Ottaviani, who posed to me this problem.

So let

$$Q_5 = \{z_0^2 + z_1 z_2 + z_3 z_4 + z_5 z_6 = 0\} \subset \mathbb{P}^6$$

be the 5-dimensional smooth quadric over an algebraically closed field k. We denote by ξ (respectively η, ζ) the generator of $A^1(\mathbb{P}^5)$ (respectively of $A^1(Q_5)$, $A^3(Q_5)$), so that

$$A(\mathbb{P}^5) = \mathbb{Z}[\xi]/(\xi^6) \quad A(Q_5) = \mathbb{Z}[\eta, \zeta]/(\eta^3 - 2\zeta, \eta^6)$$

2000 Mathematics Subject Classification Primary 14F05; Secondary 14-04,14J60.

The author was partially supported by Italian Ministery funds and the EAGER contract HPRN-CT-2000-00099.

On the coordinate ring $S(\mathbb{P}^5)$ we use variables x_i's while on $S(Q_5)$ we use z_j's.

2. The bundle on $\mathbb{P}^5$

Let k be an algebraically closed field. On $Q_5 = G_2/P(\alpha_1)$ we have the Cayley bundle C, coming from the standard representation of the semisimple part of the parabolic group $P(\alpha_1)$, where α_1 is the shortest root in the Lie algebra of G_2, the exceptional Lie group. C is irreducible G_2-homogeneous with maximal weight $\lambda_2 - 2\lambda_1$. $C(2)$ (weight λ_2) is globally generated and $\mathrm{h}^0(C(2)) = 14$.
C is the cohomology of a monad:

$$\mathcal{O}(-1) \longrightarrow S \longrightarrow \mathcal{O}$$

where S is the spinor bundle. C has rank 2 and Chern classes $(-1, 1)$. The only non-vanishing intermediate cohomology groups are $\mathrm{H}^1(C) = \mathrm{H}^4(C(-4)) = k$. All this is done in [Ottaviani 2] and follows easily from [Jantzen, Proposition 5.4] in any characteristic.
When $\mathrm{char}(k) = 2$ we have a map $f \colon \mathbb{P}^5 \to Q_5$ having the expression

$$f : (x_0 : \ldots : x_5) \mapsto (x_0x_1 + x_2x_3 + x_4x_5, x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2)$$

and, letting φ be the Frobenius and π the projection from $(1 : 0 : 0 : 0 : 0 : 0 : 0)$ we have:

$$\begin{tikzcd} & Q_5 \arrow[d, "\pi"] \\ \mathbb{P}^5 \arrow[ur, "f"] \arrow[r, "\varphi"'] & \mathbb{P}^5 \end{tikzcd} \qquad \varphi^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^5}(2) \tag{1}$$

$$\pi^*(\xi) = \eta \qquad \pi^*(\xi^2) = \eta^2 \qquad \pi^*(\xi^3) = \eta^3 = 2\zeta$$

We define $T = f^*(C(1))$. The rank-2 vector bundle T is the main subject of this paper.

By Hirzebruch-Riemann-Roch theorem we have the following easy observation

Observation 2.1. T has Chern classes $(2, 4)$ and

$$\chi(T(t)) = \frac{1}{60}t^5 + \frac{1}{3}t^4 + \frac{25}{12}t^3 + \frac{11}{3}t^2 - \frac{51}{10}t - 14$$

The proof of the following theorem on the cohomology of T is left at the end of this section.

Theorem 2.2. *The non-zero cohomology of T is the following:*

$\mathrm{h}^0(T(t)) = \chi(T(t))$ *for* $t \geq 2$

$\mathrm{h}^1(T(-2)) = 1 \quad \mathrm{h}^1(T(-1)) = 7 \quad \mathrm{h}^1(T) = 14 \quad \mathrm{h}^1(T(1)) = 13 \quad \mathrm{h}^1(T(2)) = 1$

$\mathrm{h}^2(T(-3)) = 1$

and their Serre-dual $\mathrm{h}^{5-i}(T(t)) = \mathrm{h}^i(T(-t-8))$. $T' = T(-4)$ *is the only twist for which all cohomology groups vanish.*

For this we first need an observation on the map f.

Observation 2.3. For the map f above we have

$$f_*(\mathcal{O}_{\mathbb{P}^5}) = \mathcal{O} \oplus \mathcal{O}(-1)^{14} \oplus \mathcal{O}(-2)$$

$$f_*(\mathcal{O}_{\mathbb{P}^5}(1)) = S \oplus \mathcal{O}^6 \oplus \mathcal{O}(-1)^6 \tag{2}$$

Proof. Let $\mathcal{F} = f_*(\mathcal{O}_{\mathbb{P}^5}), \mathcal{G} = f_*(\mathcal{O}_{\mathbb{P}^5}(1))$. The map f is a 16: 1 cover, because the Frobenius is 32: 1 and the projection π is 2: 1. Then $\mathcal{F}$ and $\mathcal{G}$ are rank-16 vector bundles, whose cohomology one can read from the Leray degenerate spectral sequence. Indeed since $R^i(f_*) = 0$ (for $i > 0$) we have

$$\mathrm{H}^i(Q_5, \mathcal{F}(t)) = \mathrm{H}^i(\mathbb{P}^5, f^*(\mathcal{O}_{Q_5}(t)) = \mathrm{H}^i(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2t))$$
$$\mathrm{H}^i(Q_5, \mathcal{G}(t)) = \mathrm{H}^i(\mathbb{P}^5, f^*(\mathcal{O}_{Q_5}(1)(t)) = \mathrm{H}^i(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2t+1))$$

for $0 \le i \le 5$ and every t. This says that $\mathcal{F}$ and $\mathcal{G}$ have no intermediate cohomology, hence by [Kapranov] or [Buchweitz-Greuel-Schreyer] they must decompose as sum of spinors S and line bundles, up to twist (although actually Kapranov's setting is over $\mathbb{C}$). For $\mathcal{G}$ this implies, by a computation on the Euler characteristic, that the only choice is the one stated while for $\mathcal{F}$ we have a priori two possibilities: the one stated above or $S \oplus S(-1) \oplus \mathcal{O} \oplus \mathcal{O}(-1)^6 \oplus \mathcal{O}(-2)$

Now the above formula says that the polynomial ring $S(\mathbb{P}^5)$ decomposes as module over $S(Q_5)$ (under the action given by f) as $S(\mathbb{P}^5)_{\text{even}} \oplus S(\mathbb{P}^5)_{\text{odd}}$ where:

$$S(\mathbb{P}^5)_{\text{even}} = \bigoplus_{t\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^5, \mathcal{O}(2t)) \qquad S(\mathbb{P}^5)_{\text{odd}} = \bigoplus_{t\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^5, \mathcal{O}(2t+1))$$

For $\mathcal{F}$ we have to compute explicitely a presentation of the $S(Q_5)-$module $S(\mathbb{P}^5)_{\text{even}}$. We need e_0 to generate $k = S(\mathbb{P}^5)_0$ and e_{ij} to generate the monomial $x_i x_j$ $(i \ne j)$ in $S(\mathbb{P}^5)_2$, thus obtaing a map

$$\Phi \colon S(Q_5) \oplus S(Q_5)(-1)^{15} \longrightarrow S(\mathbb{P}^5)_0 \oplus S(\mathbb{P}^5)_2$$

But the coordinate e_{45} is redundant, since $z_0\Phi(e_0) + \Phi(e_{01}) + \Phi(e_{23}) = \Phi(e_{45})$. Now for $S(\mathbb{P}^5)_4$: the terms containing x_i^2 already lie in the image (got by the action of z_{i-1}), and in fact we just have to fix $x_0x_1x_2x_3$ because, e.g. $x_0x_1x_2x_4 = z_3\Phi(e_{34}) + z_5\Phi(e_{25}) + z_0\Phi(e_{24})$ and $x_0x_1x_4x_5 = x_0x_1x_2x_3 + z_1z_2\Phi(e_0) + z_0\Phi(e_{01})$. Thus we get a generator in degree 2 (and no syzygy); moreover $x_0x_1x_2x_3x_4x_5 = z_1z_2\Phi(e_{23}) + z_3z_4\Phi(e_{01}) + z_0(x_0x_1x_2x_3)$, so that $S(\mathbb{P}^5)_6$, and in fact all $S(\mathbb{P}^5)_{\text{even}}$, is also covered.

The presentation could be computed also for $S(\mathbb{P}^5)_{\text{odd}}$, where one finds as syzygy of a map $S(Q_5)^6 \oplus S(Q_5)(-1)^{14} \to S(\mathbb{P}^5)_{\text{odd}}$ the matrix giving the spinor bundle described later in (8), thus getting again (2). □

Remark 2.4. It seems likely that the observation (2.3) can be extended to any odd dimension. Here we only mention that for $f \colon \mathbb{P}^3 \longrightarrow Q_3$ we get $f_*(\mathcal{O}_{\mathbb{P}^3}) = S \oplus \mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}(-1)$ and $f_*(\mathcal{O}_{\mathbb{P}^3}(1)) = \mathcal{O}_{Q_3}^4$ which can be computed by the presentation or by Euler characteristic.

Lastly, one may notice that $\pi_*(S) = \mathcal{O}_{\mathbb{P}^5}(-1)^8$ and clearly $\pi_*(\mathcal{O}_{\mathbb{P}^5}(t)) = \mathcal{O}(t) \oplus \mathcal{O}(t-1)$ so that the extension $0 \to S(-1) \to \mathcal{O}(-1)^8 \to S \to 0$ becomes split after π_* (actually this holds in any characteristic). This agrees with the formula

$$\varphi_*(\mathcal{O}_{\mathbb{P}^5}) = \mathcal{O} \oplus \mathcal{O}(-1)^{15} \oplus \mathcal{O}(-2)^{15} \oplus \mathcal{O}(-3)$$
$$\varphi_*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}^6 \oplus \mathcal{O}(-1)^{20} \oplus \mathcal{O}(-2)^6$$

Next we compute the cohomology of $\mathrm{Sym}^2 C$, $C \otimes C$, $S \otimes C$. First notice that if V is the $SL(2)-$representation giving C, when $\mathrm{char}(k) = 2$ the representation $\mathrm{Sym}^2 V$ (having weight $2\lambda_2 - 4\lambda_1$) will not be irreducible (recall that in finite characteristic $SL(2)$ is not linearly reductive, check [Nagata]), on the contrary, letting $C^{[2]} = \varphi^*(C)$, we have the non-split exact sequence

$$0 \longrightarrow C^{[2]} \longrightarrow \mathrm{Sym}^2 C \longrightarrow \mathcal{O}(-1) \longrightarrow 0 \tag{3}$$

We also have

$$\begin{aligned} \varphi_*(\mathcal{O}_{Q_5}) &= \mathcal{O} \oplus \mathcal{O}(-1)^{20} \oplus \mathcal{O}(-2)^7 \oplus S(-1) \\ \varphi_*(\mathcal{O}_{Q_5}(1)) &= \mathcal{O}^7 \oplus \mathcal{O}(-1)^{20} \oplus \mathcal{O}(-2) \oplus S \end{aligned} \tag{4}$$

Now again by Borel-Bott-Weil theorem ([Jantzen, Proposition 5.4]) we know $\mathrm{h}^i(\mathrm{Sym}^2 C(3)) = 0$, $\mathrm{Sym}(4)$ is globally generated and:

$$\mathrm{h}^0(\mathrm{Sym}(t)) = \chi(\mathrm{Sym}^2 C(t)) = \frac{1}{20}t^5 + \frac{3}{8}t^4 - \frac{27}{8}t^2 - \frac{81}{20}t \quad \text{for } t \geq 4$$
$$\mathrm{h}^1(\mathrm{Sym}^2 C(2)) = 14 \quad \mathrm{h}^1(\mathrm{Sym}^2 C(1)) = 7 \quad \mathrm{h}^0(\mathrm{Sym}^2 C(-1)) = 1$$

where in the above twists these are the only non-vanishing H^i's. By Serre duality we only miss $\mathrm{H}^i(\mathrm{Sym}^2 C)$. Now since $R^i(\varphi_*) = 0$, by Leray spectral sequence, (3) and (4) we get:

$$\begin{aligned} \mathrm{H}^2(\mathrm{Sym}^2 C) = \mathrm{H}^2(C^{[2]}) &\overset{\text{Leray}}{=} \mathrm{H}^2(C \otimes S(-1)) \overset{\text{Leray}}{=} \\ &= \mathrm{H}^2(C^{[2]}(-1)) = \mathrm{H}^2(\mathrm{Sym}^2 C(-1)) \overset{\text{Bott}}{=} k \end{aligned}$$

and by the same argument $\mathrm{h}^1(\mathrm{Sym}^2 C) = 1$, while the remaining h^i are zero. The same procedure yields the values of $\mathrm{h}^i(S \otimes C(t))$

$$\begin{aligned} &\mathrm{h}^1(S \otimes C) = 1 && \mathrm{h}^1(S \otimes C(1)) = 6 \\ &\mathrm{h}^2(S \otimes C(-1)) = 1 && \mathrm{h}^3(S \otimes C(-2)) = 1 \\ &\mathrm{h}^4(S \otimes C(-4)) = 6 && \mathrm{h}^4(S \otimes C(-5)) = 1 \end{aligned} \tag{5}$$

Now we can prove theorem (2.2)

Proof of theorem (2.2). Since $R^i(f_*) = 0$ for $i > 0$ we use the Leray degenerate spectral sequence that here reads:

$$\begin{aligned} \mathrm{H}^i(\mathbb{P}^5, T(2t)) &= \mathrm{H}^i(Q_5, f_*(\mathcal{O}_{\mathbb{P}^5}) \otimes C(1+t)) \\ \mathrm{H}^i(\mathbb{P}^5, T(2t+1)) &= \mathrm{H}^i(Q_5, f_*(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes C(1+t)) \end{aligned} \tag{6}$$

The even part gives $\mathrm{H}^i(\mathbb{P}^5, T(2t)) = \mathrm{H}^i(Q_5, C(1+t)) \oplus \mathrm{H}^i(Q_5, C(t))^{14} \oplus \mathrm{H}^i(Q_5, C(t-1))$, while the odd part gives $\mathrm{H}^i(\mathbb{P}^5, T(2t+1)) = \mathrm{H}^i(Q_5, C(1+t))^6 \oplus \mathrm{H}^i(Q_5, C(t))^6 \oplus \mathrm{H}^i(Q_5, S \otimes C(1+t))$ and these are known by the above formulas. $\square$

Now we can write the Beilinson table of the normalized $T(-1)$ (rather than of T) that reads:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we write $T(-1)$ as cohomology of a monad:

$$\mathcal{O}(-1) \oplus \Omega^4(4) \longrightarrow \Omega^2(2) \oplus \Omega^1(1) \longrightarrow \mathcal{O}^7$$

The following remark shows that our situation is more "rigid" than one may expect.

Remark 2.5. We notice that we have the diagram:

$$\begin{tikzcd}
& Q_5 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Q_5 \\
\mathbb{P}^5 \arrow[ur, "f"] \arrow[r, "\varphi"'] & \mathbb{P}^5 \arrow[ur, "f"'] &
\end{tikzcd}$$

Then, as Edoardo Ballico pointed out to us, we may relate this framework to a description given by Ekedahl in [Ekedahl, proposition 2.5] of finite morfisms ψ (with Y smooth) that factor through the Frobenius φ in characteristic $p > 0$:

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"'] & Y \\
\mathbb{P}^n \arrow[ur] &
\end{tikzcd}$$

Ekedahl shows that in such a situation Y is Q_n, n is odd, ψ is the projection from a point external to Q_n and the characteristic is 2. That is, precisely our setup.

Remark 2.6. The values $\mathrm{h}^2(\mathrm{Sym}^2 C) = \mathrm{h}^1(\mathrm{Sym}^2 C) = 1$ exhibit non-standard cohomology for the representation $\mathrm{Sym}^2 V$. Indeed $3\lambda_2 - 3\lambda_1$ is singular $((3\lambda_2 - 3\lambda_1, 3\alpha_1 + \alpha_2) = 0)$ so standard Borel-Bott-Weil theorem (i.e. in characteristic 0) would give $\mathrm{h}^i(\mathrm{Sym}^2 C) = 0$.
Of course, we would have no such sequence as (3). Still, by tensoring the monad defining C by $C(t)$ we get

$$\begin{array}{c}
0 \longrightarrow G^\vee \otimes C(t) \longrightarrow S \otimes C(t) \longrightarrow C(t) \longrightarrow 0 \\
0 \longrightarrow C(t-1) \longrightarrow G^\vee \otimes C(t) \longrightarrow C \otimes C(t) \longrightarrow 0
\end{array}$$

whence we derive the values of $\mathrm{h}^i(S \otimes C(t))$ from those of $\mathrm{Sym}^2 C$, since if $\mathrm{char}(k) \neq 2$ $C \otimes C = \mathrm{Sym}^2 C \oplus \mathcal{O}(-1)$. This way we would get the same values as in (5) if $\mathrm{char}(k) = 0$.
Finally, one can compute on `Macaulay2` the values of the cohomology of $C^{[2]}$ and check the correcteness of the above result.

3. The Cayley bundle and Tango's equations

In this sections we work over any algebraically closed field k and prove that Tango's equations in [Tango 1] give $C(1)$.

First we complete Tango's 3×6 matrix by stacking 9 rows to get A having everywhere rank 3 over the quotient ring $S(Q_4) = k[z_0, \dots, z_6]/(z_0^2 + z_1 z_2 + z_3 z_4 + z_5 z_6)$

(7)
$$A = \begin{pmatrix}
z_0^2 & 0 & 0 & z_1^2 & z_1 z_3 + z_0 z_6 & -z_0 z_4 + z_1 z_5 \\
0 & z_0^2 & 0 & z_1 z_3 - z_0 z_6 & z_3^2 & z_0 z_2 + z_3 z_5 \\
0 & 0 & z_0^2 & z_0 z_4 + z_1 z_5 & -z_0 z_2 + z_3 z_5 & z_5^2 \\
0 & z_0 z_2 - z_3 z_5 & z_3^2 & -z_0 z_3 - z_2 z_6 & 0 & z_2^2 \\
z_5^2 & 0 & z_0 z_4 - z_1 z_5 & z_4^2 & -z_2 z_4 - z_0 z_5 & 0 \\
-z_3^2 & z_1 z_3 + z_0 z_6 & 0 & -z_6^2 & 0 & -z_0 z_3 + z_2 z_6 \\
0 & z_5^2 & -z_0 z_2 - z_3 z_5 & -z_2 z_4 + z_0 z_5 & z_2^2 & 0 \\
z_1 z_3 - z_0 z_6 & -z_1^2 & 0 & 0 & -z_6^2 & z_0 z_1 + z_4 z_6 \\
z_0 z_4 + z_1 z_5 & 0 & -z_1^2 & 0 & -z_0 z_1 + z_4 z_6 & -z_4^2 \\
-z_2^2 & -z_2 z_4 - z_0 z_5 & z_0 z_3 - z_2 z_6 & z_0^2 & 0 & 0 \\
z_2 z_4 - z_0 z_5 & z_4^2 & z_0 z_1 + z_4 z_6 & 0 & z_0^2 & 0 \\
-z_0 z_3 - z_2 z_6 & z_0 z_1 - z_4 z_6 & -z_6^2 & 0 & 0 & -z_0^2
\end{pmatrix}$$

indeed we have:

$$\begin{aligned}
&\operatorname{sat}(A) = \operatorname{sat}((\wedge^2 A)) = \operatorname{sat}((\wedge^3 A)) = (1) \\
&\wedge^4 A = (0)
\end{aligned}$$

Now let H be the sheaf $(\operatorname{Im}(A)^{\sim})^{\vee}(-1)$. Tango's bundle F is defined as the cokernel $0 \to \mathcal{O} \to H \to F \to 0$.

One can ask `Macaulay2` for the cohomology groups $\mathrm{H}^i(H)$ and the result is:

$$\begin{aligned}
&\mathrm{h}^i(Q_5, H(t)) = 0 \qquad \forall\, t, \quad 0 < i < 5, \text{ except:} \\
&\mathrm{h}^1(Q_5, H(-1)) = 1
\end{aligned}$$

Therefore we have:

$$\operatorname{Ext}^1(\mathcal{O}, H(-1)) = k$$

the associated extension is:

$$0 \longrightarrow H(-1) \longrightarrow W \longrightarrow \mathcal{O} \longrightarrow 0$$

where W is a 4-bundle whose intermediate cohomology is forced to be zero. Then by [Buchweitz-Greuel-Schreyer] or [Kapranov] and Euler characteristic we get $W = S$ and $H(-1) = G^{\vee}$ so that $F = C(1)$.

A different method would be proving that $C(1)$ and F have the same resolution. The resolution of $C(1)$ can be obtained on the computer by in the following way. We first need a matrix for the spinor bundle S and we derive it from [Beauville]. We know that S is a rank-4 bundle with no intermediate cohomology, $S^{\vee} = S(1)$, and $\mathrm{h}^0(S(1)) = \mathrm{h}^5(S(-5)) = 8$.
Then by Beilinson theorem $S(1)$ extended by zero to $\mathbb{P}^6$ has the resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^6}(-1)^8 \xrightarrow{B} \mathcal{O}_{\mathbb{P}^6}^8 \longrightarrow S(1) \longrightarrow 0$$

where now B, by the observations in [Beauville], is an antisymmetric matrix whose determinant is the equation of the quadric, to the power 4. This is done by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & -z_3 & 0 & -z_1 & z_5 & -z_0 \\ 0 & 0 & z_3 & 0 & z_1 & 0 & -z_0 & -z_6 \\ 0 & -z_3 & 0 & 0 & -z_5 & z_0 & 0 & -z_2 \\ z_3 & 0 & 0 & 0 & z_0 & z_6 & z_2 & 0 \\ 0 & -z_1 & z_5 & -z_0 & 0 & 0 & 0 & z_4 \\ z_1 & 0 & -z_0 & -z_6 & 0 & 0 & -z_4 & 0 \\ -z_5 & z_0 & 0 & -z_2 & 0 & z_4 & 0 & 0 \\ z_0 & z_6 & z_2 & 0 & -z_4 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

Incidentally, we mention that by matrices written in such a fashion one can obtain the spinor bundles over the quadric of any dimension. Here by a standard mapping cone construction we get an infinite 2-periodic resolution of $C(1)$ of the form:

$$R^\bullet \xrightarrow{\delta} \mathcal{O}_{Q_5}(-4)^{55} \to \mathcal{O}_{Q_5}(-3)^{49} \to \mathcal{O}_{Q_5}(-2)^{34} \to \mathcal{O}_{Q_5}(-1)^{14} \to C(1) \to 0$$

where

$$R^\bullet = \cdots \xrightarrow{d} \mathcal{O}_{Q_5}(-i-3)^{56} \xrightarrow{e} \mathcal{O}_{Q_5}(-i-2)^{56} \xrightarrow{d} \mathcal{O}_{Q_5}(-i-1)^{56} \xrightarrow{e} \cdots$$

and this coincides with what we get by Tango's original construction.

We remark that the periodicity here is a standard behaviour, by [Eisenbud]. The rank of the kernel of δ is 28 and again it must have no intermediate cohomology. Then Euler characteristic shows that it must be $S(-5)^{\oplus 7}$, i.e. the resolution actually reads:

$$0 \to S(-5)^{\oplus 7} \to \mathcal{O}_{Q_5}(-4)^{55} \to \mathcal{O}_{Q_5}(-3)^{49} \to \mathcal{O}_{Q_5}(-2)^{34} \to \mathcal{O}_{Q_5}(-1)^{14} \to C(1) \to 0$$

This agrees with [Ottaviani-Szurek, pag. 197] (here we consider the dual spectral sequence) i.e. the above is a *Kapranov sequence* for $C(1)$.

Yet another method is the following. First one proves the analogous for Q_5 of [Tango 1, lemma 1], i.e.

Lemma 3.1. *Let $\rho\colon Q_n \longrightarrow \mathbb{G}(\mathbb{P}^k, \mathbb{P}^n)$ be a non-constant morphism, with n odd and k even, and let $\mathcal{E}$ be the pull-back on Q_n of the dual universal sub-bundle $U^\vee$ on $\mathbb{G}(\mathbb{P}^k, \mathbb{P}^n)$. Then $k = \frac{n-1}{2}$, and:*

$$c_i(\mathcal{E}) = 2a^i$$

for some positive integer a.

Here $\rho^*(U)(1) = H$ defined above, where ρ is given by the matrix A (7): a row of $A(x)$ is a point in $\mathbb{P}^6$ and $\mathrm{rk}(A(x)) = 3$, hence $A(x)$ represents a $\mathbb{P}^2 \subset \mathbb{P}^6$. But we know that $\rho^*(U)$ contains the sub-line-bundle $\mathcal{O}(-1)$, so that $a = 1$. So $H_{\text{norm}} = H(-1)$. Then `Macaulay2` gives $\mathrm{H}^0(H_{\text{norm}}) = \mathrm{H}^0(\wedge^2 H_{\text{norm}}) = 0$ thus H is stable by Hoppe's criterion. Hence we can conclude by [Ottaviani 1] that $\rho^*(U) = G^\vee$ and $F = C(1)$.

Proof of lemma (3.1). The proof is almost identical to the case of $\mathbb{P}^n$, the only difference being that we have to work in $H^*(Q_n)$, where $\eta^{\frac{n+1}{2}} = 2\zeta$.
We can suppose k even and $k \leq \frac{n-1}{2}$ because $\mathbb{G}(\mathbb{P}^k, \mathbb{P}^n) \cong \mathbb{G}(\mathbb{P}^{n-k-1}, \mathbb{P}^n)$ and we

put $a_i = c_i(\rho^*(\mathcal{E}))$, $b_i = c_i(\rho^*(Q))$ where Q is the universal quotient bundle. We have, in the ring $A(Q_5)[t]$ the relation on Chern polynomials

$$c_{\mathcal{E}^\vee}(t) \cdot c_{\rho^*(Q)}(t) = 1 \tag{9}$$

Now we can think of the coefficients in (9) as integers times some η^r, taking care to replace ζ by $\frac{1}{2}\eta^{\frac{n+1}{2}}$, that is, replacing a_i (and b_i) by $a'_i = \frac{1}{2}a_i$ (by $b'_i = \frac{1}{2}b_i$) whenever $i \geq \frac{n+1}{2}$.
Then one proceeds exactly as in [Tango 1, lemma 1] and [Tango 2, lemma 3.3], and finds:

$$\begin{aligned} k &= \frac{n-1}{2} \\ a'_i &= 2a^i \qquad \text{for } i = 1, \dots, \tfrac{n-1}{2} = k \\ a'_{k+1} &= a^{k+1} \end{aligned}$$

and so $a_i = 2a^i$, for all i's, as only for $i = k+1$ we have to substitute $a_{k+1} = 2a'_{k+1}$. □

4. Further remarks

Now we turn back to $\mathbb{P}^5$ and $\mathrm{char}(k) = 2$. As we have the equations for T, `Macaulay2` provides the following resolution

$$0 \longrightarrow \begin{matrix} \mathcal{O}^7(-7) \\ \oplus \\ \mathcal{O}(-8) \end{matrix} \longrightarrow \mathcal{O}^{49}(-6) \longrightarrow \mathcal{O}^{98}(-5) \longrightarrow \mathcal{O}^{76}(-4) \longrightarrow \begin{matrix} \mathcal{O}^7(-3) \\ \oplus \\ \mathcal{O}^{14}(-2) \end{matrix} \longrightarrow T \longrightarrow 0$$

One may check that this gives back the Chern classes $2, 4$.
As Wolfram Decker pointed out to me, another way to get Tango's bundle is by Horrocks bundle $\mathcal{H}$ in characteristic 2. Concretely, Horrocks becomes a non-split extension $0 \to T(-1) \to \mathcal{H} \to \mathcal{O} \to 0$.
This allows to compute the cohomology of $\mathcal{H}$ in terms of T.

Moreover in [Decker-Manolache-Schreyer] one finds an explicit description of the maps in the Beilinson monad. This provides also a check of computations. Denoting with $e_0, \dots, e_5$ the canical basis of E (the exterior algebra over V), and using the natural isomorphism $\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) = \wedge^{i-j}V = E_{j-i}$, T is the cohomology of the maps α and β:

$$\beta = \begin{pmatrix} e_0 & e_4e_5 \\ e_1 & e_3e_5 \\ e_2 & e_3e_4 \\ e_3 & e_1e_2 \\ e_4 & e_0e_2 \\ e_5 & e_0e_1 \\ 0 & e_0e_3 + e_1e_4 + e_2e_5 \end{pmatrix}$$

$$\alpha = \begin{pmatrix} e_0e_1e_2 + e_3e_4e_5 & e_0e_1e_3e_4 + e_0e_2e_3e_5 + e_1e_2e_4e_5 & 0 \\ e_0e_3 + e_1e_4 + e_2e_5 & e_0e_1e_2 + e_3e_4e_5 & e_1e_2e_4e_5 \end{pmatrix}$$

If we compute the resolution we get back the same result, thus restating what is said in [Decker-Manolache-Schreyer, proposition 1.8.].

Applying cohomology algorithms in `Macaulay2` developed by Decker Eisenbud and Schreyer one may also obtain a full table of the cohomology, which I write in

`Macaulay2` notation:

(10)

total:	573	260	92	27	14	7	2	2	7	14	27	92	260	573
-6:	573	260	91	14	.	.	.	.	.	.	.	.	.	.
-5:	.	.	1	13	14	7	1	.	.	.	.	.	.	.
-4:	.	.	.	.	.	.	1	.	.	.	.	.	.	.
-3:	.	.	.	.	.	.	.	1	.	.	.	.	.	.
-2:	.	.	.	.	.	.	.	1	7	14	13	1	.	.
-1:	.	.	.	.	.	.	.	.	.	.	14	91	260	573

Reading the table along one antidiagonal gives the list of cohomology groups of a single twist. Here the list for T starts from the up-right corner, while starting from a shift to the left means reading the list for a $(-1)-$twist.
One can check that table (10) agrees with theorem 2.2.

References

[Beauville] Beauville, Arnaud. *Determinantal hypersurfaces.* Dedicated to William Fulton on the occasion of his 60th birthday. Michigan Math. J. 48 (2000), 39–64.

[Buchweitz-Greuel-Schreyer] Buchweitz, Ragnar-Olaf; Greuel, Gert-Martin; Schreyer, Frank-Olaf. *Cohen-Macaulay modules on hypersurface singularities. II.* Invent. Math. 88, 165-182 (1987).

[Decker-Manolache-Schreyer] Decker, Wolfram; Manolache, Nicolae; Schreyer, Frank-Olaf. *Geometry of the Horrocks bundle on P^5.* Complex projective geometry (Trieste, 1989/Bergen, 1989), 128–148, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[Eisenbud] Eisenbud, David. *Homological algebra on a complete intersection, with an application to group representations.* Trans. Amer. Math. Soc. 260 (1980), no. 1,35–64.

[Ekedahl] Ekedahl, Torsten. *Foliations and inseparable morphisms.* Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 139–149. Proc. Sympos. Pure Math., 46, Part 2, Amer. Math. Soc., Providence, RI, 1987.

[Horrocks] Horrocks, Geoffrey. *Examples of rank three vector bundles on five-dimensional projective space.* J. London Math. Soc. (2) 18 (1978), no. 1, 15–27

[Jantzen] Jantzen, Jens Carsten *Representations of algebraic groups.* Pure and Applied Mathematics, 131. Academic Press, Inc., Boston, MA, 1987.

[Kapranov] Kapranov, Mikhail. *On the derived categories of coherent sheaves on some homogeneous spaces.* Invent. Math. 92, No.3, 479-508 (1988).

[Nagata] Nagata, Masayoshi *Complete reducibility of rational representations of a matric group.* J. Math. Kyoto Univ. 1 1961/1962 87–99.

[Ottaviani 1] Ottaviani, Giorgio. *Spinor bundles on quadrics.* Trans. Amer. Math. Soc. 307 (1988), no. 1, 301–316.

[Ottaviani 2] Ottaviani, Giorgio. *On Cayley bundles on the five-dimensional quadric.* Boll. Un. Mat. Ital. A (7) 4 (1990), no. 1, 87–100.

[Ottaviani-Szurek] Ottaviani, Giorgio; Szurek, Michał. *On moduli of stable 2-bundles with small Chern classes on Q_3.* With an appendix by Nicolae Manolache. Ann. Mat. Pura Appl. (4) 167 (1994), 191–241.

[Tango 1] Tango, Hiroshi. *On morphisms from projective space P^n to the Grassmann variety $Gr(n, d)$.* J. Math. Kyoto Univ. 16 (1976), no. 1, 201–207.

[Tango 2] Tango, Hiroshi. *On $(n-1)$-dimensional projective spaces contained in the Grassmann variety* $\mathrm{Gr}(n, 1)$. J. Math. Kyoto Univ. 14 (1974), 415–460.

E-mail address: `faenzi@math.unifi.it`

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, I-50134, Florence, Italy

URL: `http://www.math.unifi.it/~faenzi/`