Locally nilpotent derivations

Daniel Daigle

Lecture notes for the "September School" of Algebraic Geometry
Łukęcin, Poland, September 2003

Conventions

- The word *"ring"* means *commutative ring with a unity element.*
- The group of units of a ring A is denoted A^*.
- We write $A \leq B$ to indicate that A is a subring of B. If $A \leq B$, the phrase "B is affine over A" means that B is finitely generated as an A-algebra.
- A *"domain"* is an integral domain. If $A \leq B$ are domains, then the transcendence degree of $\mathrm{Frac}(B)$ over $\mathrm{Frac}(A)$ is denoted $\mathrm{trdeg}_A(B)$.
- If A is a ring and $n \geq 0$ an integer, $A^{[n]}$ denotes any A-algebra isomorphic to the polynomial ring in n variables over A.

1. Derivations of a ring

A *derivation* D of a ring B is a map $D : B \to B$ satisfying

$$D(x+y) = D(x) + D(y) \quad \text{and} \quad D(xy) = D(x)y + xD(y), \qquad \text{for all } x, y \in B.$$

Given a derivation D of a ring B, define the set $B^D = \ker D = \{x \in B \mid D(x) = 0\}$ and note that this is a subring of B. If $\Bbbk \leq B$ are rings and D is a derivation of B satisfying $D(\Bbbk) = \{0\}$, we call D a $\Bbbk$*-derivation* of B; in this case we have $\Bbbk \leq \ker(D) \leq B$. We use the notations:

$$\mathrm{Der}(B) = \text{ set of all derivations of } B, \quad \mathrm{Der}_{\Bbbk}(B) = \text{ set of all } \Bbbk\text{-derivations of } B.$$

Note that $\mathrm{Der}(B)$ is a B-module and that $\mathrm{Der}_{\Bbbk}(B)$ is a B-submodule of $\mathrm{Der}(B)$.

1.1. **Example.** Let $\Bbbk$ be a ring and $B = \Bbbk[X_1, \dots, X_n] = \Bbbk^{[n]}$. Here are two ways to define a $\Bbbk$-derivation of B.

(1) Given $(f_1, \dots, f_n) \in B^n$, there is a unique $D \in \mathrm{Der}_{\Bbbk}(B)$ satisfying $D(X_i) = f_i$ for all $i \in \{1, \dots, n\}$ (namely, $D = \sum_{i=1}^n f_i \, \partial/\partial X_i$). So $\mathrm{Der}_{\Bbbk}(B)$ is a free B-module with basis $\{\partial/\partial X_1, \dots, \partial/\partial X_n\}$.

(2) Given $f = (f_1, \dots, f_{n-1}) \in B^{n-1}$, define the *jacobian derivation* $\Delta_f \in \mathrm{Der}_{\Bbbk}(B)$ by $\Delta_f(g) = \det\left(\frac{\partial(f_1,\dots,f_{n-1},g)}{\partial(X_1,\dots,X_n)}\right)$, for each $g \in B$. Note that $\Bbbk[f_1, \dots, f_{n-1}] \subseteq \ker(\Delta_f)$.

Exercise 1.1. Let B be a ring, $D \in \mathrm{Der}(B)$, $f \in B[T]$ and $b \in B$. Show that

$$D\big(f(b)\big) = f^{(D)}(b) + f'(b)D(b),$$

where $f' \in B[T]$ is the T-derivative of f and where $f^{(D)} = \sum_i D(b_i)T^i \in B[T]$ (where $f = \sum_i b_i T^i$, $b_i \in B$). More generally, if $f \in B[T_1, \dots, T_n]$ and $b_1, \dots, b_n \in B$ then

$$D\big(f(b_1, \dots, b_n)\big) = f^{(D)}(b_1, \dots, b_n) + \textstyle\sum_{i=1}^n f_{T_i}(b_1, \dots, b_n)\, D(b_i),$$

where $f_{T_i} = \frac{\partial f}{\partial T_i} \in B[T_1, \dots, T_n]$.

1.2. **Definition.** Let $A \leq B$ be rings. An element $b \in B$ is *algebraic* over A if there exists a nonzero polynomial $f \in A[T] \setminus \{0\}$ such that $f(b) = 0$ (note that f is not required to be monic); if b is not algebraic over A, we say that b is *transcendental* over A; we say that A is *algebraically closed in B* if each element of $B \setminus A$ is transcendental over A.

Exercise 1.2. Let $A \leq B$ be domains. The set $\overline{A} = \{ b \in B \mid \ b \text{ is algebraic over } A \}$ is called the *algebraic closure of A in B*. Show that $\overline{A} = B \cap L$ where L is the algebraic closure of $\operatorname{Frac} A$ in $\operatorname{Frac} B$. Consequently, $\overline{A}$ is a subring of B $(A \leq \overline{A} \leq B)$.

1.3. **Lemma.** *If B is a domain of characteristic zero and $D \in \operatorname{Der}(B)$ then $\ker D$ is algebraically closed in B.*

Proof. Let $A = \ker D$ and consider $b \in B$ algebraic over A. Let $f \in A[T]$ be a nonzero polynomial of minimal degree such that $f(b) = 0$. Then

$$0 = D(f(b)) = f^{(D)}(b) + f'(b)D(b) = f'(b)D(b).$$

We have $f' \neq 0$, so $f'(b) \neq 0$ by minimality of $\deg f$, so $D(b) = 0$. $\square$

We mention (without proof) a related result:

1.4. **Theorem** (Nowicki). *Let B be an affine domain over a field $\Bbbk$ of characteristic zero. Then, for a $\Bbbk$-subalgebra A of B, tfae:*

(1) *A is algebraically closed in B*
(2) *$A = \ker(D)$ for some $D \in \operatorname{Der}_{\Bbbk}(B)$.*

Exercise 1.3. If $B = \oplus_{i=0}^{\infty} B_i$ is an $\mathbb{N}$-graded domain, B_0 is algebraically closed in B.

Exercise 1.4. Let $A \leq B$ be domains.

(1) If $\operatorname{Frac}(A)$ is algebraically closed in $\operatorname{Frac}(B)$ and $B \cap \operatorname{Frac}(A) = A$, then A is algebraically closed in B. (The converse does not hold, by part (2).)
(2) Let $A = \mathbb{Q}$ and $B = \mathbb{Q}[X,Y]/(X^2 + Y^2)$. Use exercise 1.3 to show that A is algebraically closed in B; show that $\operatorname{Frac} A$ is not algebraically closed in $\operatorname{Frac} B$.

Exercise 1.5. Let $B = \mathbb{C}[X,Y] = \mathbb{C}^{[2]}$ and $A = \mathbb{C}[XY]$.

(1) Show that $\operatorname{Frac}(B)$ is a purely transcendental extension of $\operatorname{Frac}(A)$.
(2) Use exercise 1.4 to show that A is algebraically closed in B.
(3) Consider the jacobian derivation $D = \begin{vmatrix} \frac{\partial(XY)}{\partial X} & \frac{\partial(XY)}{\partial Y} \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} \end{vmatrix} = Y\frac{\partial}{\partial Y} - X\frac{\partial}{\partial X} \in \operatorname{Der}_{\mathbb{C}}(B)$. Note that $D \neq 0$ and $D(XY) = 0$; conclude that $\ker(D) = A$.

1.5. **Definition.** Given a ring B and $D \in \operatorname{Der}(B)$, define the set

$$\operatorname{Nil}(D) = \{ x \in B \mid \exists_{n \in \mathbb{N}} \ D^n(x) = 0 \}.$$

By exercise 1.7 this is a subring of B, so we have: $\ker(D) \leq \operatorname{Nil}(D) \leq B$.

1.6. **Example.** Let $B = \mathbb{Q}[[T]]$ and $D = d/dT : B \to B$. Then $\ker(D) = \mathbb{Q}$ and $\mathrm{Nil}(D) = \mathbb{Q}[T]$. Note that $\mathrm{Nil}(D)$ is not algebraically closed in B (not even integrally closed in B): Let $b = \sqrt{1+T} \in B$, then $b \notin \mathrm{Nil}(D)$ but $b^2 \in \mathrm{Nil}(D)$.

Exercise 1.6. Prove **Leibnitz Rule**: *If B is a ring, $D \in \mathrm{Der}(B)$, $x, y \in B$ and $n \in \mathbb{N}$,*

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(x) D^i(y).$$

Exercise 1.7. Use Leibnitz Rule to show that $\mathrm{Nil}(D)$ is closed under multiplication.

2. Locally nilpotent derivations

2.1. **Definition.** Let B be any ring.

(1) A derivation $D : B \to B$ is *locally nilpotent* if it satisfies $\mathrm{Nil}(D) = B$, i.e., if $\forall_{b \in B} \exists_{n \in \mathbb{N}}\, D^n(b) = 0$.

(2) **Notations:**

$$\begin{aligned} \mathrm{LND}(B) &= \text{set of locally nilpotent derivations } B \to B \\ \mathrm{KLND}(B) &= \{\ker D \mid D \in \mathrm{LND}(B) \text{ and } D \neq 0\}. \end{aligned}$$

If $\mathbb{k} \leq B$,

$$\begin{aligned} \mathrm{LND}_{\mathbb{k}}(B) &= \mathrm{LND}(B) \cap \mathrm{Der}_{\mathbb{k}}(B) \\ \mathrm{KLND}_{\mathbb{k}}(B) &= \{\ker D \mid D \in \mathrm{LND}_{\mathbb{k}}(B) \text{ and } D \neq 0\}. \end{aligned}$$

2.2. **Examples.** Let $\mathbb{k}$ be a ring and $B = \mathbb{k}[X_1, \dots, X_n] = \mathbb{k}^{[n]}$.

(1) For each i, the partial derivative $\frac{\partial}{\partial X_i} : B \to B$ belongs to $\mathrm{LND}_{\mathbb{k}}(B)$.

(2) A $\mathbb{k}$-derivation $D : B \to B$ is *triangular* if

$$\forall\, i \quad D(X_i) \in \mathbb{k}[X_1, \dots, X_{i-1}] \quad (\text{in particular } D(X_1) \in \mathbb{k}).$$

Every triangular $\mathbb{k}$-derivation is locally nilpotent. Indeed, if D is triangular then $\mathbb{k} \subseteq \mathrm{Nil}(D)$ and it is easy to see (by induction on i) that $\forall\, i \quad X_i \in \mathrm{Nil}(D)$; so $\mathrm{Nil}(D) = B$, i.e., D is locally nilpotent.

The sets $\mathrm{LND}(B)$ and $\mathrm{LND}_{\mathbb{k}}(B)$ are not closed under addition and not closed under multiplication by elements of B. For instance, let $B = \mathbb{Q}[X, Y] = \mathbb{Q}^{[2]}$, $D_1 = Y\frac{\partial}{\partial X}$ and $D_2 = X\frac{\partial}{\partial Y}$; then $D_1, D_2 \in \mathrm{LND}(B)$ (because they are triangular) but $D_1 + D_2 \notin \mathrm{LND}(B)$ (because $(D_1 + D_2)^2(X) = X$). Also, $\frac{\partial}{\partial X} \in \mathrm{LND}(B)$ but $X\frac{\partial}{\partial X} \notin \mathrm{LND}(B)$. However:

2.3. **Lemma.** *Let B be a ring. If $D_1, D_2 \in \mathrm{LND}(B)$ satisfy $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \mathrm{LND}(B)$.*

Proof. Let $D_1, D_2 \in \mathrm{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$ and let $b \in B$. Choose $m, n \in \mathbb{N}$ such that $D_1^m(b) = 0 = D_2^n(b)$. The hypothesis $D_2 \circ D_1 = D_1 \circ D_2$ has the following three consequences:

$$\forall_{i\in\mathbb{N}}\ \forall_{j\geq n}\quad (D_1^i \circ D_2^j)(b) = D_1^i(0) = 0,$$

$$\forall_{i\geq m}\ \forall_{j\in\mathbb{N}}\quad (D_1^i \circ D_2^j)(b) = (D_2^j \circ D_1^i)(b) = D_2^j(0) = 0,$$

$$(D_1 + D_2)^{m+n-1} = \textstyle\sum_{i+j=m+n-1} \binom{m+n-1}{i} D_1^i \circ D_2^j,$$

so $(D_1 + D_2)^{m+n-1}(b) = 0$. Hence, $D_1 + D_2 \in \text{LND}(B)$. $\square$

Exercise 2.1. Let B be a ring, $D \in \text{LND}(B)$ and $A = \ker D$.

(1) If $a \in A$ then $aD \in \text{LND}(B)$. (First show that $(aD)^n = a^n D^n$ holds for all $n \in \mathbb{N}$.)
(2) If $S \subset A$ is a multiplicatively closed set, then $S^{-1}D : S^{-1}B \to S^{-1}B$ belongs to $\text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = S^{-1}A$.
(3) Let T be an indeterminate and $f \in B[T]$. Then D has a unique extension $\Delta \in \text{Der}(B[T])$ such that $\Delta(T) = f$. If $f \in B$, then $\Delta \in \text{LND}(B[T])$.

Exercise 2.2. If A is a ring and $B = A[T] = A^{[1]}$, then $\left\{ a\frac{d}{dT} \;\middle|\; a \in A \right\} \subseteq \text{LND}_A(B)$. Show that equality holds whenever A is a domain of characteristic zero. Find an example where the inclusion is strict.

2.4**. Definition.** Let B be a ring and $D \in \text{LND}(B)$. Define a map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ by $\deg_D(x) = \max\{n \in \mathbb{N} \mid D^n x \neq 0\}$ for $x \in B \setminus \{0\}$, and $\deg_D(0) = -\infty$. Note that $\ker D = \{x \in B \mid \deg_D(x) \leq 0\}$. We will see in 2.14 that $\deg_D$ has good properties when B is a domain of characteristic zero.

2.5**. Definition.** Let B be a ring and $D \in \text{LND}(B)$. A *slice* of D is an element $s \in B$ satisfying $D(s) = 1$. A *preslice* of D is an element $s \in B$ satisfying $D(s) \neq 0$ and $D^2(s) = 0$ (i.e., $\deg_D(s) = 1$).

It is clear that if $D \in \text{LND}(B)$ and $D \neq 0$ then D has a preslice. However:

2.6**. Example.** Let $\mathbb{k}$ be a field, $B = \mathbb{k}[X, Y, Z] = \mathbb{k}^{[3]}$ and consider the $\mathbb{k}$-derivation $D = X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z}$. Since D is triangular, it is locally nilpotent. Since $D(B) \subseteq (X, Y)B$, D does not have a slice.

The next fact has many consequences for locally nilpotent derivations:

2.7**. Proposition.** *Consider rings $B \leq C \geq \mathbb{Q}$. If $D \in \text{LND}(B)$ and $\gamma \in C$ then the map*

$$B \longrightarrow C, \qquad b \longmapsto \sum_{n\in\mathbb{N}} \frac{1}{n!} D^n(b)\, \gamma^n$$

is a homomorphism of A-algebras, where $A = \ker(D)$.

Proof. It is clear that the given map preserves addition and restricts to the identity map on A. So it suffices to verify that

$$\left(\sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)\,\gamma^i\right)\left(\sum_{j\in\mathbb{N}} \frac{1}{j!} D^j(y)\,\gamma^j\right) = \sum_{n\in\mathbb{N}} \frac{1}{n!} D^n(xy)\,\gamma^n \tag{1}$$

holds for all $x, y \in B$. In the left hand side of (1), the coefficient of γ^n is

$$\sum_{i+j=n} \frac{1}{i!\,j!} D^i(x)D^j(y) = \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!\,j!} D^i(x)D^j(y),$$

which is equal to $\frac{1}{n!}D^n(xy)$ by Leibnitz Rule. $\square$

Locally nilpotent derivations of $\mathbb{Q}$-algebras

Exercise 2.3. If B is a $\mathbb{Q}$-algebra then $\mathrm{Der}(B) = \mathrm{Der}_{\mathbb{Q}}(B)$.

The following result is Proposition 2.1 of [19]:

2.8. **Theorem.** *Let B be a $\mathbb{Q}$-algebra. If $D \in \textsc{lnd}(B)$ and $s \in B$ satisfy $Ds = 1$ then $B = A[s] = A^{[1]}$, where $A = \ker(D)$.*

Proof. Consider $f(T) = \sum_{i=0}^{n} a_i T^i \in A[T] \setminus \{0\}$ (where $n \geq 0$, $a_i \in A$ and $a_n \neq 0$). Then $D^j(f(s)) = f^{(j)}(s)$ for all $j \geq 0$, where $f^{(j)}(T) \in A[T]$ denotes the j-th derivative of f; so $D^n(f(s)) = n!\,a_n \neq 0$ and in particular $f(s) \neq 0$. So s is transcendental over A, i.e., $A[s] = A^{[1]}$.

To show that $B = A[s]$, consider the homomorphism of A-algebra $\xi : B \to B$, $\xi(x) = \sum_{j=0}^{\infty} \frac{D^j x}{j!}(-s)^j$ (use 2.7 with $B = C$ and $\gamma = -s$). For each $x \in B$,

$$D\big(\xi(x)\big) = \sum_{j=0}^{\infty} \frac{D^{j+1}x}{j!}(-s)^j + \sum_{j=0}^{\infty} \frac{D^j x}{j!} j(-s)^{j-1}(-1) = 0,$$

so $\xi(B) \subseteq A$; since ξ is a A-homomorphism, $\xi(B) = A$.

By induction on $\deg_D(x)$, we show that $\forall_{x\in B}\ x \in A[s]$. This is clear if $\deg_D(x) \leq 0$, so assume that $\deg_D(x) \geq 1$. Since $x = \xi(x) + (x - \xi(x))$ where $x - \xi(x) \in sB$,

$$x = a + x's, \quad \text{for some } a \in A \text{ and } x' \in B. \tag{2}$$

This implies that $Dx = D(x')s + x'$ and it easily follows that

$$\forall_{m\geq 1}\ \ D^m(x) = D^m(x')s + mD^{m-1}(x'). \tag{3}$$

Choose $m \geq 1$ such that $D^{m-1}(x') \neq 0$ and $D^m(x') = 0$ (such an m exists because $\deg_D(x) \geq 1$, so $x \notin A$, so $x' \neq 0$). Then (3) gives $D^m(x) = mD^{m-1}(x') \neq 0$ and $D^{m+1}(x) = 0$, so $\deg_D(x') = \deg_D(x) - 1$. By the inductive hypothesis we have $x' \in A[s]$; then (2) gives $x \in A[s]$. $\square$

2.9. **Corollary.** *Let B be a $\mathbb{Q}$-algebra, $D \in \textsc{lnd}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds \neq 0$ and $D^2 s = 0$, then $B_\alpha = A_\alpha[s] = A_\alpha^{[1]}$ where $\alpha = Ds \in A \setminus \{0\}$.*

Proof. Let $S = \{1, \alpha, \alpha^2, \dots\}$ and consider $S^{-1}D : S^{-1}B \to S^{-1}B$. By exercise 2.1, $S^{-1}D \in \text{LND}(S^{-1}B)$, $\ker(S^{-1}D) = S^{-1}A$ and $(S^{-1}D)(s/\alpha) = 1$, so the result follows from 2.8. $\square$

Exercise 2.4. Let $B = \mathbb{Z}[X,Y] = \mathbb{Z}^{[2]}$ and $D = \frac{\partial}{\partial Y} + Y\frac{\partial}{\partial X}$. Since D is triangular, we have $D \in \text{LND}(B)$. Moreover, $DY = 1$. Show that $\ker D = \mathbb{Z}[2X - Y^2]$ and that B is not a polynomial ring over $\ker D$. (So in 2.8 the hypothesis that B is a $\mathbb{Q}$-algebra is not superfluous.)

Exercise 2.5. Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker D$. Show:

$$D : B \to B \text{ is surjective} \iff D(B) \cap A^* \neq \varnothing \iff D \text{ has a slice} \implies B = A^{[1]}.$$

Locally nilpotent derivations of integral domains

Recall the notation $\text{KLND}(B) = \{\ker D \mid D \in \text{LND}(B) \text{ and } D \neq 0\}$.

2.10. **Lemma.** *Let B be a domain of characteristic zero.*

(1) *If $A \in \text{KLND}(B)$ then $S^{-1}B = (\operatorname{Frac} A)^{[1]}$, where $S = A \setminus \{0\}$; in particular, $\operatorname{trdeg}_A(B) = 1$.*
(2) *If $A, A' \in \text{KLND}(B)$ and $A \subseteq A'$, then $A = A'$.*
(3) *Let $A \in \text{KLND}(B)$ and let D and D' be nonzero elements of $\text{LND}_A(B)$. Then there exist $a, a' \in A \setminus \{0\}$ such that $aD = a'D'$. In particular, $D \circ D' = D' \circ D$.*
(4) *If $A \in \text{KLND}(B)$ then $\text{LND}_A(B)$ is an A-module.*

Proof. Let $A \in \text{KLND}(B)$; consider $D \in \text{LND}(B)$, $D \neq 0$, such that $\ker D = A$. If we write $S = A \setminus \{0\}$ and $K = \operatorname{Frac}(A)$ then exercise 2.1 gives $S^{-1}D \in \text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = K$; it is clear that $S^{-1}D$ has a slice, i.e., there exists $t \in S^{-1}B$ such that $(S^{-1}D)(t) = 1$; then 2.8 implies that $S^{-1}B = K[t] = K^{[1]}$, which proves assertion (1).

If $A, A' \in \text{KLND}(B)$ then $\operatorname{trdeg}_A(B) = 1 = \operatorname{trdeg}_{A'}(B)$ by part (1). If also $A \subseteq A'$, it follows that A' is algebraic over A; as A is algebraically closed in B by 1.3, we have $A = A'$, so (2) is true.

Let $A \in \text{KLND}(B)$ and $S = A \setminus \{0\}$. By part (1), $S^{-1}B = K[t] = K^{[1]}$ for some $t \in S^{-1}B$. If D and D' are nonzero elements of $\text{LND}_A(B)$ then $S^{-1}D$ and $S^{-1}D'$ are nonzero elements of $\text{LND}_K(K[t])$. By exercise 2.2, each nonzero element of $\text{LND}_K(K[t])$ has the form $\lambda\frac{d}{dt}$ for some $\lambda \in K^*$; it follows that $S^{-1}D' = \lambda S^{-1}D$ for some $\lambda \in K^*$ and consequently $aD = a'D'$ for some $a, a' \in A \setminus \{0\}$. It easily follows that $D \circ D' = D' \circ D$, which proves (3). In view of 2.3, it follows that $\text{LND}_A(B)$ is closed under addition, so (4) is true. $\square$

2.11. **Definition.** Let $A \leq B$ be domains. We say that A is *factorially closed* in B if:

$$\forall\, x, y \in B \quad xy \in A \setminus \{0\} \implies x, y \in A.$$

Exercise 2.6. Suppose that A is a factorially closed subring of a domain B. Then:

(1) A is algebraically closed in B and $A^* = B^*$.

(2) An element of A is irreducible in A iff it is irreducible in B.

(3) If B is a UFD then so is A.

2.12**. Definition.** A *degree function* on a ring B is a map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ satisfying:

(1) $\forall\, x \in B \quad \deg x = -\infty \iff x = 0$

(2) $\forall\, x, y \in B \quad \deg(xy) = \deg x + \deg y$

(3) $\forall\, x, y \in B \quad \deg(x + y) \le \max(\deg x, \deg y)$.

Note that if B admits a degree function then it is a domain, by (1) and (2).

2.13**. Lemma.** *If* $\deg$ *is a degree function on a domain* B *then* $\{x \in B \mid \deg x \le 0\}$ *is a factorially closed subring of* B.

Proof. Obvious. $\square$

2.14**. Proposition.** *Let* B *be a domain of characteristic zero and* $D \in \text{LND}(B)$*. Then the map*

$$\deg_D : B \to \mathbb{N} \cup \{-\infty\}$$

(defined in 2.4*) is a degree function.*

Proof. Consider the ring $C = (S^{-1}B)[T]$, where $S = \mathbb{Z} \setminus \{0\}$ and T is an indeterminate. Then $B \le C \ge \mathbb{Q}$, so 2.7 (with $\gamma = T \in C$) implies that $\xi : B \to C$, $\xi(b) = \sum_{i=0}^{\infty} \frac{D^n(b)}{n!} T^n$, is a ring homomorphism. Moreover, ξ is injective because setting $T = 0$ in $\xi(b)$ gives b. Now $\deg_D$ is the composite $B \xrightarrow{\xi} (S^{-1}B)[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$, which is a degree function on B. $\square$

2.15**. Corollary.** *Let* B *be a domain of characteristic zero,* $D \in \text{LND}(B)$ *and* $A = \ker(D)$*. Then* A *is a factorially closed subring of* B*. In particular* $A^* = B^*$*, and if* $\mathbb{k}$ *is any field contained in* B *then* D *is a* $\mathbb{k}$*-derivation.*

Proof. $A = \{x \in B \mid \deg_D(x) \le 0\}$ is clear, so A is factorially closed in B by 2.14 and 2.13. It follows that $A^* = B^*$ and consequently every field contained in B is in fact contained in A. $\square$

2.16**. Theorem.** *Let* B *be a domain of characteristic zero and* $0 \ne D \in \text{Der}(B)$*.*

(1) *Let* $b \in B \setminus \{0\}$ *and consider* $bD \in \text{Der}(B)$*. Then*

$$bD \in \text{LND}(B) \iff D \in \text{LND}(B) \text{ and } b \in \ker(D).$$

(2) *Let* $S \subset B$ *be a multiplicatively closed set and consider the derivation* $S^{-1}D : S^{-1}B \to S^{-1}B$*. Then*

$$S^{-1}D \in \text{LND}(S^{-1}B) \iff D \in \text{LND}(B) \text{ and } S \subset \ker(D).$$

Moreover, if $S \subset \ker(D)$ *then* $\ker(S^{-1}D) = S^{-1}\ker(D)$*.*

(3) *Let* T *be an indeterminate, let* $f \in B[T]$ *and consider the unique extension* $\Delta \in \text{Der}(B[T])$ *of* D *such that* $\Delta(T) = f$*. Then*

$$\Delta \in \text{LND}(B[T]) \iff D \in \text{LND}(B) \text{ and } f \in B.$$

Proof. In each case (1), (2), (3), we prove ($\Rightarrow$); see exercise 2.1 for ($\Leftarrow$).

(1) Assume that bD is locally nilpotent; since it is also nonzero, there exists $s \in B$ such that $(bD)(s) \neq 0$ and $(bD)^2(s) = 0$. Then $bD(s)$ belongs to the factorially closed subring $\ker(bD)$ of B, so $b \in \ker(bD) = \ker(D)$. It follows that $(bD)^n = b^n D^n$ for all n, so D is locally nilpotent and ($\Rightarrow$) is true. This proves (1).

(2) Assume that $S^{-1}D$ is locally nilpotent. Since D is a restriction of $S^{-1}D$, D is locally nilpotent and $B \cap \ker(S^{-1}D) = \ker D$. Also, $S \subseteq (S^{-1}B)^* \subset \ker(S^{-1}D)$ by 2.15, so $S \subseteq B \cap \ker(S^{-1}D) = \ker D$ and ($\Rightarrow$) is true.

If $S \subset \ker(D)$ then $(S^{-1}D)(b/s) = (Db)/s$ for all $b \in B$ and $s \in S$, so $\ker(S^{-1}D) = S^{-1}\ker(D)$.

(3) Assume that Δ is locally nilpotent. Then its restriction D is locally nilpotent. Consider $g \in B[T] \setminus B$ such that $\Delta(g) \in B$; then

$$B \ni \Delta(g) = g^{(D)}(T) + g'(T)f(T)$$

and it follows that $\deg_T(f) \leq 1$. Write $f = aT + b$ (with $a, b \in B$) and denote the leading term of g by αT^n (with $n > 0$, $\alpha \in B \setminus \{0\}$). Then

$$0 = (\text{coefficient of } T^n \text{ in } \Delta(g)) = D(\alpha) + na\alpha,$$

so $\deg_D(na\alpha) = \deg_D(D(\alpha)) < \deg_D(\alpha)$. Since $\deg_D(na\alpha) = \deg_D(a) + \deg_D(\alpha)$, we have $\deg_D(a) < 0$ so $a = 0$. Hence, $f \in B$. ☐

Exercise 2.7. In exercise 1.5, observe that $A = \ker D$ is not factorially closed in B.

Exercise 2.8. Let B be a domain of characteristic zero and suppose that $D \in \operatorname{Der}(B)$ satisfies $D^n = 0$ for some $n > 0$. Show that $D = 0$.

Exercise 2.9. Let B be a domain such that: (1) B has transcendence degree 1 over some field $\Bbbk_0 \leq B$ of characteristic zero; (2) $\text{LND}(B) \neq \{0\}$. Show that $B = \Bbbk^{[1]}$ for some field $\Bbbk$ contained in B.

Exercise 2.10. Consider the subring $B = \mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$. Show that the only locally nilpotent derivation $B \to B$ is the zero derivation.

Exercise 2.11. Let X, Y be indeterminates, $\Bbbk = \mathbb{Q}(X)$ and $B = \Bbbk[Y]/(Y^2)$. If $y \in B$ denotes the residue class of Y then $B = \Bbbk[y]$, $y \neq 0$ and $y^2 = 0$. Show that there exists $D \in \operatorname{Der}_{\mathbb{Q}}(B)$ such that $D(X) = y$ and $D(y) = 0$. Show that $D^2 = 0$, so $D \in \text{LND}(B)$. However, D is not a $\Bbbk$-derivation! (Compare with 2.15.)

Exercise 2.12. Let B be a domain of characteristic zero and $D, D' \in \text{LND}(B)$.

(1) Show: $\ker D = \ker D' \iff \deg_D = \deg_{D'}$ (equality of functions).
(*Hint:* Use part (3) of 2.10.)

(2) Assume that D and D' have the same kernel A and that $s \in B$ is a preslice of D. Show that $D(s), D'(s) \in A \setminus \{0\}$ and $D'(s)D = D(s)D'$.
(*Hint:* Consider $D'(s)D - D(s)D' \in \operatorname{Der}(B)$.)

Irreducible derivations

2.17. **Definition.** Let B be a ring. A derivation $D : B \to B$ is *irreducible* if the only principal ideal of B which contains $D(B)$ is B.

Exercise 2.13. Let B be a domain and $D \in \operatorname{Der}(B)$. Show that D is irreducible if and only if:

$$D = a\Delta,\ a \in B,\ \Delta \in \operatorname{Der}(B) \quad \implies \quad a \in B^*.$$

Exercise 2.14. Let $\Bbbk$ be a field, $B = \Bbbk[X_1, \dots, X_n] = \Bbbk^{[n]}$ and $D \in \operatorname{Der}_{\Bbbk}(B)$. Show that D is irreducible if and only if $\gcd(DX_1, \dots, DX_n) = 1$.

Exercise 2.15. Let B be a domain containing $\mathbb{Q}$ and $D \in \operatorname{LND}(B)$. Show that if D is irreducible then:

$$D \text{ is surjective} \iff D \cap B^* \neq \varnothing \iff D \text{ has a slice} \iff B = (\ker D)^{[1]}.$$

We say that a ring B satisfies the *ACC for principal ideals* if every strictly increasing sequence $I_1 \subset I_2 \subset \cdots$ of principal ideals of B is a finite sequence, or equivalently if every nonempty collection of principal ideals of B has a maximal element.[1] Note that every UFD and every noetherian ring satisfies this condition.

2.18. **Lemma.** *Let B be a domain and let $D \in \operatorname{Der}(B)$, $D \neq 0$.*

(1) *If B satisfies the ACC for principal ideals, then there exists an irreducible derivation $D_0 \in \operatorname{Der}(B)$ such that $D = aD_0$ for some $a \in B$.*
(2) *If B is a UFD then the D_0 in part (1) is unique up to multiplication by a unit.*

Proof. To prove (1), we may assume that D is not irreducible (otherwise the claim is trivial). Then the set $\mathcal{A} = \{a \in B \setminus B^* \mid D(B) \subseteq aB\}$ is nonempty. Fix $x \in B$ such that $D(x) \neq 0$ and consider the following (nonempty) collection of principal ideals of B:

$$\Sigma = \left\{ \left(\tfrac{Dx}{a}\right) B \;\middle|\; a \in \mathcal{A} \right\}.$$

By our assumption on B, we may choose $a \in \mathcal{A}$ in such a way that $I = \left(\frac{Dx}{a}\right) B$ is a maximal element of Σ. As $D(B) \subseteq aB$ and B is a domain, $x \mapsto a^{-1}D(x)$ defines a map $D_0 : B \to B$. It is easily seen that $D_0 \in \operatorname{Der}(B)$ and, obviously, $D = aD_0$. To show that D_0 is irreducible, consider $b \in B$ such that $D_0(B) \subseteq bB$; we have to show that $b \in B^*$. We have $D(B) \subseteq abB$, so $ab \in \mathcal{A}$ and consequently $J = \left(\frac{Dx}{ab}\right) B \in \Sigma$. As $I \subseteq J$, we have $I = J$ because I is a maximal element of Σ, so $\frac{Dx}{ab} \in \left(\frac{Dx}{a}\right) B$ and consequently $b \in B^*$. This proves assertion (1).

To prove (2), suppose that B is a UFD and that $a_1D_1 = a_2D_2$, where $D_1, D_2 \in \operatorname{Der}(B)$ are irreducible and $a_1, a_2 \in B \setminus \{0\}$; we have to show that $D_1 = uD_2$ for some $u \in B^*$. We may assume that $\gcd(a_1, a_2) = 1$. Suppose that $a_1 \notin B^*$. Then there exists a prime element p of B such that $p \mid a_1$; for every $x \in B$ we have $p \mid a_2D_2(x)$, so $p \mid D_2(x)$; this means that $D_2(B) \subseteq pB$, which contradicts the fact that D_2 is irreducible. Thus a_1 is a unit and (by symmetry) so is a_2. $\square$

[1] We are not saying that this maximal element is a maximal ideal!

2.19. **Corollary.** *Let B be a domain of characteristic zero satisfying ACC for principal ideals, let $A \in \text{KLND}(B)$ and consider the set*

$$S = \{ D \in \text{LND}_A(B) \mid \ D \text{ is an irreducible derivation } \}.$$

Then $S \neq \varnothing$ and $\text{LND}_A(B) = \{ aD \mid \ a \in A \text{ and } D \in S \ \}$.

Proof. By 2.18, each nonzero element of $\text{LND}_A(B)$ has the form aD where $a \in B \setminus \{0\}$ and $D \in \text{Der}_A(B)$ is an irreducible derivation. By 2.16, we have $D \in \text{LND}_A(B)$ and $a \in A$. □

2.20. **Corollary.** *Let B be a UFD of characteristic zero and let $A \in \text{KLND}(B)$. Then $\text{LND}_A(B)$ contains an irreducible derivation D, unique up to multiplication by a unit. Moreover, for any such D we have $\text{LND}_A(B) = \{ aD \mid a \in A \}$.*

In view of the above facts, we may make the following comments:

2.21. **Statement of the problem.** Given a ring B, the problem of describing $\text{LND}(B)$ splits into two parts:

(I) *Describe* $\text{KLND}(B)$. In other words, answer the question:

Which subrings of B are kernels of locally nilpotent derivations $B \to B$?

(II) *For each $A \in \text{KLND}(B)$, describe $\text{LND}_A(B)$.*

- If B is a UFD of characteristic zero, it suffices to give the unique (2.20) irreducible element of $\text{LND}_A(B)$.
- If B is a noetherian domain, it suffices (2.19) to give all irreducible elements of $\text{LND}_A(B)$.

Usually, step (I) is more difficult and more interesting than step (II). However the following exercises show that step (II) is sometimes problematic. In ex. 2.16, B is a noetherian domain of characteristic zero (and hence satisfies ACC for principal ideals); in ex. 2.17, B is a domain of characteristic zero which does not satisfy ACC for principal ideals.

Exercise 2.16. Let R be the subring $\mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$ and let $B = R[X, Y] = R^{[2]}$. Let $L = X + TY$ (and note that $L \notin B$). For each integer $n \geq 0$, define an R-derivation $D_n : B \to B$ by $D_n(X) = -T^3 L^n$ and $D_n(Y) = T^2 L^n$.

(1) Verify that $D_n^2(X) = 0 = D_n^2(Y)$, so $D_n \in \text{LND}_R(B)$.
(2) Show that D_n is irreducible.
(3) Fix $N > 0$ and consider $D = T^2 D_N \in \text{LND}(B)$. Show that for each $n \in \{0, \dots, N\}$ there exists $\alpha_n \in B$ such that $D = \alpha_n D_n$ (compare with part (2) of 2.18, i.e., note that uniqueness does not hold here). Show that the D_n all have the same kernel.

Let A denote the kernel of any D_n. By the above, $\text{LND}_A(B)$ contains the infinite family $\{ D_n \mid n \in \mathbb{N}\}$ of irreducible derivations. Actually $\text{LND}_A(B)$ contains many more irreducible derivations. It is possible to describe the set S of 2.19, but we will not do it here.

Exercise 2.17. Let a, u, v be indeterminates over $\mathbb{C}$ and let R be the $\mathbb{C}$-subalgebra of $\mathbb{C}(a, u, v)$ generated by $\{a\} \cup \{ u/a^n \mid n \in \mathbb{N}\} \cup \{ v/a^n \mid n \in \mathbb{N}\}$. Equivalently, R is the $\mathbb{C}$-vector space with basis the monomials $a^i u^j v^k$ such that $(i,j,k) \in \mathbb{Z} \times \mathbb{N}^2$ and $(i,j,k) \notin \{-1,-2,\dots\} \times \{(0,0)\}$.

(1) Show that $aR \neq R$ and that if $f,g \in R$ satisfy $fv = gu$, then $f,g \in aR$.
(2) Let $B = R[X,Y] = R^{[2]}$. Deduce from (1) that $aB \neq B$ and that if $f,g \in B$ satisfy $fv = gu$, then $f,g \in aB$.
(3) Show that if an R-derivation $\Delta : B \to B$ satisfies $\Delta(vX - uY) = 0$, then Δ is not irreducible.
(4) Consider the R-derivation $D = u\frac{\partial}{\partial X} + v\frac{\partial}{\partial Y} : B \to B$. Show that $D \in \text{LND}_R(B)$. Define $A = \ker D$, thus $A \in \text{KLND}_R(B)$. Show that no element of $\text{Der}_A(B)$ is an irreducible derivation (hence no element of $\text{LND}_A(B)$ is irreducible).

A remark in the factorial case

2.22. **Lemma.** *Let B be a UFD containing $\mathbb{Q}$. If $A \in \text{KLND}(B)$ and p is a prime element of A then the following hold:*

(1) *p is a prime element of B and $A \cap pB = pA$. Consequently, we have the inclusion $A/pA \leq B/pB$ of domains.*
(2) *The algebraic closure of A/pA in B/pB is an element of* $\text{KLND}(B/pB)$.

Proof. Assertion (1) easily follows from the fact that A is factorially closed in B. To prove (2), consider the transcendence degree d of B/pB over A/pA. By 2.20, we may consider an irreducible $D \in \text{LND}(B)$ such that $\ker D = A$. In particular $D(B) \nsubseteq pB$, so the "induced" locally nilpotent derivation $D/p : B/pB \to B/pB$ is nonzero; it follows that B/pB has transcendence degree 1 over $\ker(D/p)$ and since $A/pA \leq \ker(D/p)$ we get $d > 0$.

Let $\pi : B \to B/pB$ be the canonical epimorphism. Given any $f,g \in B$, there exists $F(T_1,T_2) \in A[T_1,T_2] \setminus \{0\}$ such that $F(f,g) = 0$ and we may arrange that some coefficient of F is not in pA; then $F^{(\pi)}(T_1,T_2) \in A/pA[T_1,T_2]$ is not the zero polynomial and satisfies $F^{(\pi)}(\pi(f),\pi(g)) = \pi(F(f,g)) = 0$. This shows that any two elements of B/pB are algebraically dependent over A/pA, so $d \leq 1$.

It follows that the algebraic closure of A/pA in B/pB is $\ker(D/p)$. □

2.23. **Proposition.** *Let B be a UFD containing $\mathbb{Q}$, $D \in \text{LND}(B)$ and $A = \ker D$. Suppose that some nonzero element of $A \cap D(B)$ is a product of prime elements p of A satisfying:*

$$A/pA \text{ is algebraically closed in } B/pB.$$

Then $B = A^{[1]}$.

Proof. By 2.20, we have $D = \alpha D_0$ for some $\alpha \in A \setminus \{0\}$ and some irreducible $D_0 \in \text{LND}(B)$. Clearly, $\ker D_0 = A$ and D_0 satisfies the hypothesis of the proposition. So, to prove the proposition, we may assume that D is irreducible. With this assumption, we show that D has a slice. Let $E \subset B$ be the set of elements $s \in B$ which satisfy $Ds \in A \setminus \{0\}$ and: *Ds is a product of prime elements p of A such that A/pA is algebraically closed in B/pB.* By assumption, we have $E \neq \varnothing$. Given $s \in E$, write $Ds = p_1 \cdots p_n$ where each p_i is a prime element of A; then $s \mapsto n$ is a well-defined map $\ell : E \to \mathbb{N}$ and it suffices to show that $\ell(s) = 0$ for some $s \in E$. Consider $s \in E$ such that $\ell(s) > 0$, write $Ds = p_1 \cdots p_n$ as before, let $p = p_n$ and note that $s + pB$ belongs to the kernel of $D/p : B/pB \to B/pB$. Since A/pA is algebraically closed in B/pB, 2.22 gives $\ker(D/p) = A/pA$, so there exists $a \in A$ such that $s - a \in pB$; define $s_1 = (s-a)/p$, then $s_1 \in B$ and $Ds_1 = p_1 \cdots p_{n-1}$, so $s_1 \in E$ and $\ell(s_1) < \ell(s)$. Hence, there exists $s' \in E$ such that $\ell(s') = 0$, i.e., $Ds' \in B^*$. By 2.8, we get $B = A^{[1]}$. $\square$

3. Automorphisms

3.1. **Lemma.** *Let B be a $\mathbb{Q}$-algebra and $f(T) \in B[T]$, where T is an indeterminate. If $\{n \in \mathbb{Z} \mid f(n) = 0\}$ is an infinite set, then $f(T) = 0$.*

Proof. By induction on $\deg_T(f)$. The result is trivial if $\deg_T(f) \leq 0$, so assume that $\deg_T(f) > 0$. Pick $n \in \mathbb{Z}$ such that $f(n) = 0$; since $T - n \in B[T]$ is a monic polynomial, $f(T) = (T-n)g(T)$ for some $g(T) \in B[T]$ such that $\deg_T(g) < \deg_T(f)$. If $m \in \mathbb{Z} \setminus \{n\}$ is such that $f(m) = 0$, then $(m-n)g(m) = 0$ and $m - n \in B^*$, so $g(m) = 0$. So $g(m) = 0$ holds for infinitely many $m \in \mathbb{Z}$ and, by the inductive hypothesis, $g(T) = 0$. It follows that $f(T) = 0$. $\square$

The following is another consequence of 2.7.

3.2. **Proposition.** *Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker D$. The map*

$$e^D : B \to B, \qquad b \longmapsto \textstyle\sum_{n \in \mathbb{N}} \frac{D^n(b)}{n!}$$

is an automorphism of B as an A-algebra and satisfies $A = \big\{ b \in B \mid e^D(b) = b \big\}$. Moreover, if $D_1, D_2 \in \text{LND}(B)$ are such that $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \text{LND}(B)$ and

$$e^{D_1 + D_2} = e^{D_1} \circ e^{D_2} = e^{D_2} \circ e^{D_1}. \tag{4}$$

Proof. Applying 2.7 with $C = B$ and $\gamma = 1$, we obtain that $e^D : B \to B$ is a homomorphism of A-algebras, which is part of the assertion. We begin by proving equation (4). Consider $D_1, D_2 \in \text{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$. By 2.3, $D_1 + D_2 \in \text{LND}(B)$ so it makes sense to consider the ring homomorphism $e^{D_1+D_2} : B \to B$. If $b \in B$,

$$(e^{D_1} \circ e^{D_2})(b) = e^{D_1} \Big(\textstyle\sum_{j \in \mathbb{N}} \frac{D_2^j(b)}{j!} \Big) = \sum_{j \in \mathbb{N}} \frac{e^{D_1}\big(D_2^j(b)\big)}{j!} = \sum_{j \in \mathbb{N}} \frac{1}{j!} \Big(\sum_{i \in \mathbb{N}} \frac{D_1^i\big(D_2^j(b)\big)}{i!} \Big)$$

$$= \textstyle\sum_{i,j \in \mathbb{N}} \frac{(D_1^i \circ D_2^j)(b)}{i!j!} = \sum_{n \in \mathbb{N}} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} (D_1^i \circ D_2^j)(b).$$

Since $D_2 \circ D_1 = D_1 \circ D_2$, we have $(D_1 + D_2)^n = \sum_{i+j=n} \binom{n}{i} D_1^i \circ D_2^j$ for each $n \in \mathbb{N}$ and consequently

$$(e^{D_1} \circ e^{D_2})(b) = \textstyle\sum_{n \in \mathbb{N}} \frac{1}{n!} (D_1 + D_2)^n(b) = e^{D_1 + D_2}(b).$$

So $e^{D_1} \circ e^{D_2} = e^{D_1+D_2}$, which proves equation (4).

Consider $D \in \text{LND}(B)$ and let $A = \ker(D)$. Since $(-D) \circ D = D \circ (-D)$, equation (4) gives $e^D \circ e^{-D} = e^{-D} \circ e^D = e^0 = \text{id}_B$, so e^D is an A-automorphism of B.

There remains to prove that $A = \{ b \in B \mid e^D(b) = b \}$, where "$\subseteq$" is clear. Consider $b \in B$ such that $e^D(b) = b$. Then for every integer $n > 0$ we have

$$b = (e^D)^n(b) = e^{nD}(b) = \sum_{j=0}^{\infty} \tfrac{1}{j!}(nD)^j(b) = \sum_{j=0}^{\infty} \tfrac{1}{j!} D^j(b) n^j = b + f(n),$$

where we define $f(T) \in B[T]$ by $f(T) = \sum_{j=1}^{\infty} \frac{1}{j!} D^j(b) T^j$. By 3.1 we have $f(T) = 0$, so in particular $D(b) = 0$. $\square$

3.3. **Lemma.** *Given rings $\mathbb{Q} \le \Bbbk \le B$, consider the subgroup $\langle E \rangle$ of $\mathrm{Aut}_{\Bbbk}(B)$ generated by the set $E = \{ e^D \mid D \in \mathrm{LND}_{\Bbbk}(B) \}$. Then $\langle E \rangle$ is a normal subgroup of $\mathrm{Aut}_{\Bbbk}(B)$.*

Proof. If $\theta \in \mathrm{Aut}_{\Bbbk}(B)$ and $D \in \mathrm{LND}_{\Bbbk}(B)$, then $\theta^{-1} \circ D \circ \theta \in \mathrm{Der}_{\Bbbk}(B)$ and $(\theta^{-1} \circ D \circ \theta)^n = \theta^{-1} \circ D^n \circ \theta$, so $\theta^{-1} \circ D \circ \theta \in \mathrm{LND}_{\Bbbk}(B)$. It is easily verified that $\theta^{-1} \circ e^D \circ \theta = e^{\theta^{-1} \circ D \circ \theta}$, so $\theta^{-1} E \theta \subseteq E$ holds for all $\theta \in \mathrm{Aut}_{\Bbbk}(B)$. It follows that $\langle E \rangle \lhd \mathrm{Aut}_{\Bbbk}(B)$. $\square$

Exercise 3.1. Let B be a domain containing $\mathbb{Q}$, let $D \in \mathrm{LND}(B)$ and consider $e^D : B \to B$. Show that if $\Bbbk$ is any field contained in B then e^D is a $\Bbbk$-automorphism of B.

Exercise 3.2. With B and D as in 2.6, consider $e^D : B \to B$. Note that e^D is a $\Bbbk$-automorphism of B. Compute $e^D(X)$, $e^D(Y)$ and $e^D(Z)$.

Exercise 3.3. Consider rings $\mathbb{Q} \le \Bbbk \le B$ and let $D \in \mathrm{LND}_{\Bbbk}(B)$. Show that $\lambda \mapsto e^{\lambda D}$ is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(B)$ with kernel $\{ \lambda \in \Bbbk \mid \lambda D = 0 \}$.

4. G_a-actions

The simple minded viewpoint

4.1. **Definition.** Let $\Bbbk$ be an algebraically closed field of characteristic zero. Then the symbol $G_a(\Bbbk)$ denotes the group $(\Bbbk, +)$ viewed as an algebraic group. If X is a $\Bbbk$-variety, an algebraic action of $G_a(\Bbbk)$ on X is a morphism $\alpha : \Bbbk \times X \to X$ which satisfies:

(1) $\alpha(0, x) = x$ for all $x \in X$

(2) $\alpha(a + b, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in \Bbbk$ and $x \in X$.

In other words, an action is a morphism $\alpha : \Bbbk \times X \to X$ satisfying:

(1+2) The map $a \mapsto \alpha(a, \_\_)$ is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(X)$.

4.2. Let $\Bbbk$ be an algebraically closed field of characteristic zero and B a $\Bbbk$-algebra. We claim that there is a bijection

$$\mathrm{LND}_{\Bbbk}(B) \longrightarrow \text{ set of actions of } G_a(\Bbbk) \text{ on } \mathrm{Spec}(B). \tag{5}$$

Indeed, fix $D \in \mathrm{LND}_{\Bbbk}(B)$; then (exercise 3.3) we have the group homomorphism

$$(\Bbbk, +) \longrightarrow \mathrm{Aut}_{\Bbbk}(B), \qquad \lambda \longmapsto e^{\lambda D};$$

applying the functor Spec, we obtain the group homomorphism

$$(\Bbbk, +) \longrightarrow \mathrm{Aut}_{\Bbbk}(\mathrm{Spec}\, B), \qquad \lambda \longmapsto \mathrm{Spec}(e^{\lambda D}).$$

To conclude that we have an action, there remains to verify that the map

$$\alpha \;:\; \Bbbk \times \operatorname{Spec} B \longrightarrow \operatorname{Spec} B, \qquad (\lambda, x) \longmapsto (\operatorname{Spec} e^{\lambda D})(x)$$

is a morphism in the sense of algebraic geometry. Note that we may identify $\Bbbk \times \operatorname{Spec} B$ with $\operatorname{Spec}\big(\Bbbk[T] \otimes_{\Bbbk} B\big) = \operatorname{Spec}(B[T])$ where T is an indeterminate. By 2.7, D determines the homomorphism of $\Bbbk$-algebras $\xi : B \to B[T]$, $\xi(b) = \sum_{j\in\mathbb{N}} \frac{D^j b}{j!} T^j$, and one can verify that $\operatorname{Spec}(\xi) = \alpha$; so α is a morphism.

This shows that (5) is a well-defined map. The fact that it is bijective will be shown in 4.12, below.

4.3. **Example.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X\frac{\partial}{\partial Y} + (Y^2 + XY)\frac{\partial}{\partial Z} \in \mathrm{LND}_{\mathbb{C}}(B)$. Then D determines an action $\alpha : \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ which we now compute. We have

$$\begin{aligned} e^{\lambda D}(Z) = \textstyle\sum_{n=0}^{\infty} \frac{(\lambda D)^n(Z)}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} D^n(Z) \\ = Z + \lambda(Y^2 + XY) + \frac{\lambda^2}{2}(2XY + X^2) + \frac{\lambda^3}{6}(2X^2), \end{aligned}$$

and similarly $e^{\lambda D}(X) = X$ and $e^{\lambda D}(Y) = Y + (\lambda D)(Y) = Y + \lambda X$. So, given $\lambda \in \mathbb{C}$ and $(x, y, z) \in \mathbb{C}^3$,

$$\alpha \;:\; \big(\lambda, (x, y, z)\big) \;\longmapsto\; \Big(x,\; y + \lambda x,\; z + \lambda(y^2 + xy) + \frac{\lambda^2}{2}(2xy + x^2) + \frac{\lambda^3}{3}x^2\Big).$$

4.4. If a group G acts on a ring B,

$$G \times B \longrightarrow B, \qquad (g, b) \longmapsto gb,$$

then one defines the *ring of invariants* $B^G = \{ b \in B \mid \forall_{g\in G}\; gb = b \}$. In the situation described in 4.2, we fix $D \in \mathrm{LND}_{\Bbbk}(B)$ and we let the group $G_a = (\Bbbk, +)$ act on the $\Bbbk$-algebra B,

$$G_a(\Bbbk) \times B \longrightarrow B, \qquad (\lambda, b) \longmapsto e^{\lambda D}(b).$$

For any $b \in B$ we have

$$b \in B^{G_a} \iff \forall_{\lambda\in\Bbbk}\; e^{\lambda D}(b) = b \overset{3.2}{\iff} \forall_{\lambda\in\Bbbk}\; b \in \ker(\lambda D) \iff b \in \ker(D),$$

so $B^{G_a} = \ker(D)$. Note that this is a genuine equality, not just an isomorphism.

Next, we describe the fixed points of a G_a-action on $\operatorname{Spec}(B)$.

4.5. **Proposition.** *Let $\mathbb{Q} \le \Bbbk \le B$ be rings, let $D \in \mathrm{LND}_{\Bbbk}(B)$ and let $\mathfrak{m}$ be a maximal ideal of B. Then tfae:*

(1) *For all $\lambda \in \Bbbk$, $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$*

(2) *$\mathfrak{m} \supseteq D(B)$.*

Proof. Suppose that (2) holds. Given $\lambda \in \Bbbk$ and $b \in \mathfrak{m}$, we have $D^j(b) \in \mathfrak{m}$ for all $j \in \mathbb{N}$, so $e^{\lambda D}(b) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!}\lambda^j \in \mathfrak{m}$; this shows that $e^{\lambda D}(\mathfrak{m}) \subseteq \mathfrak{m}$, and since $e^{\lambda D}$ is an automorphism we must have $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$. So (2) implies (1).

Conversely, suppose that (1) holds. The first step is to prove that

$$D(\mathfrak{m}) \subseteq \mathfrak{m}. \tag{6}$$

Let $b \in \mathfrak{m}$. Define $f(T) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!} T^j \in B[T]$ and note that $f(\lambda) = e^{\lambda D}(b)$ for all $\lambda \in \mathbb{k}$. Since (1) holds, we have $f(\lambda) \in \mathfrak{m}$ for all $\lambda \in \mathbb{k}$, so in particular this holds for all $\lambda \in \mathbb{Q}$. Consider the field $\kappa = B/\mathfrak{m}$, the canonical epimorphism $\pi : B \to \kappa$ and the polynomial $f^{(\pi)} \in \kappa[T]$. Then $\mathbb{Q} \subseteq \kappa$ and $f^{(\pi)}(\lambda) = 0$ for all $\lambda \in \mathbb{Q}$; so $f^{(\pi)} = 0$, i.e., all coefficients of $f(T)$ belong to $\mathfrak{m}$. In particular $D(b) \in \mathfrak{m}$, which proves (6).

By (6), $\delta(b + \mathfrak{m}) = D(b) + \mathfrak{m}$ is a well-defined locally nilpotent derivation $\delta : \kappa \to \kappa$. By 2.15, $\delta = 0$; this means that $D(B) \subseteq \mathfrak{m}$, i.e., (2) holds. □

In view of 4.5, the following is natural:

4.6. **Definition.** Let B be a ring and $D \in \mathrm{LND}(B)$. The elements of the set

$$\mathrm{Fix}(D) = \{\mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{p} \supseteq D(B)\}$$

are called the *fixed points* of D. Note that $\mathrm{Fix}(D)$ is a closed subset of $\mathrm{Spec}(B)$.

The rigorous approach

We prove that (5) is a bijective map in a more general setting, i.e., when $\mathbb{k}$ is any $\mathbb{Q}$-algebra. Some parts of the following discussion are even valid for any ring $\mathbb{k}$.

4.7. **Definition.** For an arbitrary ring $\mathbb{k}$, one defines the group scheme $G_a(\mathbb{k})$ as follows. Let $G_a(\mathbb{k}) = G_a = \mathrm{Spec}(\mathbb{k}[T])$ as a scheme over $\mathbb{k}$, where T is an indeterminate, and let the group operation be the morphism

$$G_a \xleftarrow{\mu} G_a \times G_a$$

which corresponds to the $\mathbb{k}$-homomorphism

$$\begin{array}{rcl} \mathbb{k}[T] & \longrightarrow & \mathbb{k}[X, Y] \\ T & \longmapsto & X + Y. \end{array}$$

Remark. We write $G_a \times G_a$ as an abbreviation of $G_a \times_{\mathrm{Spec}\,\mathbb{k}} G_a$, the fibered product over $\mathrm{Spec}(\mathbb{k})$. The same remark applies to all products below.

4.8. **Definition.** Let $\mathbb{k} \le B$ be rings and let $X = \mathrm{Spec}\, B$. An *algebraic action of* $G_a(\mathbb{k})$ *on* X (or simply a G_a-action on X) is a morphism over $\mathbb{k}$

$$\alpha : G_a(\mathbb{k}) \times X \to X$$

satisfying the following two conditions:

(1) The composition $X \xrightarrow{\epsilon} G_a \times X \xrightarrow{\alpha} X$ is 1_X, where ϵ is defined as follows. Let $\mathrm{ev}_0 : B[T] \to B$ be the B-homomorphism which maps T to 0; then $\epsilon = \mathrm{Spec}(\mathrm{ev}_0)$.

(2) The diagram:

$$\begin{array}{ccc} G_a \times G_a \times X & \xrightarrow{1_{G_a} \times \alpha} & G_a \times X \\ {\scriptstyle \mu \times 1_X} \big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\ G_a \times X & \xrightarrow{\alpha} & X \end{array}$$

is commutative.

4.9**. Definition.** Let B be a ring. Given a ring homomorphism $\varphi : B \to B[T]$ and an element h of $B[X,Y]$, let $\varphi^h : B[X,Y] \to B[X,Y]$ be the unique ring homomorphism satisfying $\varphi^h(X) = X$ and $\varphi^h(Y) = Y$ and making the diagram

$$\begin{tikzcd} B[X,Y] \arrow[r, "\varphi^h"] & B[X,Y] \\ B \arrow[u, "\nu"] \arrow[r, "\varphi"'] & B[T] \arrow[u, "\mathrm{ev}_h"'] \end{tikzcd}$$

commute, where ev_h is the B-homomorphism mapping T on h and ν is the inclusion map.

4.10**. Proposition.** *Let* $\Bbbk$ *be a ring,* B *a* $\Bbbk$*-algebra and* $\varphi : B \to B[T]$ *a* $\Bbbk$*-homomorphism. Then the following are equivalent.*

1. $\mathrm{Spec}(\varphi) : G_a(\Bbbk) \times \mathrm{Spec}\, B \to \mathrm{Spec}\, B$ *is an action.*
2. $\varphi^0 = \mathrm{id}_{B[X,Y]}$ *and* $\varphi^{X+Y} = \varphi^Y \circ \varphi^X$.
3. *The assignment* $a \mapsto \varphi^a$ *gives a group homomorphism from* $(\Bbbk[X,Y], +)$ *to* $\mathrm{Aut}_{\Bbbk[X,Y]}\, B[X,Y]$.

Proof. Let $\varphi : B \to B[T]$ be a $\Bbbk$-homomorphism and $\mathrm{Spec}(\varphi) : G_a(\Bbbk) \times \mathrm{Spec}\, B \to \mathrm{Spec}\, B$ the corresponding morphism. Condition (1) of 4.8 is equivalent to the composition

$$B \xrightarrow{\varphi} B[T] \xrightarrow{\mathrm{ev}_0} B$$

being the identity of B, which is equivalent to $\varphi^0 = \mathrm{id}_{B[X,Y]}$.

On the other hand, condition (2) of 4.8 is equivalent to the diagram

$$\begin{tikzcd} B[X,Y] & B[T] \arrow[l, "\psi"'] \\ B[T] \arrow[u, "\mathrm{ev}_{X+Y}"] & B \arrow[u, "\varphi"'] \arrow[l, "\varphi"] \end{tikzcd} \tag{7}$$

being commutative, where ev_{X+Y} is the B-homomorphism which maps T to $X+Y$ and where ψ is defined by $\psi(T) = X$ and, for $b \in B$, $\psi(b) = \varphi^Y(b)$.

Note that the composite map

$$B[T] \xrightarrow{\mathrm{ev}_X} B[X,Y] \xrightarrow{\varphi^Y} B[X,Y]$$

maps T to X and, for each $b \in B$, b to $\varphi^Y(b)$. So $\varphi^Y \circ \mathrm{ev}_X = \psi$. Consequently,

$$\psi \circ \varphi = \varphi^Y \circ \mathrm{ev}_X \circ \varphi = \varphi^Y \circ \varphi^X \circ \nu, \tag{8}$$

where $\nu : B \hookrightarrow B[X,Y]$ is the inclusion homomorphism. On the other hand,

$$\mathrm{ev}_{X+Y} \circ \varphi = \varphi^{X+Y} \circ \nu \tag{9}$$

by definition of φ^{X+Y}. By (8) and (9), commutativity of diagram (7) is equivalent to

$$\varphi^{X+Y} \circ \nu = \varphi^Y \circ \varphi^X \circ \nu. \tag{10}$$

Since φ^{X+Y} (resp. $\varphi^Y \circ \varphi^X$) maps X to X and Y to Y, equation (10) is equivalent to $\varphi^{X+Y} = \varphi^Y \circ \varphi^X$.

This proves the equivalence of the first two conditions, in the statement of the proposition. The implication ($3 \implies 2$) being obvious, there remains only to show that ($2 \implies 3$). So assume that (2) holds.

We first show that, given $u, v \in \mathbb{k}[X,Y]$, $\varphi^{u+v} = \varphi^v \circ \varphi^u$. Since φ^{u+v} (resp. $\varphi^v \circ \varphi^u$) maps X on X and Y on Y, it's enough to show that $\varphi^{u+v}(b) = (\varphi^v \circ \varphi^u)(b)$ for all $b \in B$. Fix $b \in B$ and write

$$\varphi(b) = \sum_{i \in \mathbb{N}} b_i T^i$$

and, for each $i \in \mathbb{N}$,

$$\varphi(b_i) = \sum_{j \in \mathbb{N}} b_{ij} T^j.$$

Also, let $E : B[X,Y] \to B[X,Y]$ be the B-homomorphism satisfying $E(X) = u$ and $E(Y) = v$. Since φ^v is a $\mathbb{k}[X,Y]$-homomorphism, we have $\varphi^v(u) = u$ and this allows us to write

$$\varphi^v(\varphi^u(b)) = \varphi^v\left(\sum_{i \in \mathbb{N}} b_i u^i\right) = \sum_{i \in \mathbb{N}} \varphi^v(b_i) u^i = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} b_{ij} v^j u^i.$$

On the other hand,

$$\varphi^Y(\varphi^X(b)) = \varphi^Y\left(\sum_{i \in \mathbb{N}} b_i X^i\right) = \sum_{i \in \mathbb{N}} \varphi^Y(b_i) X^i = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} b_{ij} Y^j X^i,$$

so we have

$$\begin{aligned}\varphi^v(\varphi^u(b)) = E\big((\varphi^Y \circ \varphi^X)(b)\big) &= E\big(\varphi^{X+Y}(b)\big) \\ &= E\left(\sum_{n \in \mathbb{N}} b_n (X+Y)^n\right) = \sum_{n \in \mathbb{N}} b_n (u+v)^n = \varphi^{u+v}(b).\end{aligned}$$

Hence, $\varphi^{u+v} = \varphi^v \circ \varphi^u$. Since (2) is assumed to hold, we also have $\varphi^0 = \mathrm{id}_{B[X,Y]}$. It follows that, for each $u \in \mathbb{k}[X,Y]$, $\varphi^u \circ \varphi^{-u} = \mathrm{id}_{B[X,Y]} = \varphi^{-u} \circ \varphi^u$, which shows that $\varphi^u \in \mathrm{Aut}_{\mathbb{k}[X,Y]} B[X,Y]$. $\square$

4.11. **Assume that $\mathbb{k}$ is a $\mathbb{Q}$-algebra** and let B be a $\mathbb{k}$-algebra. We show that the concept of a $G_a(\mathbb{k})$-action on $\operatorname{Spec} B$ is equivalent to that of a locally nilpotent $\mathbb{k}$-derivation $B \to B$.

By 4.10, $\varphi \mapsto \operatorname{Spec}(\varphi)$ is a bijection from

$$\Sigma \stackrel{\text{def}}{=} \big\{ \varphi \in \mathrm{Hom}_{\mathbb{k}}(B, B[T]) \mid \varphi^0 = \mathrm{id}_{B[X,Y]} \text{ and } \varphi^{X+Y} = \varphi^Y \circ \varphi^X \big\}$$

to the set of $G_a(\mathbb{k})$-actions on $\operatorname{Spec} B$. We now proceed to define bijections $\Sigma \to \mathrm{LND}_{\mathbb{k}}(B)$ and $\mathrm{LND}_{\mathbb{k}}(B) \to \Sigma$ which are inverse of each other.

4.11.1. For this part, we may let $\mathbb{k}$ be any ring. Given $\varphi \in \Sigma$, let $D_\varphi : B \to B$ be the composition

$$B \xrightarrow{\varphi} B[T] \xrightarrow{d/dT} B[T] \xrightarrow{\mathrm{ev}_0} B,$$

where d/dT is the usual T-derivative. We show that $D_\varphi \in \text{LND}_{\Bbbk}(B)$. Begin by observing that φ satisfies

$$\operatorname{ev}_0 \circ \varphi = \operatorname{id}_B, \tag{11}$$

since this is equivalent to $\varphi^0 = \operatorname{id}_{B[X,Y]}$, which holds by assumption.

Clearly, D_φ preserves addition and, given $x, y \in B$,

$$\begin{aligned} D_\varphi(xy) = \operatorname{ev}_0\left(\frac{d}{dT}(\varphi(xy))\right) &= \operatorname{ev}_0\left(\frac{d}{dT}(\varphi(x)\varphi(y))\right) \\ &= \operatorname{ev}_0\left(\frac{d}{dT}(\varphi(x))\cdot\varphi(y) + \varphi(x)\cdot\frac{d}{dT}(\varphi(y))\right) \\ = \operatorname{ev}_0\left(\frac{d}{dT}(\varphi(x))\right)\operatorname{ev}_0(\varphi(y)) + \operatorname{ev}_0(\varphi(x))\operatorname{ev}_0\left(\frac{d}{dT}(\varphi(y))\right) &\overset{(11)}{=} D_\varphi(x)y + xD_\varphi(y). \end{aligned}$$

Thus, $D_\varphi \in \operatorname{Der}_{\Bbbk}(B)$. Next, we claim that the diagram

$$\begin{array}{ccc} B[T] & \xrightarrow{d/dT} & B[T] \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} \\ B & \xrightarrow{D_\varphi} & B \end{array} \tag{12}$$

is commutative. To see this, consider $b \in B$ and write

$$\varphi(b) = \sum_{i\in\mathbb{N}} b_i T^i \in B[T].$$

Then

$$\begin{aligned} \sum_{i\in\mathbb{N}} \varphi^Y(b_i)X^i = \varphi^Y\left(\sum_{i\in\mathbb{N}} b_i X^i\right) &= \varphi^Y(\varphi^X(b)) = \varphi^{X+Y}(b) = \sum_{n\in\mathbb{N}} b_n(X+Y)^n \\ &= \sum_{n\in\mathbb{N}} b_n \sum_{i+j=n}\binom{n}{i}X^iY^j = \sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}} b_{i+j}\binom{i+j}{i}Y^j\right)X^i \end{aligned}$$

and consequently

$$\varphi(b_i) = \sum_{j\in\mathbb{N}} b_{i+j}\binom{i+j}{i}T^j \qquad (\text{for all } i \in \mathbb{N}). \tag{13}$$

On the other hand, we have $D_\varphi(b) = b_1$ by definition of D_φ, so

$$\varphi(D_\varphi(b)) = \varphi(b_1) \overset{(13)}{=} \sum_{i\in\mathbb{N}} b_{i+1}(i+1)T^i = \frac{d}{dT}\sum_{i\in\mathbb{N}} b_i T^i = \frac{d}{dT}\big(\varphi(b)\big),$$

which shows that (12) is a commutative diagram. It follows that, for each $n \in \mathbb{N}$,

$$\begin{array}{ccc} B[T] & \xrightarrow{(d/dT)^n} & B[T] \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} \\ B & \xrightarrow{D_\varphi^n} & B \end{array} \tag{14}$$

is commutative. Since d/dT is locally nilpotent and φ is injective (by (11)), D_φ is locally nilpotent. Thus we have a well-defined map

$$\begin{array}{rcl} \Sigma & \longrightarrow & \mathrm{LND}_{\Bbbk}(B) \\ \varphi & \longmapsto & D_\varphi. \end{array}$$

4.11.2. Assume that $\Bbbk$ is a $\mathbb{Q}$-algebra. Given $D \in \mathrm{LND}_{\Bbbk}(B)$, consider the map

$$\begin{array}{rcl} \varphi : B & \longrightarrow & B[T] \\ b & \longmapsto & \sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!} T^n \end{array}$$

and note that φ is a homomorphism of $\Bbbk$-algebras (see 2.7). In order to show that $\varphi \in \Sigma$, consider $\Delta : B[X,Y] \to B[X,Y]$ defined by $\Delta(f) = f^{(D)}$. Clearly,

$$\Delta \in \mathrm{LND}_{\Bbbk[X,Y]} B[X,Y].$$

For each $h \in \Bbbk[X,Y]$, we have $h \in \ker \Delta$ and consequently $h\Delta \in \mathrm{LND}_{\Bbbk[X,Y]} B[X,Y]$. By 3.2, we may consider $e^{h\Delta} \in \mathrm{Aut}_{\Bbbk[X,Y]} B[X,Y]$, and in fact we claim that

$$e^{h\Delta} = \varphi^h : B[X,Y] \to B[X,Y]. \tag{15}$$

To show this, we have to verify that $e^{h\Delta}$ satisfies the definition of φ^h, i.e., the following three conditions: (i) $e^{h\Delta}(X) = X$; (ii) $e^{h\Delta}(Y) = Y$; and (iii) the diagram

$$\begin{array}{ccc} B[X,Y] & \xrightarrow{e^{h\Delta}} & B[X,Y] \\ \uparrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \mathrm{ev}_h} \\ B & \xrightarrow[\varphi]{} & B[T] \end{array} \tag{16}$$

is commutative. Now (i) and (ii) are trivial and, for each $b \in B$,

$$(e^{h\Delta}\circ\nu)(b) = e^{h\Delta}(b) = \sum_{n\in\mathbb{N}} \frac{(h\Delta)^n(b)}{n!} = \sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!} h^n = \mathrm{ev}_h\left(\sum_{n\in\mathbb{N}} \frac{D^n(b)}{n!} T^n\right) = \mathrm{ev}_h(\varphi(b)),$$

so (16) commutes and (15) holds.

So $\varphi^0 = e^0 = \mathrm{id}_{B[X,Y]}$ and 3.2 implies

$$\varphi^{X+Y} = e^{X\Delta + Y\Delta} = e^{Y\Delta} \circ e^{X\Delta} = \varphi^Y \circ \varphi^X$$

because $(X\Delta) \circ (Y\Delta) = (Y\Delta) \circ (X\Delta)$. So $\varphi \in \Sigma$.

4.11.3. We show that the maps $\mathrm{LND}_{\Bbbk} B \stackrel{4.11.2}{\longrightarrow} \Sigma$ and $\Sigma \stackrel{4.11.1}{\longrightarrow} \mathrm{LND}_{\Bbbk} B$ are inverse of each other.

If $D \in \mathrm{LND}_{\Bbbk} B$ then define $\varphi : B \to B[T]$ as in 4.11.2; then D_φ (defined as in 4.11.1) is immediately seen to be equal to D.

Conversely, let $\varphi \in \Sigma$, define D_φ as in 4.11.1 and let $\Phi : B \to B[T]$ be the map

$$\Phi(b) = \sum_{n\in\mathbb{N}} \frac{D_\varphi^n(b)}{n!} T^n.$$

To verify that $\Phi = \varphi$, consider $b \in B$ and write

$$\varphi(b) = \sum_{n \in \mathbb{N}} b_n T^n.$$

Since $\varphi \in \Sigma$, diagram (14) is commutative (for all $n \in \mathbb{N}$) and in particular the constant term of $(\frac{d}{dT})^n\big(\varphi(b)\big)$ is equal to that of $\varphi\big(D_\varphi^n(b)\big)$. In other words we have $n!\, b_n = D_\varphi^n(b)$, so $b_n = \frac{D_\varphi^n(b)}{n!}$ (for all $n \in \mathbb{N}$) and

$$\varphi(b) = \sum_{n \in \mathbb{N}} \frac{D_\varphi^n(b)}{n!} T^n = \Phi(b).$$

Thus $\Phi = \varphi$, which means that the composition $\Sigma \to \mathrm{LND}_{\Bbbk} B \to \Sigma$ is id_Σ.

Proposition 4.10 and paragraph 4.11 prove the following:

4.12. **Theorem.** *Let $\Bbbk$ be a $\mathbb{Q}$-algebra and B a $\Bbbk$-algebra. Given $D \in \mathrm{LND}_{\Bbbk} B$, let $\varphi : B \to B[T]$ be the $\Bbbk$-homomorphism defined in 4.11.2 and let $\alpha_D = \mathrm{Spec}(\varphi) : G_a(\Bbbk) \times \mathrm{Spec}\, B \to \mathrm{Spec}\, B$. Then*

$$\begin{array}{rcl} \mathrm{LND}_{\Bbbk} B & \longrightarrow & \text{set of } G_a(\Bbbk)\text{-actions on } \mathrm{Spec}\, B \\ D & \longmapsto & \alpha_D \end{array}$$

is a well-defined bijection.

5. Variables and coordinate systems

5.1. **Proposition.** *Let R be any ring and consider the polynomial algebra $R[X_1, \dots, X_n]$ in n variables over R. If $f_1, \dots, f_n \in R[X_1, \dots, X_n]$ satisfy $R[f_1, \dots, f_n] = R[X_1, \dots, X_n]$, then $f_1, \dots, f_n$ are algebraically independent over R.*

The significance of 5.1 is that $(f_1, \dots, f_n)$ can then be used as a new set of variables for the polynomial ring, i.e., it is as good as $(X_1, \dots, X_n)$. For the proof of 5.1 we need:

5.1.1. *Let B be a noetherian ring and $\varphi : B \to B$ a surjective ring homomorphism. Then φ is an automorphism of B.*

Proof. Suppose that φ is not injective and pick $y \in \ker \varphi$, $y \neq 0$. If n is any positive integer then $\varphi^n : B \to B$ is surjective, so there exists $x_n \in B$ such that $\varphi^n(x_n) = y$; then $\varphi^n(x_n) \neq 0$ and $\varphi^{n+1}(x_n) = 0$, which shows that all inclusions are strict in the infinite sequence of ideals $\ker(\varphi) \subset \ker(\varphi^2) \subset \ker(\varphi^3) \subset \cdots$. This contradicts the assumption that B is noetherian. $\square$

We prove the following statement, which is equivalent to 5.1.

5.1.2. *Let R be any ring and consider the polynomial algebra $R[X_1, \dots, X_n]$ in n variables over R. If $\varphi : R[X_1, \dots, X_n] \to R[X_1, \dots, X_n]$ is a surjective homomorphism of R-algebras, then φ is an automorphism of $R[X_1, \dots, X_n]$.*

Proof. Let $h \in \ker \varphi$; we show that $h = 0$. Write $B = R[X_1, \dots, X_n]$. Choose $g_1, \dots, g_n \in B$ such that $\varphi(g_i) = X_i$. There exists a finite subset of R which contains all coefficients of $\varphi(X_i)$ and g_i for $1 \le i \le n$, and all coefficients of h. Thus there exists a noetherian ring

$R_0 \leq R$ such that $\varphi(X_i)$, g_i and h all belong to $B_0 = R_0[X_1, \dots, X_n]$. Then φ restricts to a surjective ring homomorphism $\varphi_0 : B_0 \to B_0$ satisfying $\varphi_0(h) = 0$. Since B_0 is noetherian, φ_0 is injective by 5.1.1; so $h = 0$. □

5.2. **Definition.** Suppose that $R \leq B$ are rings and $B = R^{[n]}$. A *variable of B over R* is an element $f \in B$ satisfying $B = R[f, f_2, \dots, f_n]$ for some $f_2, \dots, f_n \in B$. A *coordinate system of B over R* is an ordered n-tuple $(f_1, \dots, f_n)$ of elements of B satisfying $B = R[f_1, \dots, f_n]$.

Exercise 5.1. Let $\mathbb{k}$ be a field, $B = \mathbb{k}[X, Y] = \mathbb{k}^{[2]}$, $R_1 = \mathbb{k}[X]$ and $R_2 = \mathbb{k}[Y]$ and note that $B = R_1^{[1]}$ and $B = R_2^{[1]}$. Let $f = X + Y^2 \in B$. Show that f is a variable of B over R_2 but not a variable of B over R_1.

5.3. **Definition.** Suppose that B is a polynomial ring over some field $\mathbb{k}$. Then, by a *variable of B*, we mean a variable of B over $\mathbb{k}$; by a *coordinate system of B*, we mean a coordinate system of B over $\mathbb{k}$. This makes sense because $\mathbb{k} = \{0\} \cup B^*$ is uniquely determined by B, i.e., there is only one field over which B is a polynomial ring.

Exercise 5.2. Let $\mathbb{k}$ be a field, $B = \mathbb{k}[T, X, Y] = \mathbb{k}^{[3]}$ and $R = \mathbb{k}[T]$ and note that $B = R^{[2]}$. Let $f = TX + Y^2 \in B$. Show that f is not a variable of B over R, but that it is a variable of $\mathbb{k}(T)[X, Y]$.

6. R-derivations of $R[X, Y]$

In this section R is a domain containing $\mathbb{Q}$, $B = R[X, Y] = R^{[2]}$ and $K = \operatorname{Frac}(R)$. To what extent can we describe $\operatorname{LND}_R(B)$? (Keep in mind paragraph 2.21.)

Recall from 1.1 that, given $P \in B$, we may define an R-derivation $\Delta_P : B \to B$ by

$$\Delta_P = -P_Y \frac{\partial}{\partial X} + P_X \frac{\partial}{\partial Y}, \quad \text{or equivalently} \quad \Delta_P(h) = \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix} \quad \text{for all } h \in B.$$

Then we have $R[P] \leq \ker(\Delta_P)$.

Exercise 6.1. Show that for any P_1, $P_2 \in B$, $\quad \Delta_{P_1} = \Delta_{P_2} \Leftrightarrow P_1 - P_2 \in R$.

Exercise 6.2. Let $D = \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial X} : \mathbb{Z}[X, Y] \to \mathbb{Z}[X, Y]$. Show that D is locally nilpotent but is not of the form Δ_P with $P \in \mathbb{Z}[X, Y]$. So, in 6.2 (see below), the hypothesis that R contains $\mathbb{Q}$ is needed.

6.1. **Definition.** An element P of B is *generically univariate* if the following equivalent conditions hold:

- $P \in K[U]$, for some variable U of $K[X, Y]$
- there exists a coordinate system (U, V) of $K[X, Y]$ such that $P \in K[U]$.

6.2. **Lemma.**

(1) $\operatorname{KLND}_R(B) = \{\, B \cap K[U] \mid U$ *is a variable of* $K[X, Y] \,\}$.

(2) *For each* $A \in \operatorname{KLND}_R(B)$, *we have* $\operatorname{LND}_A(B) = \{\, \Delta_P \mid P \in A \,\}$.

(3) $\text{LND}_R(B) = \{\, \Delta_P \mid P \in B \text{ } \textit{is generically univariate}\}$.

Proof. Given $0 \neq D \in \text{LND}_R(B)$, consider the locally nilpotent derivation $\delta = S^{-1}D : K[X,Y] \to K[X,Y]$, where $S = R \setminus \{0\}$. By Rentschler's Theorem, there exists a coordinate system (U,V) of $K[X,Y]$ and $f(U) \in K[U]$ such that $\delta = f(U)\frac{\partial}{\partial V}$. Let $F(U) \in K[U]$ be such that $F'(U) = f(U)$ and $a_{00} = 0$, where $F(U) = \sum_{ij} a_{ij}X^iY^j$ ($a_{ij} \in K$). Let $\lambda = \left|\begin{smallmatrix} U_X & U_Y \\ V_X & V_Y \end{smallmatrix}\right| \in K^*$ and define $P = \lambda^{-1}F(U) \in K[X,Y]$. Then

$$\left|\begin{smallmatrix} P_X & P_Y \\ U_X & U_Y \end{smallmatrix}\right| = \lambda^{-1}F'(U)\left|\begin{smallmatrix} U_X & U_Y \\ U_X & U_Y \end{smallmatrix}\right| = 0 = \delta(U)$$

and

$$\left|\begin{smallmatrix} P_X & P_Y \\ V_X & V_Y \end{smallmatrix}\right| = \lambda^{-1}F'(U)\left|\begin{smallmatrix} U_X & U_Y \\ V_X & V_Y \end{smallmatrix}\right| = f(U) = \delta(V),$$

so $\delta(H) = \left|\begin{smallmatrix} P_X & P_Y \\ H_X & H_Y \end{smallmatrix}\right|$ for any $H \in K[X,Y]$. In particular $P_X = D(Y)$ and $P_Y = -D(X)$, so $P_X, P_Y \in B$; together with $\mathbb{Q} \subset R$ and $a_{00} = 0$, this gives $P \in B$ and consequently $D = \Delta_P$, proving "$\subseteq$" in each of assertions (3) and (2). As $\ker D = B \cap \ker\delta = B \cap K[U]$, "$\subseteq$" of (1) is also proved.

Next, let U be any variable of $K[X,Y]$ and let $P \in B \cap K[U]$, $P \notin R$; we show that $\Delta_P : B \to B$ is locally nilpotent and that $\ker\Delta_P = B \cap K[U]$. This will imply "$\supseteq$" in all three assertions.

Write $P = \Phi(U)$ where Φ is a polynomial in one variable with coefficients in K; choose V such that (U,V) is a coordinate system of $K[X,Y]$. Consider the K-derivation $\delta = S^{-1}\Delta_P$ of $K[X,Y]$, where $S = R \setminus \{0\}$. For any $H \in K[X,Y]$ we have $\delta(H) = \left|\begin{smallmatrix} P_X & P_Y \\ H_X & H_Y \end{smallmatrix}\right| = \Phi'(U)\left|\begin{smallmatrix} U_X & U_Y \\ H_X & H_Y \end{smallmatrix}\right|$, so $\delta(U) = 0$ and $\delta(V) \in K[U]$; thus δ is locally nilpotent and so is its restriction Δ_P. As $P \notin R$, we have $\Phi'(U) \neq 0$ and hence $\delta \neq 0$. Note that $K[U] \subseteq \ker\delta$ and, by Rentschler's Theorem, $K[X,Y] = (\ker\delta)^{[1]}$; as $K[X,Y] = K[U]^{[1]}$, we get $\ker\delta = K[U]$ and consequently $\ker\Delta_P = B \cap \ker\delta = B \cap K[U]$. $\square$

Comments. Result 6.2 is a partial solution to problems (I) and (II) of 2.21, but not a very satisfactory solution. For instance, consider an element A of $\text{KLND}_R(B)$. Then 6.2 does not say what are the irreducible derivations in $\text{LND}_A(B)$ (this is a hard question, in view of exercises 2.16 and 2.17); neither does it describe A a an R-algebra. Regarding this last question, we know that A has transcendence degree 1 over R (by 2.10 we have $\text{trdeg}_A(B) = 1$, so $\text{trdeg}_R(A) = 1$). Is A always finitely generated as an R-algebra? Is $A = R^{[1]}$ always true?

- By 6.3, A is not necessarely a finitely generated R-algebra.
- By 6.5, if R is a UFD then $A = R^{[1]}$.
- If we assume that R is a noetherian normal domain containing $\mathbb{Q}$ then Bhatwadekar and Dutta [2] give a complete description of A. In particular, they show that A is not necessarely a finitely generated R-algebra (even with R normal).

6.3. **Example.** Let R be the subring $\mathbb{C}[T^2,T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$, let $B = R[X,Y] = R^{[2]}$, $P = T^2X + T^3Y \in B$ and consider $\Delta_P : B \to B$. Note that P is a variable of $K[X,Y]$, where $K = \text{Frac}(R) = \mathbb{C}(T)$, so by 6.2 we have $\Delta_P \in \text{LND}_R(B)$ and $\ker\Delta_P = B \cap K[P]$.

We show that $\ker \Delta_P$ is not finitely generated as an R-algebra. (A different proof is given in [1].)

Define an $\mathbb{N}^2$-grading on $S = \mathbb{C}[T, X, Y] = \mathbb{C}^{[3]}$ by $\deg(T) = (1, 0)$, $\deg(X) = (1, 1)$ and $\deg(Y) = (0, 1)$. Since $B = \mathbb{C}[T^2, T^3, X, Y]$ where T^2, T^3, X, Y are homogeneous elements of S it follows that B is a homogeneous subring of S, i.e., if $b \in B$ then all homogeneous components of b belong to B. So B is a graded ring, $B = \oplus_{(i,j)\in\mathbb{N}^2} B_{(i,j)}$, and it is easy to see that Δ_P is a homogeneous derivation of B; consequently $A = \ker(\Delta_P)$ is a homogeneous subring of B. The subring $R[P]$ of A is also homogeneous, because $R[P] = \mathbb{C}[T^2, T^3, P]$ where T^2, T^3 and P are homogeneous. We claim:

(17) Each homogeneous element of $R[P]$ has the form $\lambda T^i P^j$ for some $\lambda \in \mathbb{C}$ and $(i, j) \in \mathbb{N}^2$.

In fact it is clear that each element of $R[P]$ is a finite sum of terms $\lambda T^i P^j$. Since $\deg(T^i P^j) = i(1, 0) + j(3, 1)$ is an injective function of (i, j), the claim (17) is clear. Next we show:

(18) A is the $\mathbb{C}$-vector space spanned by $\{T^i(X + TY)^j \mid i \geq 2 \text{ and } j \in \mathbb{N}\}$.

Let $V = \operatorname{Span}_{\mathbb{C}} \{T^i(X + TY)^j \mid i \geq 2 \text{ and } j \in \mathbb{N}\}$, then $V \subseteq A$ is clear. Conversely, let h be a homogeneous element of A; to prove (18), it suffices to show that $h \in V$. By 6.2 we have $A = B \cap \mathbb{C}(T)[P]$ so $f(T)h \in R[P]$ for some $f(T) \in R\setminus\{0\}$. Then each homogeneous component of $f(T)h$ belongs to $R[P]$, so $T^k h \in R[P]$ for some $k \in \mathbb{N}$. By (17), we get $T^k h = \lambda T^{i_1} P^j$, for some $\lambda \in \mathbb{C}$ and $(i_1, j) \in \mathbb{N}^2$, so $h = \lambda T^i(X + TY)^j$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. Since $h \in B$, we have $i \geq 2$ and consequently $h \in V$, which proves (18).

Finally, let $g_n = T^2(X+TY)^n$ for each $n \geq 0$ and note that g_n is homogeneous of degree $(n+2, n)$. By (18) we have $A = R[g_1, g_2, \dots]$ and if A is a finitely generated R-algebra then $A = R[g_1, \dots, g_m] = \mathbb{C}[T^3, g_0, g_1, \dots, g_m]$ for some m. However, $\deg(g_{m+1})$ does not belong to the semigroup generated by $\deg(T^3)$, $\deg(g_0)$, $\dots$, $\deg(g_m)$. So $g_{m+1} \notin R[g_1, \dots, g_m]$ and A is not finitely generated. □

Exercise 6.3. Verify that $\deg(g_{m+1}) \notin \langle\, \deg(T^3),\ \deg(g_0),\ \dots,\ \deg(g_m)\, \rangle$.

The case where R is a UFD

The following useful fact (6.4) was proved in [16] and [15], and is valid without assuming that R has characteristic zero.

6.4. *Let R and A be UFD's satisfying $R \leq A \leq R^{[n]}$ for some n. If $\operatorname{trdeg}_R(A) = 1$ then $A = R^{[1]}$.*

6.5. **Proposition.** *Let R be a UFD of characteristic zero and $B = R[X, Y] = R^{[2]}$. If $A \in \mathrm{KLND}_R(B)$, then $A = R^{[1]}$.*

Proof. Since R is a UFD, so is B. Since B is a domain of characteristic zero and $A \in \mathrm{KLND}(B)$, A is factorially closed in B by 2.15; so exercise 2.6 implies that A is a UFD. Hence $R \leq A \leq B = R^{[2]}$ are UFD's. We have $\operatorname{trdeg}_A(B) = 1$ by 2.10, so $\operatorname{trdeg}_R(A) = 1$; by 6.4, we conclude that $A = R^{[1]}$. □

Remark. The MSc thesis [1] of Joost Berson contains the following result: *Let R be a UFD of characteristic zero, $B = R[X,Y] = R^{[2]}$ and $D \in \mathrm{Der}_R(B)$. If $D \neq 0$ then* $\ker D = R[P]$ *for some $P \in B$.* (That is, the kernel is either R or $R^{[1]}$.)

6.6. **Theorem.** *Let R be a UFD containing $\mathbb{Q}$, $B = R[X,Y] = R^{[2]}$ and $K = \mathrm{Frac}\, R$. Consider the set*

$$\mathcal{P} = \{\, P \in B \mid \gcd_B(P_X, P_Y) = 1 \text{ and } P \text{ is a variable of } K[X,Y] \,\}.$$

Then the following hold.

(1) *For $P \in B$, tfae:*
   (a) $P \in \mathcal{P}$
   (b) $\Delta_P : B \to B$ *is locally nilpotent and irreducible*
   (c) $R[P] \in \mathrm{KLND}_R(B)$.
   Consequently, $\mathrm{KLND}_R(B) = \{\ R[P] \ \mid\ P \in \mathcal{P}\ \}$.
(2) *If $R[P] \in \mathrm{KLND}_R(B)$, then $\mathrm{LND}_{R[P]}(B) = \{\, \alpha\Delta_P \mid \alpha \in R[P]\}$.*
(3) $\mathrm{LND}_R(B) = \{\, \alpha\Delta_P \mid P \in \mathcal{P} \text{ and } \alpha \in R[P]\}$.

Proof. (1) Let $P \in \mathcal{P}$. By 6.2, $\Delta_P : B \to B$ is locally nilpotent. If I is a principal ideal of B such that $\Delta_P(B) \subseteq I$ then $P_X = \Delta_P(Y)$ and $P_Y = -\Delta_P(X)$ belong to I, so the gcd condition implies that $I = B$; so Δ_P is irreducible and we showed that (a) implies (b). Suppose that (b) holds. By 6.5, $\ker \Delta_P = R[W]$ for some $W \in B$. Thus $P \in R[W]$ and we may write $P = f(W)$ with $f(T) \in R[T]$, T an indeterminate. Now $\Delta_P(Y) = P_X = f'(W)W_X$ and $\Delta_P(X) = -P_Y = -f'(W)W_Y$, so $\Delta_P(B) \subseteq f'(W)B$; by irreducibility of Δ_P, we get $f'(W) \in B^* = R^*$. Consequently $f(T) = uT + r$ with $u \in R^*$ and $r \in R$, so $\ker \Delta_P = R[W] = R[P]$ and (c) holds. Finally, suppose that (c) holds. Let $S = R \setminus \{0\}$, then $S^{-1}R[P] = K[P]$ belongs to $\mathrm{KLND}(K[X,Y])$ by exercise 2.1, so Rentschler's Theorem implies that P is a variable of $K[X,Y]$. Thus P_X and P_Y are relatively prime in $K[X,Y]$, which implies that $r \in R \setminus \{0\}$ where we define $r = \gcd_B(P_X, P_Y)$. Then, if $c \in R$ is the constant term of $P \in R[X,Y]$, r divides every coefficient of $P - c$ (we are using $\mathbb{Q} \subseteq R$ here). So $P - c = rP'$ for some $P' \in B$. As $R[P]$ is factorially closed in B and $rP' \in R[P] \setminus \{0\}$, we get $P' \in R[P]$. Hence $R[P] = R[P']$ and consequently $r \in R^*$ and $\gcd_B(P_X, P_Y) = 1$. So (c) implies (a) and conditions (a–c) are therefore equivalent. Then $\mathrm{KLND}_R(B) = \{\ R[P] \ \mid\ P \in \mathcal{P}\ \}$ is clear and assertion (1) is proved.

If $R[P] \in \mathrm{KLND}_R(B)$ then, by (1), Δ_P belongs to $\mathrm{LND}_{R[P]}(B)$ and is irreducible. So (2) follows from 2.20. Assertion (3) follows from (1) and (2). ☐

Exercise 6.4. Let R, B and $\mathcal{P}$ be as in 6.6.

(1) Show that $\{\, \Delta_P \mid P \in \mathcal{P}\}$ is the set of irreducible elements of $\mathrm{LND}_R(B)$.
(2) Show that if $P \in \mathcal{P}$ then $\Delta_P(B)$ contains a nonzero element of R.

Exercise 6.5. Let $\mathbb{k}$ be a field of characteristic zero, $B = \mathbb{k}[X,Y,Z] = \mathbb{k}^{[3]}$ and define $D \in \mathrm{Der}_{\mathbb{k}}(B)$ by $DX = 0$, $DY = X$ and $DZ = Y^2$. Show that D is irreducible. With $R = \mathbb{k}[X]$, verify that $D \in \mathrm{LND}_R(B)$ and find $P \in \mathcal{P}$ such that $D = \Delta_P$. What is $\ker D$?

Exercise 6.6. Let R be the subring $\mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$ and $B = R[X,Y] = R^{[2]}$. Consider the R-derivation $\Delta_W : B \to B$ where $W = T^2(X+TY)^3 \in B$. Note that W is generically univariate and hence Δ_W is locally nilpotent by 6.2. Show that Δ_W is irreducible, and is not of the form $\alpha\Delta_P$ where $P \in B$ is a variable of $K[X,Y]$ and $\alpha \in \ker \Delta_W$ (where $K = \operatorname{Frac} R = \mathbb{C}(T)$). Compare with part (3) of 6.6.

Variables and slices

We shall prove the following criterion for deciding if a polynomial is a variable:

6.7. **Theorem.** *Let R be a UFD containing $\mathbb{Q}$.*
For an element P of $B = R[X,Y] = R^{[2]}$, tfae:

(1) *P is a variable of B over R.*
(2) *$\Delta_P : B \to B$ is locally nilpotent and $(P_X, P_Y)B = B$.*

Before proving 6.7, we deduce:

6.8. **Corollary.** *Let R be a UFD containing $\mathbb{Q}$, $B = R^{[2]}$ and $D \in \operatorname{LND}_R(B)$. Tfae:*

(1) *1 belongs to the ideal of B generated by $D(B)$*
(2) *$1 \in D(B)$.*

Proof. Suppose that (1) holds. In particular, D is an irreducible derivation so (ex. 6.4) for some $P \in \mathcal{P}$ we have $D = \Delta_P$ and $\ker D = R[P]$. Result 6.7 implies that P is a variable of B over R, so $B = R[P]^{[1]} = (\ker D)^{[1]}$; then exercise 2.15 implies that (2) holds. □

Remarks.

- Condition (1) of 6.8 states that D is fix-point-free (see 4.6) and (2) says that D has a slice. So the claim is that if D is fix-point-free then it has a slice (the converse is trivial).
- Results 6.7 and 6.8 are two ways to say the same thing: We obtained 6.8 as a corollary of 6.7, but we could have done it the other way around.
- Both 6.7 and 6.8 remain valid when R is any $\mathbb{Q}$-algebra (see [17]). We restrict ourselves to the UFD case because the proof is considerably easier.

The proof of 6.7 requires some preliminaries.

6.9. **Lemma.** *Consider rings $\mathbb{Q} \le R \le S$ and $R[X,Y] \le S[X,Y]$, where X, Y are indeterminates over S. If $P \in B = R[X,Y]$ satisfies $(P_X, P_Y)B = B$, then $B \cap S[P] = R[P]$.*

In fact we prove the following more general version (Δ_f is defined in 1.1 and $(\Delta_f B)$ denotes the ideal of B generated by $\Delta_f(B)$):

6.10. **Lemma.** *Consider rings $\mathbb{Q} \le R \le S$ and $R[X_1, \dots, X_n] \le S[X_1, \dots, X_n]$, where $X_1, \dots, X_n$ are indeterminates over S. Write $B = R[X_1, \dots, X_n]$. If $f = (f_1, \dots, f_{n-1}) \in B^{n-1}$ satisfies $(\Delta_f B) = B$, then $B \cap S[f_1, \dots, f_{n-1}] = R[f_1, \dots, f_{n-1}]$.*

Proof. Since the ideal $(\Delta_f B)$ is generated by $\Delta_f(X_1), \dots, \Delta_f(X_n)$, the assumption $(\Delta_f B) = B$ implies that there exist $b_1, \dots, b_n \in B$ such that $\sum_{i=1}^n b_i \Delta_f(X_i) = 1$. Then the matrix

$$M = \begin{pmatrix} (f_1)_{X_1} & \dots & (f_1)_{X_n} \\ \vdots & & \vdots \\ (f_{n-1})_{X_1} & \dots & (f_{n-1})_{X_n} \\ b_1 & \dots & b_n \end{pmatrix}$$

has all its entries in B and has determinant 1; so M^{-1} exists and has all its entries in B.

We claim that the S-homomorphism $\mathrm{ev}_f : S[T_1, \dots, T_{n-1}] \to S[X_1, \dots, X_n]$ defined by $\Phi(T) \mapsto \Phi(f)$ satisfies

$$\mathrm{ev}_f^{-1}(B) = R[T_1, \dots, T_{n-1}]. \tag{19}$$

Suppose that (19) is false and choose $\Phi(T) \in S[T_1, \dots, T_{n-1}] \setminus R[T_1, \dots, T_{n-1}]$ of minimal total degree such that $\Phi(f) \in B$. The chain rule gives

$$\begin{aligned}\begin{pmatrix} \Phi(f)_{X_1} & \cdots & \Phi(f)_{X_n} \end{pmatrix} &= \begin{pmatrix} \Phi_{T_1}(f) & \cdots & \Phi_{T_{n-1}}(f) \end{pmatrix} \begin{pmatrix} (f_1)_{X_1} & \dots & (f_1)_{X_n} \\ \vdots & & \vdots \\ (f_{n-1})_{X_1} & \dots & (f_{n-1})_{X_n} \end{pmatrix} \\ &= \begin{pmatrix} \Phi_{T_1}(f) & \cdots & \Phi_{T_{n-1}}(f) & 0 \end{pmatrix} M,\end{aligned}$$

so:

$$\begin{pmatrix} \Phi(f)_{X_1} & \cdots & \Phi(f)_{X_n} \end{pmatrix} M^{-1} = \begin{pmatrix} \Phi_{T_1}(f) & \cdots & \Phi_{T_{n-1}}(f) & 0 \end{pmatrix}. \tag{20}$$

Since $\Phi(f)$ belongs to B, so does $\Phi(f)_{X_j}$ for every j. So the left hand side of (20) has entries in B and consequently $\Phi_{T_1}(f), \dots, \Phi_{T_{n-1}}(f) \in B$. By minimality of the degree of Φ, we must have

$$\Phi_{T_1}(T), \dots, \Phi_{T_{n-1}}(T) \in R[T_1, \dots, T_{n-1}].$$

Since $\mathbb{Q} \subseteq R$, it follows that $\Phi(T) = \lambda + \Psi(T)$ for some $\lambda \in S$ and $\Psi(T) \in R[T_1, \dots, T_{n-1}]$. Then $\lambda = \Phi(f) - \Psi(f) \in B$, i.e., $\lambda \in R$. Consequently $\Phi(T) \in R[T_1, \dots, T_{n-1}]$, a contradiction. So (19) is true and the desired result follows. $\square$

6.11. **Lemma.** *Let R be a domain containing $\mathbb{Q}$ and $P \in B = R[X, Y] = R^{[2]}$. If*

$$\Delta_P : B \to B \text{ is locally nilpotent and } (P_X, P_Y)B = B,$$

then $\ker \Delta_P = R[P]$ *and there exists $Q \in B$ such that $\Delta_P(Q) \in R \setminus \{0\}$.*

Proof. Let $K = \operatorname{Frac} R$ and $S = R \setminus \{0\}$; consider the locally nilpotent derivation $S^{-1}\Delta_P$ of $K[X, Y]$. By Rentschler's Theorem, $\ker(S^{-1}\Delta_P) = K[V]$ for some variable V of $K[X, Y]$; then the conditions $P \in K[V]$ and $1 \in (P_X, P_Y)$ imply that $K[P] = K[V]$; so P is a variable of $K[X, Y]$ and we may choose $Q \in B$ such that $K[P, Q] = K[X, Y]$; then $B \ni \Delta_P(Q) = \begin{vmatrix} P_X & P_Y \\ Q_X & Q_Y \end{vmatrix} \in K^*$, so $\Delta_P(Q) \in R \setminus \{0\}$.

Note that $R[X, Y] \cap K[P] = R[P]$ by 6.9, so $\ker \Delta_P = B \cap \ker(S^{-1}\Delta_P) = B \cap K[P] = R[P]$ and we are done. $\square$

Proof of 6.7. If (1) holds then Δ_P is locally nilpotent by 6.2; also, there exists $Q \in R[X,Y]$ such that $R[P,Q] = R[X,Y]$ and any such Q satisfies $\left|\begin{smallmatrix} P_X & P_Y \\ Q_X & Q_Y \end{smallmatrix}\right| \in R^*$. So (1) implies (2).

Suppose that (2) holds; we deduce (1). By 6.11, we have:

$$\ker \Delta_P = R[P] \tag{21}$$

$$\Delta_P(Q) \in R \setminus \{0\} \quad \text{for some } Q \in B. \tag{22}$$

We shall prove a condition stronger than (22), namely:

$$\Delta_P(Q_0) \in R^* \quad \text{for some } Q_0 \in B. \tag{23}$$

Note that if (23) is true then 2.8 gives $B = (\ker \Delta_P)^{[1]} = R[P]^{[1]}$ and we are done. To prove (23) we consider the "length function" $\ell : R \setminus \{0\} \to \mathbb{N}$ defined by $\ell(p_1 \cdots p_n) = n$ for any product of prime elements $p_1, \dots, p_n$ of R (and where $\ell(u) = 0$ if $u \in R^*$). Pick $Q \in B$ such that $\Delta_P(Q) \in R \setminus \{0\}$; if $\ell\big(\Delta_P Q\big) = 0$ then we are done, so suppose that $\ell\big(\Delta_P Q\big) > 0$. Then there exists a prime element q of R which divides $\Delta_P(Q)$ in R. Let $\overline{R} = R/qR$ and let $\pi : R[X,Y] \to \overline{R}[X,Y]$ be the unique extension of the canonical homomorphism $R \to \overline{R}$ which maps $X \mapsto X$ and $Y \mapsto Y$. Let $h = \pi(P) \in \overline{R}[X,Y]$ and consider $\Delta_h : \overline{R}[X,Y] \to \overline{R}[X,Y]$. We claim:

$$\Delta_h \text{ is locally nilpotent and 1 belongs to the ideal } (h_X, h_Y) \text{ of } \overline{R}[X,Y]. \tag{24}$$

Indeed, it is clear that π is surjective and that

$$\begin{tikzcd} \overline{R}[X,Y] \arrow[r, "\Delta_h"] & \overline{R}[X,Y] \\ R[X,Y] \arrow[u, "\pi"] \arrow[r, "\Delta_P"] & R[X,Y] \arrow[u, "\pi"] \end{tikzcd} \tag{25}$$

commutes, so Δ_h is locally nilpotent; applying π to an equation $aP_X + bP_Y = 1$ $(a, b \in B)$ gives $\pi(a)h_X + \pi(b)h_Y = 1$, so $1 \in (h_X, h_Y)\overline{R}[X,Y]$ and (24) holds. Moreover, $\overline{R}$ is a domain which contains $\mathbb{Q}$ (because $\mathbb{Q} \leq R$). Applying 6.11 to $h \in \overline{R}[X,Y]$ we conclude that $\ker \Delta_h = \overline{R}[h]$, or equivalently

$$\ker \Delta_h = \pi\big(R[P]\big). \tag{26}$$

It follows that $\pi(Q) \in \pi\big(R[P]\big)$, because q divides $\Delta_P(Q)$ in R and (25) commutes. So there exists $\Phi(T) \in R[T]$ such that $Q + \Phi(P) \in \ker \pi = qB$. Define $Q' = (Q + \Phi(P))/q$; then $Q' \in B$ and $\Delta_P(Q') = \frac{1}{q}\Delta_P(Q) \in R \setminus \{0\}$, so $\ell\big(\Delta_P Q'\big) < \ell\big(\Delta_P Q\big)$. This argument shows that there exists $Q_0 \in B$ such that $\ell\big(\Delta_P Q_0\big) = 0$, and this proves (23). □

Locally nilpotent derivations of $\mathbb{k}^{[n]}$ of low rank

In this section we let $B = \mathbb{k}^{[n]}$, where $n \geq 1$ and $\mathbb{k}$ is a field of characteristic zero.

6.12. **Definition.** Let $D : B \to B$ be a $\mathbb{k}$-derivation. The *rank* of D is the least $r \in \{0, 1, \dots, n\}$ for which there exists a coordinate system $(T_1, \dots, T_{n-r}, X_1, \dots, X_r)$ of B satisfying

$$\mathbb{k}[T_1, \dots, T_{n-r}] \subseteq \ker D.$$

Given such a coordinate system, we may write

$$D = f_1(T,X)\frac{\partial}{\partial X_1} + \cdots + f_r(T,X)\frac{\partial}{\partial X_r}$$

with $f_i(T,X) \in B = \mathbb{k}[T_1,\dots,T_{n-r},X_1,\dots,X_r]$ for all i. In this sense,

The rank of D is the least number of partial derivatives needed for expressing D.

The following claims are clear:

- $\operatorname{rank} D = 0 \iff D = 0$
- If two derivations have the same kernel then they have the same rank.

6.13**. Example.** Let $B = \mathbb{k}[X,Y,Z]$ and $D = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z} : B \to B$. Since $(X-Z, Y-Z, Z)$ is a coordinate system of B and $\mathbb{k}[X-Z, Y-Z] \subseteq \ker D$, we have $\operatorname{rank} D \le 1$. Since $D \neq 0$, we conclude that $\operatorname{rank} D = 1$. Writing $(U,V,W) = (X-Z, Y-Z, Z)$, we have $D = 0\frac{\partial}{\partial U} + 0\frac{\partial}{\partial V} + \frac{\partial}{\partial W} = \frac{\partial}{\partial W}$. (*Remark:* $\frac{\partial}{\partial W} \neq \frac{\partial}{\partial Z}$, even though $W = Z$.)

Exercise 6.7. Verify the following claims.

(1) $\operatorname{rank} D = 1 \iff \ker D = \mathbb{k}^{[n-1]}$ and $B = (\ker D)^{[1]}$

(2) $\operatorname{rank} D < n \iff \ker D$ contains a variable of B.

Exercise 6.8. Let $B = \mathbb{k}[X,Y,Z] = \mathbb{k}^{[3]}$ and define $D \in \text{LND}(B)$ by $DX = 0$, $DY = X$ and $DZ = Y$. Verify that D is an irreducible derivation which does not have a slice; deduce that B is not $(\ker D)^{[1]}$ (see ex. 2.15). Conclude that $\operatorname{rank} D = 2$.

Remark. One can reformulate Rentschler's Theorem as:

If D is a locally nilpotent derivation of $\mathbb{k}[X,Y] = \mathbb{k}^{[2]}$, then $\operatorname{rank} D < 2$.

However we will see later that if $n > 2$ then there exist locally nilpotent derivations of $\mathbb{k}^{[n]}$ of rank n.

6.14**. Notation.** Given a coordinate system $\gamma = (T_1,\dots,T_{n-2},X,Y)$ of B and an element $P \in B$, define

$$\Delta_P^\gamma = -P_Y\tfrac{\partial}{\partial X} + P_X\tfrac{\partial}{\partial Y} \ : \ B \longrightarrow B.$$

Note that Δ_P^γ is a $\mathbb{k}[T_1,\dots,T_{n-2}]$-derivation of B and that $\mathbb{k}[T_1,\dots,T_{n-2},P] \le \ker \Delta_P^\gamma$.

6.15**. Corollary.** *For a $\mathbb{k}$-derivation $D \neq 0$ of $B = \mathbb{k}^{[n]}$, tfae:*

(1) *D is locally nilpotent and* $\operatorname{rank} D \le 2$

(2) *$D = \alpha\Delta_P^\gamma$ for some γ, P and α satisfying:*
- *$\gamma = (T_1,\dots,T_{n-2},X,Y)$ is a coordinate system of B*
- *$P \in B$ satisfies $\gcd_B(P_X,P_Y) = 1$ and is a variable of $\mathbb{k}(T_1,\dots,T_{n-2})[X,Y]$.*
- *α is a nonzero element of $\mathbb{k}[T_1,\dots,T_{n-2},P]$.*

Moreover, if the above two conditions hold then:

(3) $\ker D = \mathbb{k}[T_1,\dots,T_{n-2},P]$

(4) *Δ_P^γ is an irreducible locally nilpotent derivation*

(5) $\Delta_P^\gamma(B)$ *contains a nonzero element of* $\Bbbk[T_1,\dots,T_{n-2}]$.

Proof. Suppose that D satisfies condition (1). As $\operatorname{rank} D \le 2$, there exists a coordinate system $\gamma = (T_1,\dots,T_{n-2},X,Y)$ of B satisfying $\Bbbk[T_1,\dots,T_{n-2}] \subset \ker D$. Define $R = \Bbbk[T_1,\dots,T_{n-2}]$ and note that this is a UFD containing $\mathbb{Q}$; we have $B = R[X,Y] = R^{[2]}$ and $D \in \mathrm{LND}_R(B)$, so 6.6 can be applied to this situation. It follows that (2) holds. The other assertions are left to the reader. $\square$

7. Preliminaries for section 8

The following facts are needed in the next section.

7.1. **Theorem** (Miyanishi). *Let* $\Bbbk$ *be a field of characteristic zero and* $B = \Bbbk^{[3]}$. *If* $A \in \mathrm{KLND}(B)$ *then* $A = \Bbbk^{[2]}$.

The case $\Bbbk = \mathbb{C}$ of 7.1 was proved by Miyanishi in [14]; it is not difficult to show that the result remains valid if $\Bbbk$ is any field of characteristic zero.

For the next result, let $\Bbbk$ be a field of characteristic zero and $B = \Bbbk[X_0,X_1,X_2] = \Bbbk^{[3]}$. Recall that any $f,g \in B$ determine a $\Bbbk$-derivation

$$\Delta_{(f,g)} = \begin{vmatrix} f_{X_0} & f_{X_1} & f_{X_2} \\ g_{X_0} & g_{X_1} & g_{X_2} \\ \frac{\partial}{\partial X_0} & \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} \end{vmatrix} \ : \ B \to B$$

satisfying $\Bbbk[f,g] \subseteq \ker \Delta_{(f,g)}$.

7.2. **Theorem.** *Let* $\Bbbk$ *be a field of characteristic zero and* $B = \Bbbk[X_0,X_1,X_2] = \Bbbk^{[3]}$. *Let* $f,g \in B$ *be such that* $\Bbbk[f,g] \in \mathrm{KLND}(B)$. *Then the* $\Bbbk$*-derivation* $\Delta_{(f,g)} : B \to B$ *is irreducible, locally nilpotent and satisfies* $\ker \Delta_{(f,g)} = \Bbbk[f,g]$. *Consequently we have* $\mathrm{LND}_A(B) = \{\ \alpha\Delta_{(f,g)} \ \mid \alpha \in A\}$, *where* $A = \Bbbk[f,g]$.

Result 7.2 is Corollary 2.6 of [3]. The nontrivial claim in that result is the fact that $\Delta_{(f,g)}$ is an irreducible derivation. Once this is known, the other assertions follow from 2.20.

7.3. **Graded rings.** Every graded ring considered in sections 7 and 8 is either $\mathbb{Z}$-graded or $\mathbb{N}$-graded. Let $R = \bigoplus_i R_i$ be a graded ring.

(1) If $x \in R_i \setminus \{0\}$ then we write $\deg x = i$.

(2) A subring A of R is said to be *homogeneous* if $A = \bigoplus_i (A \cap R_i)$. If this is the case then we set $A_i = A \cap R_i$ and regard $A = \bigoplus_i A_i$ as a graded ring.

(3) If $D : R \to R$ is a derivation, we say that D is *homogeneous* if there exists an integer n such that $D(R_i) \subseteq R_{i+n}$ holds for all i; note that n is unique if $D \neq 0$; we say that D is homogeneous of degree n. Observe that if D is homogeneous then $\ker D$ is a homogeneous subring of R.

(4) If S is a multiplicatively closed subset of $\bigcup_i (R_i \setminus \{0\})$ then the localized ring $S^{-1}R$ inherits a $\mathbb{Z}$-grading in a natural way: If $x \in R_i \setminus \{0\}$ and $s \in S$ then x/s is declared to be homogeneous of degree $\deg x - \deg s$. If we write $S^{-1}R = \mathcal{R} = \bigoplus_{i\in\mathbb{Z}} \mathcal{R}_i$ then $\mathcal{R}_0$ is a subring of $\mathcal{R}$, called the *homogeneous localization* of R with respect to S.

If $S = \{1, f, f^2, \dots\}$ where $f \in R_i \setminus \{0\}$ for some i, we write $R_f = S^{-1}R = \mathcal{R}$ and $R_{(f)} = \mathcal{R}_0$.

Exercise 7.1. Let p, q be relatively prime positive integers and let $\mathbb{k}$ be any field. Define an $\mathbb{N}$-grading on $R = \mathbb{k}[X, Y] = \mathbb{k}^{[2]}$ by declaring that $R_0 = \mathbb{k}$, $X \in R_p$ and $Y \in R_q$. Consider the ring $R_{(XY)}$, i.e., the homogeneous localization of R with respect to the multiplicative set $\{1, XY, (XY)^2, \dots\}$. Show that $R_{(XY)} = \mathbb{k}[\xi, \xi^{-1}]$ where $\xi = X^q/Y^p$. Also show that $R_{(Y)} = \mathbb{k}[\xi] = \mathbb{k}^{[1]}$ and that $R_{(X)} = \mathbb{k}[\xi^{-1}] = \mathbb{k}^{[1]}$.

The next two facts (7.4 and 7.5) are needed in the proof of 8.8:

7.4. **Lemma.** *Let $R = \bigoplus R_n$ be a $\mathbb{Z}$-graded UFD satisfying:*

$$\textit{For all } n \in \mathbb{Z}, \textit{ if } R_n \neq 0 \textit{ then } R_n \cap R^* \neq \varnothing.$$

Then R_0 is a UFD.

7.5. **Lemma.** *Let $R = \bigoplus R_n$ be a $\mathbb{Z}$-graded domain and Q a homogeneous subring of R satisfying:*

$$\textit{For all } n \in \mathbb{Z}, \textit{ if } R_n \neq 0 \textit{ then } Q_n \cap Q^* \neq \varnothing.$$

Then tfae:

(1) *There exists a homogeneous element v of R such that $R = Q[v] = Q^{[1]}$*
(2) $R_0 = (Q_0)^{[1]}$.

Exercise 7.2. Prove 7.4 and 7.5.

The following is needed in the proof of 7.7.1:

7.6. **Lemma.** *Let $\mathbb{k}$ be a field, $A = \mathbb{k}^{[r]}$ $(r \geq 1)$ and let $A = \oplus_{i \in \mathbb{N}} A_i$ be a grading such that $A_0 = \mathbb{k}$. If $f_1, \dots, f_n$ are homogeneous elements of A satisfying $\mathbb{k}[f_1, \dots, f_n] = A$, then there is a subset $\{g_1, \dots, g_r\}$ of $\{f_1, \dots, f_n\}$ satisfying $A = \mathbb{k}[g_1, \dots, g_r]$.*

Proof. Consider a subset $\{g_1, \dots, g_s\}$ of $\{f_1, \dots, f_n\}$ satisfying $A = \mathbb{k}[g_1, \dots, g_s]$ and minimal with respect to this property (in particular, $\deg(g_i) > 0$ for all i). Let $R = \mathbb{k}[T_1, \dots, T_s] = \mathbb{k}^{[s]}$, with grading $R = \oplus_{i \in \mathbb{N}} R_i$ determined by $R_0 = \mathbb{k}$ and $\deg(T_i) = \deg(g_i)$. Then the surjective $\mathbb{k}$-homomorphism $e : R \to A$, $e(\varphi) = \varphi(g_1, \dots, g_s)$, is homogeneous of degree zero. It suffices to show that the prime ideal $\mathfrak{p} = \ker e$ is zero. Assume the contrary. Note that $(T_1, \dots, T_s) \supseteq \mathfrak{p}$, i.e., the variety $V(\mathfrak{p}) \subseteq \mathbb{A}^s$ passes through the origin. Since the origin is a smooth point (A is smooth over $\mathbb{k}$), and since $\mathfrak{p}$ is generated by its homogeneous elements, the jacobian condition implies that some homogeneous $\varphi \in \mathfrak{p}$ contains a term λT_j $(\lambda \in \mathbb{k}^*)$. Since φ is homogeneous and $\deg(T_i) > 0$ for all i, $\varphi - \lambda T_j \in \mathbb{k}[T_1, \dots, T_{j-1}, T_{j+1}, \dots T_s]$, so $g_j \in \mathbb{k}[g_1, \dots, g_{j-1}, g_{j+1}, \dots g_s]$, contradicting minimality of $\{g_1, \dots, g_s\}$. $\square$

7.7. Fix a field $\mathbb{k}$ of characteristic zero and a triple $\omega = (a_0, a_1, a_2)$ of positive integers. Let $B = \mathbb{k}[X_0, X_1, X_2] = \mathbb{k}^{[3]}$. The symbol (B, ω) means B regarded as an $\mathbb{N}$-graded ring, $B = \oplus_{i \in \mathbb{N}} B_i$, where $B_0 = \mathbb{k}$ and $X_i \in B_{a_i}$ for $i \in \{0, 1, 2\}$. Consider the following subsets of $\text{LND}(B)$ and $\text{KLND}(B)$ respectively:

$$\text{LND}(B,\omega) = \{ D \in \text{LND}(B) \mid D \text{ is homogeneous with respect to the grading of } (B,\omega) \}$$
$$\text{KLND}(B,\omega) = \{ \ker D \mid D \in \text{LND}(B,\omega) \text{ and } D \neq 0 \}.$$

7.7.1. **Lemma.** *For a subalgebra A of B, tfae:*

(1) $A \in \text{KLND}(B,\omega)$
(2) $A \in \text{KLND}(B)$ *and A is a homogeneous subring of B*
(3) $A \in \text{KLND}(B)$ *and $A = \Bbbk[f,g]$ for some homogeneous f,g.*

Moreover, if $A = \Bbbk[f,g]$ satisfies condition (3) then

$$\text{LND}_A(B,\omega) = \left\{ \alpha\Delta_{(f,g)} \;\middle|\; \alpha \text{ is a homogeneous element of } A \right\}.$$

Proof. It is obvious that (1) implies (2). If (2) holds then Miyanishi's theorem 7.1 tells us that $A = \Bbbk^{[2]}$; then (3) follows from 7.6.

It is easy to see that if $f, g \in B$ are homogeneous then so is $\Delta_{(f,g)} : B \to B$. In view of 7.2, we obtain that (3) implies both (1) and the description of $\text{LND}_A(B,\omega)$. □

Remark. Zurkowski [21] gives a direct proof (i.e. a proof which does not rely on Miyanishi's result) of the fact that if $A \in \text{KLND}(B,\omega)$ then $A = \Bbbk[f,g]$ for some homogeneous f,g. A simplified version of Zurkowski's argument is given in Daan Holtackers'MSc thesis [12].

7.8. **Weighted projective planes.** Fix an algebraically closed field $\Bbbk$ and a triple $\omega = (a_0, a_1, a_2)$ of positive integers. Consider the $\mathbb{N}$-graded ring $B = \Bbbk[X_0, X_1, X_2] = \Bbbk^{[3]}$ where $B_0 = \Bbbk$ and (for all $i \in \{0,1,2\}$) $X_i \in B_{a_i}$. The Proj of that graded ring is denoted $\mathbb{P}_\omega$ and is called the *weighted projective plane* determined by weights $\omega = (a_0, a_1, a_2)$. One can see that $\mathbb{P}_\omega$ is an algebraic surface which is projective and normal.

Concretely, define an equivalence relation $\sim$ on the set $\Bbbk^3 \setminus \{(0,0,0)\}$ by declaring that $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if for some $t \in \Bbbk^*$ we have $(y_0, y_1, y_2) = (t^{a_0}x_0, t^{a_1}x_1, t^{a_2}x_2)$. Then $\mathbb{P}_\omega$ is the set of equivalence classes, and each equivalence class is called a point of $\mathbb{P}_\omega$. The equivalence class of (x_0, x_1, x_2) is denoted $(x_0 : x_1 : x_2)$.

If $h \in B$ is homogeneous with respect to the above grading, then the zero set of h is well-defined: $V(h) = \{ (x_0 : x_1 : x_2) \in \mathbb{P}_\omega \mid h(x_0 : x_1 : x_2) = 0 \}$.

Note that if $\omega = (1,1,1)$ then $\mathbb{P}_\omega = \mathbb{P}^2$ is the usual projective plane.

8. Homogeneous derivations of $\Bbbk^{[3]}$

The references for this section are: [4], [5], [9], [10], [6]. In this section, $\Bbbk$ is an algebraically closed field of characteristic zero, $B = \Bbbk[X_0, X_1, X_2] = \Bbbk^{[3]}$ and $\omega = (a_0, a_1, a_2)$, where a_0, a_1, a_2 are positive integers. The symbol (B,ω) means B regarded as an $\mathbb{N}$-graded ring, $B = \oplus_{i \in \mathbb{N}} B_i$, where $B_0 = \Bbbk$ and $X_i \in B_{a_i}$ for $i \in \{0,1,2\}$. Our goal is to describe the sets $\text{LND}(B,\omega)$ and $\text{KLND}(B,\omega)$ defined in 7.7. By result 7.7.1, we may formulate our problem as follows (where homogeneity of f,g is relative to the grading of (B,ω)):

8.1. *Which homogeneous elements $f, g \in B$ are such that $\Bbbk[f,g] \in \text{KLND}(B,\omega)$?*

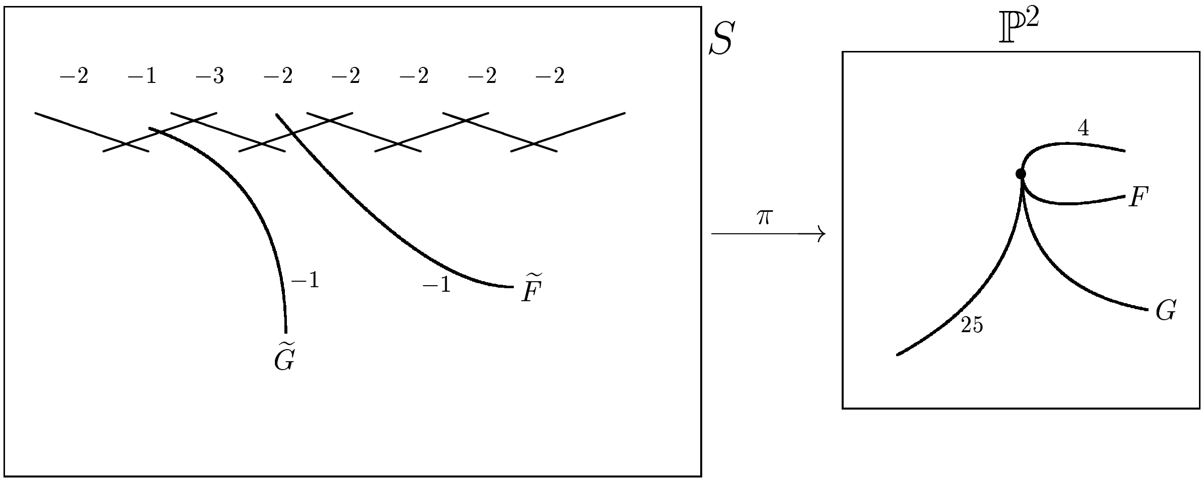
8.2. **Example.** Consider Freudenburg's first example of a rank 3 locally nilpotent derivation of $\Bbbk^{[3]}$, namely, $\Bbbk[f,g] \in \text{KLND}(B)$ where

$$f = X_0X_2 - X_1^2, \qquad g = X_0^5 + 2X_0^3X_1X_2 - 2X_0^2X_1^3 + X_0^2X_2^3 - 2X_0X_1^2X_2^2 + X_1^4X_2.$$

In fact we have $\Bbbk[f,g] \in \text{KLND}(B,\omega)$ where $\omega = (1,1,1)$. In view of question 8.1, we want to understand the properties of f, g, or equivalently the properties of the curves $F = V(f)$ and $G = V(g)$ in $\mathbb{P}^2$. We ask:

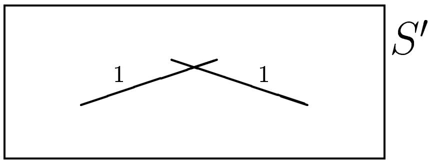
What is the affine surface $\mathbb{P}^2 \setminus (F \cup G)$ *?*

To identify that surface, blow-up $\mathbb{P}^2$ 8 times, as follows:



$\pi^{-1}(F \cup G) =$ union of the 10 curves pictured in S

Then $\mathbb{P}^2 \setminus (F \cup G) \;\cong\; S$ minus those 10 curves. Further blowing-up and blowing-down transforms "S minus the 10 curves" into:



Thus $S' = \mathbb{P}^2$ and $\mathbb{P}^2 \setminus (F \cup G) \cong \mathbb{P}^2$ minus two lines. So we conclude:

(27) *The surface* $\mathbb{P}^2 \setminus (F \cup G)$ *is isomorphic to* $\mathbb{A}^1_* \times \mathbb{A}^1$

where $\mathbb{A}^1_*$ denotes $\mathbb{A}^1$ minus a point. We will see in 8.8 that every element $\Bbbk[f,g]$ of $\text{KLND}(B,\omega)$ satisfies (27), and conversely.

We now return to the general situation, i.e., let (B,ω) be as in the introduction of the present section. Note that if $d = \gcd(a_0,a_1,a_2)$ and $\omega' = (a_0/d, a_1/d, a_2/d)$ then a derivation of B is ω-homogeneous if and only if it is ω'-homogeneous; so $\text{LND}(B,\omega) = \text{LND}(B,\omega')$ and $\text{KLND}(B,\omega) = \text{KLND}(B,\omega')$ and consequently we may assume throughout:

(28) $$\gcd(a_0,a_1,a_2) = 1.$$

Assumption (28) is in effect until the end of section 8. The problem splits into two cases:

"Easy" case: a_0, a_1, a_2 are not pairwise relatively prime.

Hard case: a_0, a_1, a_2 are pairwise relatively prime.

Before discussing how to answer question 8.1 in each case, we define:

8.3. A *homogeneous coordinate system* of B is an ordered triple (u_0, u_1, u_2) of homogeneous elements of B satisfying $\mathbb{k}[u_0, u_1, u_2] = B$.

Exercise 8.1. If (u_0, u_1, u_2) is any homogeneous coordinate system of B then the triple $(\deg u_0, \deg u_1, \deg u_2)$ is a permutation of (a_0, a_1, a_2).

THE "EASY" CASE

Assume that $\omega = (a_0, a_1, a_2)$ satisfies (28) and:

(29) $$a_0, a_1, a_2 \textit{ are not pairwise relatively prime.}$$

8.4. **Example.** Suppose that $\omega = (4, 6, 7)$. Note that $\mathbb{k}[X_0, X_1] \in \text{KLND}(B, \omega)$ and that $\gcd(\deg X_0, \deg X_1) = 2$; thus:

$$\mathbb{k}[f, g] \in \text{KLND}(B, \omega) \quad \not\Rightarrow \quad \gcd(\deg f, \deg g) = 1.$$

Compare with 8.7.

Exercise 8.2. With $\omega = (4, 6, 7)$, verify that $\mathbb{k}[X_0,\ X_0^2 X_1 + X_2^2] \in \text{KLND}(B, \omega)$.

Under assumption (29), one can show that all elements of $\text{LND}(B, \omega)$ have rank < 3. More precisely, the main result is as follows:

8.5. **Theorem.** *Let $A \in \text{KLND}(B, \omega)$. Then there exists a homogeneous coordinate system (X_0, X_1, X_2) of B such that one of the following conditions holds:*

1. $A = \mathbb{k}[X_0, X_1]$.
2. $\gcd(\deg X_0, \deg X_2) = 1$ *and* $A = \mathbb{k}[X_0, X_0^e X_1 + \psi(X_0, X_2)]$, *for some* $e \in \mathbb{N}$ *and some* $\psi(X_0, X_2) \in \mathbb{k}[X_0, X_2]$ *such that* $X_0^e X_1 + \psi(X_0, X_2)$ *is homogeneous and irreducible.*
3. $\gcd(\deg X_0, \deg X_1) = 1 = \gcd(\deg X_0, \deg X_2)$ *and* $A = k[X_0, P]$ *for some homogeneous* $P \in B$ *which satisfies* $\gcd_B(P_{X_1}, P_{X_2}) = 1$ *and which is a variable of* $\mathbb{k}(X_0)[X_1, X_2]$.

Refer to [6] for the proof of 8.5 and also for that of the following:

8.6. **Corollary.** *If* $\gcd(a_i, a_j) > 1$ *for all* $\{i, j\} \subset \{0, 1, 2\}$, *then:*

$$\text{KLND}(B, \omega) = \big\{\ \mathbb{k}[X_0, X_1],\ \mathbb{k}[X_0, X_2],\ \mathbb{k}[X_1, X_2]\ \big\}.$$

THE HARD CASE

Until the end of section 8, we assume that $\omega = (a_0, a_1, a_2)$ satisfies:

(30) $$a_0, a_1, a_2 \textit{ are pairwise relatively prime.}$$

Then the first result is:

8.7. **Proposition.** *Suppose that* $\mathbb{k}[f,g]$ *is an element of* $\mathrm{KLND}(B,\omega)$*, where* f,g *are homogeneous. Then* $\gcd(\deg f, \deg g) = 1$.

Proof. This is a corollary to Theorem 3.7 of [4]. A different proof is given in [5]. □

8.8. **Theorem** ([4], Theorem 3.5). *For homogeneous elements* $f,g \in B$*, tfae:*

(1) $\mathbb{k}[f,g] \in \mathrm{KLND}(B,\omega)$

(2) f,g *are irreducible and* $\mathbb{P}_\omega \setminus V(fg) \cong \mathbb{A}^1_* \times \mathbb{A}^1$ *(isomorphism of algebraic surfaces).*

See 7.8 for the definition of $\mathbb{P}_\omega$. Note that 8.8 replaces the problem of describing $\mathrm{KLND}(B,\omega)$ by a problem of geometry, namely:

What are all pairs of curves C_1, C_2 *in* $\mathbb{P}_\omega$ *such that* $\mathbb{P}_\omega \setminus (C_1 \cup C_2) \cong \mathbb{A}^1_* \times \mathbb{A}^1$ *?*

Proof of 8.8. Note that condition 8.8(2) is equivalent to:

8.8(2′) f,g *are irreducible and* $B_{(fg)} = \mathbb{k}[\zeta,\zeta^{-1}]^{[1]}$ *for some* $\zeta \in B_{(fg)}$ *such that* $\zeta \notin \mathbb{k}$,

where $B_{(fg)}$ denotes the homogeneous localization of B at the set $\{1, fg, (fg)^2, (fg)^3, \dots\}$. We assume throughout that at least one of conditions 8.8(1), 8.8(2) (or 8.8(2′)) holds. Let $p = \deg f$, $q = \deg g$ and $A = \mathbb{k}[f,g]$. We claim:

(31) $\gcd(p,q) = 1$ and f,g are irreducible and not associates.

Indeed, if 8.8(1) holds then $\gcd(p,q) = 1$ by 8.7 and f,g are not associates as $\mathrm{trdeg}_{\mathbb{k}}(A) = 2$ by 2.10; as f,g are irreducible in A and A is factorially closed in B, it follows that f,g are irreducible in B. On the other hand, if 8.8(2) holds then the Picard group of $\mathbb{P}_\omega \setminus V(fg)$ is trivial; as this group is $\mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(p,q)$, we get $\gcd(p,q) = 1$. If f,g are associates then $B_{(fg)} = B_{(f^2)} = B_{(f)}$ and it is easily verified that $B^*_{(f)} = \mathbb{k}^*$, but this is impossible because 8.8(2′) implies that some unit of $B_{(fg)}$ does not belong to $\mathbb{k}$; so f,g are not associates. This shows that (31) is true. It follows that if we define $\xi = f^q/g^p \in B_{(fg)}$ then

(32) $$A_{(fg)} = \mathbb{k}[\xi,\xi^{-1}].$$

Suppose that 8.8(1) holds; we shall now prove that 8.8(2′) is satisfied. In view of (31) and (32), it suffices to show that

(33) $$B_{(fg)} = (A_{(fg)})^{[1]}.$$

By 8.8(1), $A = \ker D$ for some $0 \neq D \in \mathrm{LND}(B,\omega)$; then A_{fg} is the kernel of the localization $D_{fg} : B_{fg} \to B_{fg}$ of D. By (31), we may choose $i,j \in \mathbb{Z}$ such that $pi + qj + \deg(D) = 0$; define $\mathcal{D} = f^i g^j D_{fg}$, then $\mathcal{D} : B_{fg} \to B_{fg}$ is locally nilpotent, homogeneous of degree zero and has kernel A_{fg}; the restriction $\mathcal{D}_0 : B_{(fg)} \to B_{(fg)}$ of $\mathcal{D}$ is locally nilpotent and $\ker(\mathcal{D}_0) = A_{fg} \cap B_{(fg)} = A_{(fg)}$. We claim that

(34) $B_{(fg)}$ is a UFD

and that each irreducible element π of $A_{(fg)}$ satisfies:

(35) $A_{(fg)}/\pi A_{(fg)}$ is algebraically closed in $B_{(fg)}/\pi B_{(fg)}$.

Indeed, note that the $\mathbb{Z}$-graded factorial domain $R = B_{fg}$ satisfies the hypothesis of 7.4 because $\gcd(p,q) = 1$; thus (34) holds. Since $\Bbbk$ is an algebraically closed field by assumption, we have $A_{(fg)}/\pi A_{(fg)} = \Bbbk$ by (32) and hence (35) holds. From (34), (35) and 2.23, we obtain (33). This shows that 8.8(1) implies 8.8(2′).

Conversely, suppose that 8.8(2′) holds. In order to prove 8.8(1), it suffices to show that $A \in \text{KLND}(B)$ (then condition (3) of 7.7.1 is satisfied).

As ζ is a unit of $B_{(fg)}$, we have $\zeta = \lambda f^i g^j$ for some $\lambda \in \Bbbk^*$ and $i, j \in \mathbb{Z}$; we also have $0 = \deg \zeta = pi + qj$, and it follows from $\gcd(p,q) = 1$ that $\zeta = \lambda \xi^n$ for some $n \in \mathbb{Z}$. We also have $\xi \in B^*_{(fg)} = \Bbbk[\zeta, \zeta^{-1}]^*$, so $\xi = \mu \zeta^m$ for some $\mu \in \Bbbk^*$ and $m \in \mathbb{Z}$. We conclude that $mn = 1$ and that $\Bbbk[\zeta, \zeta^{-1}] = \Bbbk[\xi, \xi^{-1}] = A_{(fg)}$. Thus 8.8(2′) implies that (33) holds; by 7.5 we obtain $B_{fg} = A^{[1]}_{fg}$, so in particular A_{fg} is the kernel of some $\mathcal{D} \in \text{LND}(B_{fg})$. Consider $\mathcal{D}' = f^u g^v \mathcal{D} \in \text{LND}(B_{fg})$, where $u, v \in \mathbb{N}$ are such that $f^u g^v \mathcal{D}(X_i) \in B$ for all $i \in \{0, 1, 2\}$ (recall that $B = \Bbbk[X_0, X_1, X_2]$). Then $\mathcal{D}'(B) \subseteq B$ and the restriction $D : B \to B$ of $\mathcal{D}'$ satisfies $D \in \text{LND}(B)$ and $\ker D = A_{fg} \cap B$. So there remains only to show that $A_{fg} \cap B = A$. As a first step, we prove:

$$A \cap fB = fA. \tag{36}$$

It suffices to verify that if $h \in A \cap fB$ and h is homogeneous then $h \in fA$. As $h \in A = \Bbbk[f, g]$, we have $h = f\alpha + \beta$ where $\alpha \in A$ and $\beta \in \Bbbk[g]$; by homogeneity of h, we have in fact $\beta = \lambda g^n$ for some $\lambda \in \Bbbk$ and $n \in \mathbb{N}$. Then $\lambda g^n \in fB$. We have $f \nmid g^n$ by (31), so $\lambda = 0$ and $h = f\alpha \in fA$. This proves (36) and, by symmetry, we also have $A \cap gB = gA$. It follows by induction that $A \cap f^i g^j B = f^i g^j A$ for all $i, j \in \mathbb{N}$, and consequently $A_{fg} \cap B = A$. Thus $A \in \text{KLND}(B)$ and we have shown that 8.8(2′) implies 8.8(1). □

Affine rulings and locally nilpotent derivations

There is a rich interplay between the theory of algebraic surfaces and homogeneous locally nilpotent derivations of $\Bbbk^{[3]}$. As an example, [6] contains the following result:

8.9. **Theorem.** *Consider (B, ω), where ω is any triple of positive integers. Given any $A, A' \in \text{KLND}(B, \omega)$, there exists a finite sequence of local slice constructions which transforms A into A'.*

The statement of 8.9 is purely algebraic, but we don't know how to give a direct, algebraic proof of it. In the following paragraphs, we indicate how geometry can be used to prove the "hard case" of that result, i.e., the case where ω is pairwise relatively prime. (We will not discuss the "easy case", whose proof does not require geometry.)

8.10. **Definition.** Fix an algebraic surface X which is complete, normal and rational (for instance, $X = \mathbb{P}_\omega$). An *explicit affine ruling* of X is a morphism $\rho : U \to \mathbb{P}^1$ satisfying:

(1) The image of ρ is an open subset $\Gamma \neq \varnothing$ of $\mathbb{P}^1$

(2) $U \neq \varnothing$ is an open subset of X such that $U \cong \Gamma \times \mathbb{A}^1$

(3) ρ is the composition $U \xrightarrow{\cong} \Gamma \times \mathbb{A}^1 \xrightarrow{\text{proj.}} \Gamma \hookrightarrow \mathbb{P}^1$.

8.11. **Example.** Consider (B,ω) and $\mathbb{P}_\omega$ as before, where $\omega = (a_0,a_1,a_2)$ is pairwise relatively prime. Consider an element $\Bbbk[f,g]$ of $\textsc{klnd}(B,\omega)$, where f,g are homogeneous. We show that the ordered pair (f,g) determines an explicit affine ruling of $\mathbb{P}_\omega$.

Let $p = \deg f$, $q = \deg g$ and $\xi = f^q/g^p$, then $\Bbbk[\xi,\xi^{-1}]$ is a subring of $B_{(fg)}$. Applying the functor Spec to the inclusion homomorphism $\Bbbk[\xi,\xi^{-1}] \hookrightarrow B_{(fg)}$ yields a morphism $U \to \Gamma$, where $U = \operatorname{Spec} B_{(fg)} = \mathbb{P}_\omega \setminus (V(f) \cup V(g))$ and $\Gamma = \operatorname{Spec} \Bbbk[\xi,\xi^{-1}] = \mathbb{P}^1$ minus two points. As shown in the proof of 8.8, we have $B_{(fg)} = \Bbbk[\xi,\xi^{-1}]^{[1]}$; this means that the composition $U \to \Gamma \hookrightarrow \mathbb{P}^1$ (which we denote $\rho : U \to \mathbb{P}^1$) is an explicit affine ruling of $\mathbb{P}_\omega$. Moreover, the map ρ is described by:

$$\begin{array}{rcl} \rho \,:\, U & \longrightarrow & \mathbb{P}^1 \\ (x_0 : x_1 : x_2) & \longmapsto & \big(f(x_0,x_1,x_2)^q : g(x_0,x_1,x_2)^p\big) \end{array} \tag{37}$$

8.12. Let X be a surface as in 8.10 and let $\rho : U \to \mathbb{P}^1$ be an explicit affine ruling of X.

(1) If $\Gamma' \neq \varnothing$ is an open subset of the image of ρ then the restriction $\rho' : \rho^{-1}(\Gamma') \to \mathbb{P}^1$ of ρ is an explicit affine ruling of X; we write $\rho' \leq \rho$ in this situation.

(2) If $\theta : \mathbb{P}^1 \to \mathbb{P}^1$ is an automorphism then the composite $\rho' : U \xrightarrow{\rho} \mathbb{P}^1 \xrightarrow{\theta} \mathbb{P}^1$ is an explicit affine ruling of X; we write $\rho \asymp \rho'$ in this case.

Consider the set S of explicit affine rulings of X. Two elements $\rho, \rho' \in S$ are *equivalent* if there exists a finite sequence $\rho_0, \dots, \rho_n$ of elements of S satisfying $\rho_0 = \rho$, $\rho_n = \rho'$ and

$$\forall_{i<n} \qquad \rho_i \leq \rho_{i+1} \quad \text{or} \quad \rho_{i+1} \leq \rho_i \quad \text{or} \quad \rho_i \asymp \rho_{i+1}.$$

By an *affine ruling* of X, we mean an equivalence class of explicit affine rulings of X. We write

$$\textsc{afrul}(X) \quad = \quad \text{set of affine rulings of } X.$$

8.12.1. Each explicit affine ruling $\rho : U \to \mathbb{P}^1$ of X extends to a rational map $X \dashrightarrow \mathbb{P}^1$ defined everywhere outside a finite set of points (because X is normal); in turn, this rational map determines a linear system Λ on X without fixed components. Then $\rho \mapsto \Lambda$ is a well-defined map and one can see that two explicit affine rulings are equivalent if and only if they determine the same linear system. (The image of the map $\rho \mapsto \Lambda$ is a certain collection of linear systems on X; one may consider that these linear systems *are* the affine rulings—this is how the notion of *affine ruling* is defined in papers [9], [10] and [6].)

8.13. **Theorem.** *Consider (B,ω) and $\mathbb{P}_\omega$ as before, where $\omega = (a_0,a_1,a_2)$ is pairwise relatively prime. Then the process described in* 8.11 *determines a well-defined bijection*

$$\textsc{klnd}(B,\omega) \longrightarrow \textsc{afrul}(\mathbb{P}_\omega).$$

Comments. Let $A \in \textsc{klnd}(B,\omega)$. Then example 8.11 shows that each homogeneous coordinate system (f,g) of A determines an explicit affine ruling of $\mathbb{P}_\omega$, and it is not difficult to see that if (f,g) and (f',g') are two homogeneous coordinate systems of A then the corresponding explicit affine rulings are equivalent; thus one gets a well-defined set map $\textsc{klnd}(B,\omega) \to \textsc{afrul}(\mathbb{P}_\omega)$. It is proved in [6] that this map is bijective. $\square$

So describing $\text{KLND}(B, \omega)$ "reduces" to describing $\text{AFRUL}(\mathbb{P}_\omega)$. Paper [10] gives a geometric classification of $\text{AFRUL}(\mathbb{P}_\omega)$ and [6] derives algebraic consequences for $\text{KLND}(B, \omega)$. To conclude this section, we mention some aspects of that work; this part is very sketchy, we only give a vague idea of how this works.

8.14. (1) Fix a surface X as in 8.10. To each affine ruling $\Lambda \in \text{AFRUL}(X)$, one associates a set $\mathcal{P}(\Lambda)$ and, given $P \in \mathcal{P}(\Lambda)$, one defines an element $\Lambda * P$ of $\text{AFRUL}(X)$. The cardinality of the set $\mathcal{P}(\Lambda)$ is that of the ground field $\mathbb{k}$. The object P may be thought of as a "recipe" for modifying Λ and the affine ruling $\Lambda * P$ is obtained by modifying Λ according to P. The modification is achieved by performing a (possibly long) sequence of blowings-up and blowings-down. Given $\Lambda, \Lambda' \in \text{AFRUL}(X)$, we call Λ' a *modification* of Λ if there exists $P \in \mathcal{P}(\Lambda)$ such that $\Lambda * P = \Lambda'$. Then one shows that if Λ' is a modification of Λ then Λ is a modification of Λ'.

(2) Using the description of $\text{AFRUL}(\mathbb{P}_\omega)$ given in [10], paper [6] proves that if ω is any triple of positive integers then:

Given any two affine rulings Λ, Λ' of $\mathbb{P}_\omega$, there exists a finite sequence of modifications which transforms Λ into Λ'.

(3) Suppose that ω is pairwise relatively prime, let $A, A' \in \text{KLND}(B, \omega)$ and let $\Lambda, \Lambda' \in \text{AFRUL}(\mathbb{P}_\omega)$ be the images of A and A' respectively, under the bijection of 8.13. Then [6] also proves:

If Λ' is a modification of Λ then A' can be obtained from A by a "local slice construction".

Parts (2) and (3) of 8.14 immediately imply that 8.9 is true when ω is pairwise relatively prime. Regarding the case where ω is not pairwise relatively prime, we will only say that its proof is much easier and does not require geometry.

9. Danielewski surfaces and local slice construction

In this section, $\mathbb{k}$ is any field of characteristic zero.

We present some general facts about Danielewski surfaces and then use some of that material to clarify Freudenburg's "local slice construction". The main reference for this section is [11], but note that some of the results (notably 9.5.2 and 9.6) can also be found in the work of Makar-Limanov (see [11] for references).

9.1. **Definition.** Let B be a $\mathbb{k}$-algebra. We call B a *Danielewski surface over* $\mathbb{k}$ if B is isomorphic to

$$\mathbb{k}[X, Y, Z]/(XY - \varphi(Z)) \tag{38}$$

for some $\varphi(Z) \in \mathbb{k}[Z] \setminus \mathbb{k}$ (where X, Y, Z are indeterminates). If B is a Danielewski surface over $\mathbb{k}$ then any triple $(x, y, z) \in B^3$ satisfying $B = \mathbb{k}[x, y, z]$ and $xy \in \mathbb{k}[z] \setminus \mathbb{k}$ is called a *coordinate system* of B.

9.2. **Example.** Let $B = \mathbb{k}[U, V] = \mathbb{k}^{[2]}$, then B is a Danielewski surface over $\mathbb{k}$ (take φ of degree 1 in (38)). The triple (U, V, UV) is a coordinate system of the Danielewski surface B. The triple $(x, y, z) = (0, U, V)$ satisfies $B = \mathbb{k}[x, y, z]$ and $xy \in \mathbb{k}[z]$, but is not a coordinate system of B because $xy \in \mathbb{k}$.

9.3. **Lemma.** *Let B be a Danielewski surface over $\mathbb{k}$.*

(1) *B is a normal domain,* $\operatorname{trdeg}_{\mathbb{k}} B = 2$ *and* $B^* = \mathbb{k}^*$.
(2) *B has at least one coordinate system.*
(3) *If (x, y, z) is any coordinate system of B then there exists a unique $D \in \mathrm{LND}(B)$ such that $D(x) = 0$ and $D(z) = x$. Moreover, D is irreducible,* $\ker D = \mathbb{k}[x]$ *and* $\mathrm{LND}_{\mathbb{k}[x]}(B) = \{\alpha D \mid \alpha \in \mathbb{k}[x]\}$.
(4) *Let (x, y, z) be a coordinate system of B and let I be the principal ideal $\mathbb{k}[z] \cap xB$ of $\mathbb{k}[z]$. Then xy is a generator of I.*

Proof. We may assume that $B = \mathbb{k}[X, Y, Z]/(XY - \psi(Z))$ where $\psi(Z) \in \mathbb{k}[Z] \setminus \mathbb{k}$. As $\psi(Z) \neq 0$, $XY - \psi(Z)$ is an irreducible element of $\mathbb{k}[X, Y, Z]$; so it is clear that B is a domain and that $\operatorname{trdeg}_{\mathbb{k}}(B) = 2$. Let $\pi_1 : \mathbb{k}[X, Y, Z] \to B$ be the canonical epimorphism. Observe that if $\pi_1(X)\pi_1(Y) = \lambda \in \mathbb{k}$ then $XY - \lambda \in \ker \pi_1$, so $XY - \psi(Z)$ divides $XY - \lambda$ in $\mathbb{k}[X, Y, Z]$; this is absurd because $\deg_Z(XY - \psi(Z)) > 0$, so $\pi_1(X)\pi_1(Y) \notin \mathbb{k}$ and it follows that $(\pi_1(X), \pi_1(Y), \pi_1(Z))$ is a coordinate system of B, proving assertion (2).

Let (x, y, z) be any coordinate system of B and consider the surjective $\mathbb{k}$-homomorphism $\pi : \mathbb{k}[X, Y, Z] \to B$ which maps X, Y, Z to x, y, z respectively. As B is a domain of transcendence degree 2 over $\mathbb{k}$, $\ker \pi$ is a principal ideal generated by an irreducible element of $\mathbb{k}[X, Y, Z]$. Since $xy \in \mathbb{k}[z] \setminus \mathbb{k}$, there exists $\varphi(Z) \in \mathbb{k}[Z]$ such that $\deg_Z \varphi(Z) > 0$ and $xy = \varphi(z)$; then $P = XY - \varphi(Z)$ belongs to $\ker \pi$ and is irreducible; consequently $\ker \pi = (P)$. Thus we may assume that $B = \mathbb{k}[X, Y, Z]/(XY - \varphi(Z))$ and that π is the canonical epimorphism. Let $n = \deg_Z(P) = \deg_Z \varphi(Z)$ and recall that $n > 0$. Viewing P as a polynomial in Z with coefficients in $\mathbb{k}[X, Y]$, we note that its leading coefficient belongs to $\mathbb{k}[X, Y]^*$; so, by the division algorithm, for each $F \in \mathbb{k}[X, Y, Z]$ there exists a unique pair (Q, G) of elements of $\mathbb{k}[X, Y, Z]$ satisfying $F = PQ + G$ and $\deg_Z(G) < n$; consequently,

(39) For each $b \in B$, there exists a unique $G \in \mathbb{k}[X, Y, Z]$ such that $\deg_Z(G) < n$ and $b = G(x, y, z)$

or equivalently:

(40) x, y are algebraically independent over $\mathbb{k}$ and B is a free module over $\mathbb{k}[x, y]$ with basis $\{1, z, \dots, z^{n-1}\}$.

From (40) we deduce:

(41) $$\mathbb{k}(x) \cap B = \mathbb{k}[x].$$

Indeed, if $b \neq 0$ belongs to $\mathbb{k}(x) \cap B$ then write $b = \sum_{i<n} a_i z^i$ (with $a_i \in \mathbb{k}[x, y]$); then there exist $a, a' \in \mathbb{k}[x] \setminus \{0\}$ such that $a' = ab = \sum_{i<n} (aa_i) z^i$, which implies that $a_i = 0$ for all $i > 0$; so $b \in \mathbb{k}[x, y] \cap \mathbb{k}(x) = \mathbb{k}[x]$ and (41) is true. Note that $y = x^{-1}\varphi(z) \in \mathbb{k}(x)[z]$, so it is clear that $\mathbb{k}[x, z] \subseteq B \subseteq \mathbb{k}(x)[z]$; thus if we let $S = \mathbb{k}[x] \setminus \{0\}$ then $S^{-1}B = \mathbb{k}(x)[z] =$

$\mathbb{k}(x)^{[1]}$, from which we deduce:

(42) $\qquad$ $\mathbb{k}[x]$ is factorially closed (and hence algebraically closed) in B.

Indeed, if $b_1, b_2 \in B$ satisfy $b_1 b_2 \in \mathbb{k}[x] \setminus \{0\}$ then b_1, b_2 are units of $S^{-1}B = \mathbb{k}(x)^{[1]}$ and so belong to $B \cap \mathbb{k}(x) = \mathbb{k}[x]$. Statement (42) implies that $B^* = \mathbb{k}[x]^*$; as x is transcendental over $\mathbb{k}$ by (40), we obtain $B^* = \mathbb{k}^*$. We claim:

$$B = \mathbb{k}(x)[z] \cap \mathbb{k}(y)[z] \quad \text{(intersection in } \operatorname{Frac} B\text{)}. \tag{43}$$

Consider $\beta \in \mathbb{k}(x)[z] \cap \mathbb{k}(y)[z]$ and write $\beta = F(x,y,z)/f(x) = G(x,y,z)/g(y)$ where $F(X,Y,Z), G(X,Y,Z) \in \mathbb{k}[X,Y,Z]$, $f(X) \in \mathbb{k}[X]$ and $g(Y) \in \mathbb{k}[Y]$. By (39), we may arrange that $\deg_Z(F) < n$ and $\deg_Z(G) < n$. Then $g(y)F(x,y,z) = f(x)G(x,y,z)$ and the uniqueness claim in (39) implies that $g(Y)F(X,Y,Z) = f(X)G(X,Y,Z)$ in $\mathbb{k}[X,Y,Z]$. So $f \mid F$ in $\mathbb{k}[X,Y,Z]$, i.e., $F = fQ$ where $Q \in \mathbb{k}[X,Y,Z]$; thus $\beta = Q(x,y,z) \in B$ and (43) is true. As $\mathbb{k}(x)[z] = \mathbb{k}(x)^{[1]}$ and $\mathbb{k}(y)[z] = \mathbb{k}(y)^{[1]}$ are normal, so is their intersection B. Assertion (1) is proved.

It is easy to see that the $\mathbb{k}$-derivation $x\frac{\partial}{\partial z}$ of the field $\mathbb{k}(x,z) = \operatorname{Frac}(B)$ maps B into itself; let $D : B \to B$ be the restriction, then clearly D is locally nilpotent and is the only $\mathbb{k}$-derivation of B which maps x to 0 and z to x. We have $\mathbb{k}[x] \leq \ker D \leq B$; consideration of transcendence degrees shows that $\ker D$ is algebraic over $\mathbb{k}[x]$, so $\ker D = \mathbb{k}[x]$ by (42). Let us now prove:

$$\operatorname{LND}_{\mathbb{k}[x]}(B) = \{\, \alpha D \mid \alpha \in \mathbb{k}[x] \,\}. \tag{44}$$

Consider a nonzero element D' of $\operatorname{LND}_{\mathbb{k}[x]}(B)$ and note that $\ker D' = \mathbb{k}[x]$. Exercise 2.12 implies that $D'(z) \in \mathbb{k}[x] \setminus \{0\}$ and that $D(z)D' = D'(z)D$, so

$$xD' = D'(z)D. \tag{45}$$

By (40), we may write $D'y = \sum_{i<n} a_i z^i$ where $a_i \in \mathbb{k}[x,y]$, so

$$\textstyle\sum_{i<n} (xa_i) z^i = xD'(y) = D'(z)D(y) = D'(z)\varphi'(z). \tag{46}$$

As $D'z \in \mathbb{k}[x]$, (46) and (40) imply that $\forall_i\ xa_i = \lambda_i D'(z)$, where the $\lambda_i \in \mathbb{k}$ are defined by $\varphi'(Z) = \sum_{i<n} \lambda_i Z^i$ (and hence are not all zero). So $D'z \in \mathbb{k}[x] \cap x\mathbb{k}[x,y] = x\mathbb{k}[x]$, i.e., $D'z = x\alpha$ for some $\alpha \in \mathbb{k}[x]$. Thus (45) gives $D' = \alpha D$, which proves (44). It follows that D is irreducible. Indeed, 2.19 implies that $D = \alpha_0 D_0$ for some $\alpha_0 \in \mathbb{k}[x]$ and some irreducible $D_0 \in \operatorname{LND}_{\mathbb{k}[x]}(B)$; then (44) gives $D_0 = \alpha D$ for some $\alpha \in \mathbb{k}[x]$, so $D = \alpha_0 \alpha D$ and hence $\alpha, \alpha_0 \in \mathbb{k}^*$. So D is irreducible and assertion (3) is proved. To prove assertion (4), consider:

$$\begin{tikzcd}
\mathbb{k}[Z] \arrow[r, "u"] \arrow[d, "v"', "\cong"] & \mathbb{k}[X,Y,Z] \arrow[d, "\pi"] & \\
\mathbb{k}[z] \arrow[r, "f"] & \mathbb{k}[X,Y,Z]/(XY - \varphi(Z)) \arrow[r, "g"] & \mathbb{k}[X,Y,Z]/(X, XY - \varphi(Z))
\end{tikzcd}$$

Then it is clear that

$$\ker(g \circ \pi \circ u) = \mathbb{k}[Z] \cap (X, XY - \varphi(Z)) = \mathbb{k}[Z] \cap (X, \varphi(Z)) = \varphi(Z)\mathbb{k}[Z] \tag{47}$$

and $\ker(g \circ f) = \varphi(z)\Bbbk[z]$ follows by applying v to (47). As

$$I = \ker\big(\Bbbk[z] \hookrightarrow B \to B/xB\big) = \ker(g \circ f),$$

we conclude that $I = \varphi(z)\Bbbk[z] = xy\Bbbk[z]$. $\square$

Exercise 9.1. Suppose that B is a Danielewski surface over $\Bbbk$ and that (x, y, z) is a coordinate system of B. Show:

(1) (y, x, z) is a coordinate system of B; give an example where (z, x, y) is not a coordinate system of B. Show that, for any $\alpha, \beta, \gamma \in \Bbbk^*$, $(\alpha x, \beta y, \gamma z)$ is a coordinate system of B.

(2) Any two elements of $\{x, y, z\}$ are algebraically independent over $\Bbbk$.

(3) $\Bbbk[x], \Bbbk[y] \in \text{KLND}(B)$ and $\Bbbk[x] \cap \Bbbk[y] = \Bbbk$.

9.4. **Lemma.** *Let B be a Danielewski surface over $\Bbbk$ and let (x, y, z) and (x', y', z') be two coordinate systems of B. If $\Bbbk[x] = \Bbbk[x']$ and $\Bbbk[z] = \Bbbk[z']$, then $\Bbbk[y] = \Bbbk[y']$.*

Proof. Since (x', y', z') is a coordinate system and $\Bbbk[z] = \Bbbk[z']$, it follows that (x', y', z) is a coordinate system. By 9.3, we may consider $D, D' \in \text{LND}(B)$ such that $Dz = x$, $D'z = x'$ and $\ker D = \Bbbk[x] = \ker D'$; since D' is irreducible and $\text{LND}_{\Bbbk[x]}(B) = \{\alpha D \mid \alpha \in \Bbbk[x]\}$, we have $D' = \lambda D$ for some $\lambda \in \Bbbk^*$. Then $x' = D'z = \lambda Dz = \lambda x$ and consequenty (x, y', z) is a coordinate system. Applying part (4) of 9.3 to each of (x, y, z), (x, y', z) shows that each of xy, xy' is a generator of the principal ideal I of $\Bbbk[z]$ defined by $I = \Bbbk[z] \cap xB$. Thus $xy = \mu xy'$ for some $\mu \in \Bbbk^*$, so $y = \mu y'$ and consequently $\Bbbk[y] = \Bbbk[y']$. $\square$

Tame automorphisms of Danielewski surfaces

9.5. Fix a coordinate system $\gamma = (x, y, z)$ of a Danielewski surface B over $\Bbbk$.

9.5.1. **Definition.**

- Define $\tau \in \text{Aut}_\Bbbk(B)$ by $\tau(x) = y$, $\tau(y) = x$ and $\tau(z) = z$.
- For each $f \in \Bbbk[x]$, define $\Delta_f \in \text{Aut}_\Bbbk(B)$ by $\Delta_f(x) = x$ and $\Delta_f(z) = z + xf(x)$.
- Let G_γ be the subgroup of $\text{Aut}_\Bbbk(B)$ generated by $\{\tau\} \cup \{\Delta_f \mid f \in \Bbbk[x]\}$.

We call G_γ the *tame subgroup* of $\text{Aut}_\Bbbk(B)$.

The assignment $(\alpha, A) \longmapsto \alpha(A)$, where $\alpha \in \text{Aut}_\Bbbk(B)$ and $A \in \text{KLND}(B)$, is a left-action of the group $\text{Aut}_\Bbbk(B)$ on the set $\text{KLND}(B)$. We restrict this action to the subgroup G_γ of $\text{Aut}_\Bbbk(B)$, then the main result of [7] is:

9.5.2. **Transitivity Theorem.** *The action of G_γ on $\text{KLND}(B)$ is transitive.*

As a corollary to the Transitivity Theorem, we obtain the following generalization of Rentschler's Theorem (recall that $\Bbbk^{[2]}$ is a special case of Danielewski surface):

9.5.3. **Corollary.** *Given any $D' \in \text{LND}(B)$, there exists $\theta \in G_\gamma$ such that $\theta \circ D \circ \theta^{-1} = f(x)D$ for some $f(x) \in \Bbbk[x]$, where D is the unique element of $\text{LND}(B)$ satisfying $Dx = 0$ and $D(z) = x$.*

Isomorphisms between Danielewski surfaces

Although we don't need it for our purpose, we mention the following fact (see for instance 2.10 of [11]).

9.6. **Proposition.** *Let $\varphi, \psi \in \Bbbk[Z] \setminus \Bbbk$ and consider the Danielewski surfaces (over $\Bbbk$):*

$$B = \Bbbk[X,Y,Z]/(XY - \varphi(Z)) \qquad \text{and} \qquad B' = \Bbbk[X,Y,Z]/(XY - \psi(Z)).$$

Then B is $\Bbbk$-isomorphic to B' if and only if there exist $\theta \in \mathrm{Aut}_{\Bbbk}(\Bbbk[Z])$ and $\lambda \in \Bbbk^*$ such that $\psi = \lambda\theta(\varphi)$.

Two characterizations of Danielewski surfaces

The following results are Theorems 2.5 and 2.6 of [11].

9.7. **Theorem.** *Let B be a domain containing a field $\Bbbk$ of characteristic zero, let $z \in B$ and let $D_1, D_2 \in \mathrm{LND}(B)$. Suppose that (z, D_1, D_2) satisfies:*

(i) $\ker D_1 \neq \ker D_2$
(ii) *For each $i = 1, 2$, $\ker D_i = \Bbbk^{[1]}$ and $D_i(z) \in \ker D_i \setminus \{0\}$.*

Then B is a Danielewski surface over $\Bbbk$. Moreover, if D_1, D_2 are irreducible then one of the following holds:

(1) *$B = \Bbbk^{[2]}$ and $D_1(z), D_2(z) \in \Bbbk^*$*
(2) *$B \neq \Bbbk^{[2]}$ and $(D_1(z), D_2(z), z)$ is a coordinate system of B.*

9.8. **Theorem.** *Let B be a UFD containing a field $\Bbbk$ of characteristic zero. Suppose that $D \in \mathrm{LND}(B)$ and $z \in B$ satisfy:*

$$\ker D = \Bbbk[Dz] = \Bbbk^{[1]}.$$

Then B is a Danielewski surface over $\Bbbk$ and the following hold:

(1) *If D is irreducible then there exists $y \in B$ such that (Dz, y, z) is a coordinate system of B.*
(2) *If D is not irreducible then $B = \Bbbk[z, Dz] = \Bbbk^{[2]}$.*

Two lemmas on localization

These facts have nothing to do with Danielewski surfaces, but we need them for the discussion of local slice construction.

9.9. **Notation.** Given integral domains $R \leq B$, we write $B_R = S^{-1}B$ where $S = R \setminus \{0\}$ (so $R_R = \mathrm{Frac}\, R$). If $D : B \to B$ is a derivation, we also write $D_R = S^{-1}D : B_R \to B_R$.

9.10. **Lemma.** *Let $R \leq B$ be domains, where B is finitely generated as an R-algebra. Then $A \mapsto A_R$ is a bijection $\mathrm{KLND}_R(B) \to \mathrm{KLND}(B_R)$, with inverse $\mathcal{A} \mapsto \mathcal{A} \cap B$.*

Proof. Given $A \in \mathrm{KLND}_R(B)$, choose $D \in \mathrm{LND}_R(B) \setminus \{0\}$ such that $\ker D = A$. By exercise 2.1, $D_R : B_R \to B_R$ is locally nilpotent and has kernel A_R. Since B is a domain, D_R is an extension of D; this implies that $D_R \neq 0$ (so $A_R \in \mathrm{KLND}(B_R)$) and $A = \ker D =$

$B \cap \ker D_R = B \cap A_R$, showing that $\Lambda : \text{KLND}_R(B) \to \text{KLND}(B_R)$ $(A \mapsto A_R)$ is well-defined and injective. To show that Λ is surjective, consider $\mathcal{A} \in \text{KLND}(B_R)$. Choose $\mathcal{D} \in \text{LND}(B_R) \setminus \{0\}$ such that $\ker \mathcal{D} = \mathcal{A}$; note that $\mathcal{D}$ is an R_R-derivation, because R_R is a field contained in B_R (see 2.15). By assumption, we have $B = R[b_1, \dots, b_n]$ for some $b_1, \dots, b_n \in B$. For each $i \in \{1, \dots, n\}$, we have $\mathcal{D}(b_i) \in B_R$; so there exists $r \in R \setminus \{0\}$ satisfying $\forall_i \; r\mathcal{D}(b_i) \in B$. Since the derivation $r\mathcal{D} : B_R \to B_R$ maps R to 0 and maps each b_i in B, it maps B into itself; also, $r\mathcal{D}$ is locally nilpotent, since $r \in \ker \mathcal{D}$. Let $D : B \to B$ be the restriction of $r\mathcal{D}$, then $D \in \text{LND}_R(B)$ and $\ker D = A$, where we define $A = B \cap \mathcal{A}$. Since D has a *unique* extension to a derivation of B_R, we have $D_R = r\mathcal{D}$; in particular $D_R \neq 0$, so $D \neq 0$ and $A \in \text{KLND}_R(B)$; by exercise 2.1 the kernel of D_R is A_R, so we obtain $\mathcal{A} = A_R = \Lambda(A)$. So Λ is surjective. □

9.11. **Lemma.** *Let B be a UFD, R a factorially closed subring of B and $D : B \to B$ an irreducible R-derivation. Then $D_R : B_R \to B_R$ is irreducible.*

Proof. Assume the contrary; then there exists $b \in B_R \setminus {B_R}^*$ such that $D_R(B_R) \subseteq bB_R$. In fact, such an element b may be chosen in B. Then some prime factor $p \in B$ of b satisfies $p \notin {B_R}^*$. Since D is irreducible and $p \notin B^*$, we may choose $x \in B$ such that $Dx \notin pB$. Since $D(x) = D_R(x) \in pB_R$, there exists $r \in R \setminus \{0\}$ such that $p \mid rD(x)$ in B. Then $p \mid r$ in B; since $r \in R \setminus \{0\}$ and R is an factorially closed subring of B, $p \in R \setminus \{0\}$. Thus $p \in {B_R}^*$, a contradiction. □

Local slice construction revisited

Adopting the viewpoint of Danielewski surfaces allows us to clarify and generalize the notion of "local slice construction". We do this in two steps. The first approach (9.12) closely follows Freudenburg's method in the case $B = \mathbb{k}^{[3]}$ but only gives a partial clarification; the second approach (9.15) is simpler and more general.

Let $\mathbb{k}$ be any field of characteristic zero.

9.12. Let B be a $\mathbb{k}$-affine UFD. Suppose that (A, R, w) satisfies:

(i) $A \in \text{KLND}(B)$

(ii) R is a $\mathbb{k}$-subalgebra of A such that $A_R = K^{[1]}$ (where $K = R_R$)

(iii) $w \in B$ satisfies $A_R = K[Dw]$, where $D : B \to B$ is the unique[2] irreducible derivation with kernel A (thus $D \in \text{LND}(B)$ and $D \neq 0$; see 2.20).

Then (A, R, w) determines an element A' of $\text{KLND}(B)$ which we now proceed to define. We say that A' is obtained from (A, R, w) by "local slice construction", and we write $A' = \text{LSC}(A, R, w)$.

9.12.1. **Proposition and definition.** *Let B, (A, R, w), K and D be as in* 9.12. *Then B_R is a Danielewski surface over K and there exists $v \in B$ such that (Dw, v, w) is a coordinate system of B_R. More generally, consider any $(u, v) \in B \times B$ satisfying*

(48) $$(u, v, w) \text{ is a coordinate system of } B_R \text{ and } A_R = K[u].$$

[2] D is unique up to multiplication by an element of B^*.

Then the following hold:

(1) *The ring $K[v]$ is independent of the choice of (u,v) satisfying* (48).

(2) *The ring $K[v] \cap B$ belongs to $\text{KLND}_R(B)$.*

We define $\text{LSC}(A,R,w) = K[v] \cap B$.

Remark. $R[v] \leq \text{LSC}(A,R,w) \leq K[v]$, so $\text{LSC}(A,R,w)$ is the unique element of $\text{KLND}(B)$ which contains $R[v]$.

Proof of 9.12.1. By exercise 2.1, $D_R : B_R \to B_R$ is locally nilpotent and has kernel $A_R = K^{[1]}$. Since $(B_R)^* = (A_R)^* = K^*$, it follows that the ring $R' = B \cap K$ is factorially closed in B; thus $D_{R'} : B_{R'} \to B_{R'}$ is irreducible by 9.11. As $R \leq R' \leq K$, we have $B_R = B_{R'}$ and $D_R = D_{R'}$, so D_R is irreducible. It follows from 9.8 that B_R is a Danielewski surface over K and that, for some $v \in B_R$, (Dw, v, w) is a coordinate system of B_R. Multiplying v by a suitable element of $R \setminus \{0\}$, we may arrange that $v \in B$; then the pair $(Dw, v) \in B \times B$ satisfies (48).

Assertion (1) is an immediate consequence of 9.4. If (u,v) satisfies (48) then (u,v,w) is a coordinate system of B_R so exercise 9.1 implies that $K[v] \in \text{KLND}(B_R)$; then $K[v] \cap B \in \text{KLND}_R(B)$ by 9.10. $\square$

9.13. **Example.** We revisit Freudenburg's "$(2,5)$-example". Let $B = \mathbb{k}[X,Y,Z]$ and $f = XZ - Y^2$, then $A = \mathbb{k}[X,f] \in \text{KLND}(B)$. To perform a LSC on A, we define:

$$R = \mathbb{k}[f] \qquad \text{and} \qquad w = X^3 + Yf.$$

We claim that (A,R,w) satisfy conditions (i)–(iii) of 9.12. Indeed, the unique irreducible $D \in \text{Der}(B)$ with kernel A is $D = \Delta_{(X,f)} = -X\frac{\partial}{\partial Y} - 2Y\frac{\partial}{\partial Z}$, so $Dw = D(X^3 + Yf) = fDY = -Xf$. Writing $K = \text{Frac}\, R = \mathbb{k}(f)$, we have

$$K[Dw] = \mathbb{k}(f)[-Xf] = \mathbb{k}(f)[X] = A_R,$$

so conditions (i)–(iii) hold.

By 9.12.1, it follows that $B_R = \mathbb{k}(f)[X,Y,Z]$ is a Danielewski surface over $\mathbb{k}(f)$ and that, for a suitable $v \in B$, $(-Xf, v, w)$ is a coordinate system of B_R; consequently, (X,v,w) is a coordinate system of B_R so (X,v) satisfies condition (48). To compute v, we consider the equation $Xv = \varphi(w)$, where $\varphi(T) \in K[T]$ has positive T-degree. Replacing v and $\varphi(T)$ by rv and $r\varphi(T)$ respectively, where $r \in R \setminus \{0\}$, we may assume that $\varphi(T) \in R[T]$. In other words, we seek an irreducible $\Phi(S,T) \in \mathbb{k}[S,T]$ satisfying:

$$Xv = \Phi(f,w), \quad \deg_T \Phi > 0.$$

Following Freudenburg's technique we set $X = 0$ and find $\Phi(-Y^2, -Y^3) = 0$, from which we find $\Phi = S^3 + T^2$. So $Xv = f^3 + w^2$ and hence

$$v = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z.$$

Then $\text{LSC}(A,R,w)$ is the unique element of $\text{KLND}(B)$ containing $R[v] = \mathbb{k}[f,v]$; one can see that $\text{LSC}(A,R,w) = \mathbb{k}[f,v]$.

Remark. In the above example, we know that B_R is a Danielewski surface over $\mathbb{k}(f)$, that (X, v, w) is a coordinate system of B_R and that $Xv = f^3 + w^2$; consequently B_R is the Danielewski surface $\mathbb{k}(f)[X_1, X_2, X_3]/(X_1X_2 - X_3^2 - f^3)$.

9.14. **Example.** Let $B = \mathbb{k}[T_1, T_2, X, Y, Z] = \mathbb{k}^{[5]}$ and consider $\Delta \in \text{LND}(B)$ defined by

$$\Delta(T_1) = 0 = \Delta(T_2), \quad \Delta(X) = T_1, \quad \Delta(Y) = T_2, \quad \Delta(Z) = 1 + L, \quad \text{where } L = T_2X - T_1Y.$$

The derivation Δ was studied by Winkelman in [18]. We show that it can be obtained from $\partial/\partial Z$ by performing one LSC.

Let $D = \partial/\partial Z$, $A = \ker(D) = \mathbb{k}[T_1, T_2, X, Y]$, $R = \mathbb{k}[T_1, T_2, L] \leq A$ and $w = X(T_1Z - X(1 + L))$. Then $Dw = XT_1$. Writing $K = \operatorname{Frac} R = \mathbb{k}(T_1, T_2, L)$, we have $A_R = K[X, Y] = K[X] = K[XT_1] = K[Dw]$ so (A, R, w) satisfies conditions (i)-(iii) of 9.12. By 9.12.1, B_R is a Danielewski surface over K and, for a suitable $v \in B$, (XT_1, v, w) is a coordinate system of B_R; thus (X, v, w) is a coordinate system of B_R.

We seek v. Note that $Y \in K[X, Z]$, so $B_R = K[X, Z] = K^{[2]}$ (which is a Danielewski surface). As $B_R = K[X, Z] = K[X,\ T_1Z - X(1 + L)] = K[X, v]$, where we write $v = T_1Z - X(1 + L)$, it follows that $(X, v, Xv) = (X, v, w)$ is a coordinate system of the Danielewski surface B_R. Thus (X, v) satisfies (48). Consequently $\text{LSC}(A, R, w)$ is the unique element of $\text{KLND}(B)$ which contains $R[v] = \mathbb{k}[T_1, T_2, L, v]$. As $\mathbb{k}[T_1, T_2, L, v] \leq \ker \Delta$ is clear, we have $\text{LSC}(A, R, w) = \ker \Delta$.

Paragraph 9.15 reformulates the notion of local slice construction in such a way that the concept is now completely transparent (but the practical calculations are the same as before).

9.15. Suppose that $A \in \text{KLND}(B)$, where B is any domain of characteristic zero. To perform a LSC on A,

(1) Choose a subring $R \leq A$ such that B_R is a Danielewski surface over $K = R_R$ and B is finitely generated as an R-algebra.
(2) Choose a coordinate system (u, v, w) of B_R such that $A_R = K[u]$.
(3) Define $A' = K[v] \cap B$ and declare that A' is obtained from A by performing a local slice construction.

Comments.

- In step (1), there may not exist a ring R with the desired properties; in that case, it is impossible to perform a LSC on A. Assuming that such rings R exist, finding one may be difficult in practice. Note that the same difficulty exists in the approach of 9.12, i.e., one has to "find" a triple (A, R, w) in order to perform a LSC.
- Once we have a ring R as in step (1), 9.10 implies that A_R belongs to $\text{KLND}(B_R)$; then the theory of Danielewski surfaces implies that there exist infinitely many coordinate systems (u, v, w) of B_R satisfying $A_R = K[u]$ as in step (2).
- Result 9.10 also implies that, in step (3), $A' \in \text{KLND}(B)$ and $A' \neq A$.
- It is clear that if A' can be obtained from A by performing a local slice construction, then A can be obtained from A' by performing a local slice construction.

Remark. Of course one could further generalize the LSC by replacing, in 9.15, the class of Danielewski surfaces by any other class of rings for which we understand the locally nilpotent derivations. However:

- We don't know which class of rings would give a useful theory.
- The class of Danielewski surfaces seems to be "the right choice", and perhaps the only natural choice, if our aim is to understand the ring $B = \mathbb{k}^{[3]}$. Indeed, the study of homogeneous locally nilpotent derivations of $\mathbb{k}^{[3]}$ leads naturally to that class of rings, because the geometric modification of affine rulings turns out to be nothing else than LSC (see part (3) of 8.14). So the arbitrariness character of the LSC disappears when we consider the homogeneous theory of $\mathbb{k}^{[3]}$.

9.16. **Definition.** Given a domain B of characteristic zero, define the graph $\underline{\text{KLND}}(B)$ whose vertex-set is $\text{KLND}(B)$ and in which distinct vertices $A, A' \in \text{KLND}(B)$ are joined by an edge if one can be obtained from the other by LSC (defined as in 9.15).

More precisely, $\underline{\text{KLND}}(B)$ is a non-oriented graph such that there is at most one edge between any two vertices, and where no edge connects a vertex to itself. Note that there is a natural action of $\text{Aut}_{\mathbb{k}}(B)$ on $\underline{\text{KLND}}(B)$.

9.17. **Example.** If B is a Danielewski surface over some field $\mathbb{k}$ of characteristic zero then $\underline{\text{KLND}}(B)$ is a connected graph with $|\mathbb{k}|$ vertices. It is a tree if and only if $\deg \varphi \geq 3$, where $B = \mathbb{k}[X, Y, Z]/(XY - \varphi(Z))$.

9.18. **Corollary.** *Let $B = \mathbb{k}^{[3]}$ where $\mathbb{k}$ is a field of characteristic zero and consider two elements $\mathbb{k}[f, g]$ and $\mathbb{k}[f, h]$ of $\text{KLND}(B)$, where $\mathbb{k}[f, h]$ is obtained from $\mathbb{k}[f, g]$ by LSC (or vice-versa). Then $\text{KLND}_{\mathbb{k}[f]}(B)$ contains $|\mathbb{k}|$ elements and any two of them are related by a sequence of LSCs.*

Proof. By definition 9.15 of the LSC, there exists a ring $R \leq \mathbb{k}[f, g] \cap \mathbb{k}[f, h]$ such that B_R is a Danielewski surface over R_R. It is easy to see that $R = \mathbb{k}[f]$ has the desired property. Note that the bijection $\text{KLND}(B_R) \to \text{KLND}_R(B)$ of 9.10 preserves edges, when regarded as a map from $\underline{\text{KLND}}(B_R)$ to $\underline{\text{KLND}}(B)$; thus the assertion follows from 9.17. $\square$

9.19. **Example.** Let B be a domain of transcendence degree 2 over a field $\mathbb{k}$ of characteristic zero. Suppose that $\text{ML}(B) = \mathbb{k}$ (where $\text{ML}(B)$ is the intersection of $\ker(D)$ for all $D \in \text{LND}(B)$) and that B is not a Danielewski surface over $\mathbb{k}$. Then $\underline{\text{KLND}}(B)$ is a graph with $|\mathbb{k}|$ vertices and no edges.

10. Polynomials $f(X, Y, Z)$ whose generic fiber is a Danielewski surface

The graph $\underline{\text{KLND}}(B)$ is an invariant of the ring B and, presumably, can be used for investigating the structure of B. However 9.19 shows that, for certain rings, it is totally useless to consider that graph. In the case $B = \mathbb{k}^{[3]}$, it seems that $\underline{\text{KLND}}(B)$ contains just the right amount of edges to be interesting.

From now-on, let $B = \mathbb{k}^{[3]}$ where $\mathbb{k}$ is a field of characteristic zero. The main question is:

Question 1. What is the structure of $\underline{\text{KLND}}(B)$?

Of course, this is very difficult. A particularly intriguing aspect of question 1 is:

Question 2. Which subalgebras R of B satisfy: B_R is a Danielewski surface over R_R?

Exercise 10.1. If R is a subalgebra of B such that B_R is a Danielewski surface over R_R, then so is $R' = B \cap \operatorname{Frac}(R)$. Moreover, $B_{R'} = B_R$ and R' is factorially closed in B.

In view of this exercise, there is no loss of generality if we restrict question 2 to rings R which are factorially closed in B. In other words, question 2 should be replaced by:

Question 3. Which subalgebras R of B satisfy:

$(*)$ *B_R is a Danielewski surface over R_R and R is factorially closed in B.*

We shall now discuss question 3.

10.1. **Lemma.** *If R is a subalgebra of B satisfying $(*)$ then $R = \mathbb{k}^{[1]}$.*

Proof. Since B_R is a Danielewski surface over R_R we have $\operatorname{trdeg}_R B = 2$, so $\operatorname{trdeg}_{\mathbb{k}} R = 1$. We also have $|\operatorname{KLND}(B_R)| > 1$, so $|\operatorname{KLND}_R(B)| > 1$ by 9.10. Pick distinct $A, A' \in \operatorname{KLND}_R(B)$. By a result in Freudenburg's lectures, $A \cap A'$ is either $\mathbb{k}$ or $\mathbb{k}^{[1]}$; since $A \cap A' \supseteq R$ and $\operatorname{trdeg}_{\mathbb{k}} R = 1$, we have $A \cap A' = \mathbb{k}^{[1]}$ and $A \cap A'$ is algebraic over R. As R is factorially closed in B by assumption, it is algebraically closed in B and hence $R = A \cap A' = \mathbb{k}^{[1]}$. $\square$

10.2. **Definition.** Let $f \in B = \mathbb{k}[X,Y,Z]$ and $R = \mathbb{k}[f]$. The $\mathbb{k}(f)$-algebra $B_R = \mathbb{k}(f)[X,Y,Z]$ is called the *generic fiber* of f. If B_R is a Danielewski surface over $R_R = \mathbb{k}(f)$, we call f a polynomial "whose generic fiber is a Danielewski surface".

10.3. **Lemma.** *If $f \in B$ is a polynomial whose generic fiber is a Danielewski surface then $\mathbb{k}[f]$ is factorially closed in B.*

Proof. The fact that $\mathbb{k}(f)[X,Y,Z]$ is a Danielewski surface over $\mathbb{k}(f)$ implies that

$$\mathbb{k}(f)[X,Y,Z]^* = \mathbb{k}(f)^*$$

and it follows that $R = \mathbb{k}(f) \cap B$ is factorially closed in B; as $B_R = \mathbb{k}(f)[X,Y,Z]$ is clear, we obtain that R satisfies $(*)$. By 10.1, it follows that $R = \mathbb{k}[g]$ for some $g \in B$, so we have $\mathbb{k}[f] \subseteq \mathbb{k}[g]$ and $\mathbb{k}(f) = \mathbb{k}(g)$. Consequently, $\mathbb{k}[f] = \mathbb{k}[g]$ and hence $\mathbb{k}[f]$ is factorially closed in B. $\square$

Combining 10.1 and 10.3 gives:

10.4. **Corollary.** *The rings R which answer question 3 are exactly the $\mathbb{k}[f]$ where $f \in B$ is a polynomial whose generic fiber is a Danielewski surface.*

Note that 10.4 replaces question 3 by

Question 4. Describe the class (call it "$\mathcal{C}$") of polynomials $f \in B$ whose generic fiber is a Danielewski surface.

Let us make a few comments concerning the class $\mathcal{C}$.

(1) If the local slice construction turns out to be significant in the study of $\mathbb{k}^{[3]}$ (and at this time it seems to be an interesting idea) then the above facts suggest that the class $\mathcal{C}$ should also play a significant rôle.

(2) The class $\mathcal{C}$ contains in particular all variables: If f is a variable of B then $\mathbb{k}(f)[X,Y,Z] = \mathbb{k}(f)^{[2]}$ is a Danielewski surface, so $f \in \mathcal{C}$. Also note the converse: If $f \in B$ satisfies $\mathbb{k}(f)[X,Y,Z] = \mathbb{k}(f)^{[2]}$ then a result of Kaliman [13] implies that f is a variable of B.

(3) The polynomials $\{H_n\}_{n=1}^{\infty}$ all belong to $\mathcal{C}$ (these are the standard-homogeneous polynomials of degrees $1, 2, 5, 13, 34, \dots$ which were defined inductively in one of Freudenburg's lectures). If we assume that $\mathbb{k}$ is algebraically closed then, as a corollary to [8], one can show that the H_n are the only[3] standard-homogeneous elements of $\mathcal{C}$. The same list of polynomials has arisen in the work of several researchers (for instance Kashiwara or Gizatullin) investigating problems which have apparently nothing to do with the LSC. Also note that the zero-set of H_3 in $\mathbb{P}^2$ is Yoshihara's quintic [20].

(4) The elements of $\mathcal{C}$ which are homogeneous with respect to some positive weights $\omega = (a_0, a_1, a_2)$ are partially understood: See the discussion of Gizatullin curves below. However many elements of $\mathcal{C}$ are not homogeneous (with respect to *positive* weights). For instance, if $\varphi(Z) \in \mathbb{k}[Z]$ is any nonconstant polynomial then $XY - \varphi(Z)$ is a member of $\mathcal{C}$ (these are among the simplest members of $\mathcal{C}$).

(5) Suppose that $f \in \mathcal{C}$. Then it is not difficult to see that the *general* fiber of f is a Danielewski surface, i.e., $\mathbb{k}[X,Y,Z]/(f-\lambda)$ is a Danielewski surface for almost all $\lambda \in \mathbb{k}$. However, in most cases there does not exist an automorphism of $\mathbb{k}[X,Y,Z]$ which maps $f - \lambda$ to a polynomial of the form $XY - \varphi(Z)$. In other words, this gives Danielewski surfaces which are embedded in $\mathbb{A}^3$ in non-standard ways (this is the case for $H_n - \lambda$ when $n \geq 3$ and $\lambda \neq 0$).

From now-on, assume that $\mathbb{k}$ is an algebraically closed field of characteristic zero and let $\omega = (a_0, a_1, a_2)$, where a_0, a_1, a_2 are pairwise relatively prime positive integers. Let $B = \mathbb{k}^{[3]}$ and consider (B, ω) and $\mathbb{P}_\omega$ as in section 8.

10.5. **Definition.** Let us say that a curve C in $\mathbb{P}_\omega$ is a *Gizatullin curve* if it is irreducible, rational and such that:

($\star$) *The affine surface $\mathbb{P}_\omega \setminus C$ is completable by a zig-zag.*

10.6. **Theorem.** *Consider an irreducible $f \in B$ which is ω-homogeneous. Then tfae:*

(1) *$V(f) \subset \mathbb{P}_\omega$ is a Gizatullin curve.*

(2) *$f \in \mathcal{C}$, i.e., the generic fiber of f is a Danielewski surface.*

This result is equivalent to Proposition 7.3 of [8]. That paper also explains how to construct all Gizatullin curves of $\mathbb{P}_\omega$, for any ω. In this sense we can say that the ω-homogeneous elements of $\mathcal{C}$ are (at least partially) understood.

[3]Up to a linear automorphism of $\mathbb{k}[X,Y,Z]$.

References

1. J. Berson, *Derivations on polynomial rings over a domain*, MSc thesis, University of Nijmegen, The Netherlands, 1999.
2. S.M. Bhatwadekar and A.K. Dutta, *Kernel of Locally Nilpotent R-Derivations of* $R[X,Y]$, Trans. Amer. Math. Soc. **349** (1997), 3303–3319.
3. D. Daigle, *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra **114** (1997), 221–230.
4. ______, *Homogeneous locally nilpotent derivations of* $k[x,y,z]$, J. Pure Appl. Algebra **128** (1998), 109–132.
5. ______, *On kernels of homogeneous locally nilpotent derivations of* $k[x,y,z]$, Osaka J. Math. **37** (2000), 689–699.
6. ______, *Classification of homogeneous locally nilpotent derivations of* $\mathbf{k}[x,y,z]$, preprint, 2003.
7. ______, *On locally nilpotent derivations of* $k[X_1,X_2,Y]/(\varphi(Y)-X_1X_2)$, J. Pure and Appl. Algebra **181** (2003), 181–208.
8. D. Daigle and P. Russell, *On log $\mathbb{Q}$-homology planes and weighted projective planes*, to appear in Canadian J. of Math.
9. ______, *Affine rulings of normal rational surfaces*, Osaka J. Math. **38** (2001), 37–100.
10. ______, *On weighted projective planes and their affine rulings*, Osaka J. Math. **38** (2001), 101–150.
11. D.Daigle, *Locally nilpotent derivations and Danielewski surfaces*, to appear in Osaka J. of Math.
12. D. Holtackers, *On kernels of w-homogeneous derivations*, MSc thesis, University of Nijmegen, The Netherlands, 2003.
13. S. Kaliman, *Polynomials with general $\mathbb{C}^2$-fibers are variables*, Pacific J. of Math. **203** (2002), 161–190.
14. M. Miyanishi, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories—to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, pp. 37–51.
15. K.P. Russell and A. Sathaye, *On finding and cancelling variables in* $\mathbf{k}[x,y,z]$, J. of Algebra **57** (1979), 151–166.
16. P. Eakin S.S. Abhyankar and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.
17. A. van den Essen and P. van Rossum, *Coordinates in two variables over a $\mathbb{Q}$-algebra*, Report 0033, Department of Mathematics, University of Nijmegen, The Netherlands, December 2000.
18. J. Winkelmann, *On free holomorphic C-actions on C^n and homogeneous Stein manifolds*, Math. Ann. **286** (1990), 593–612.
19. D. Wright, *On the jacobian conjecture*, Illinois J. of Math. **25** (1981), 423–440.
20. H. Yoshihara, *On Plane Rational Curves*, Proc. Japan Acad. (Ser. A) **55** (1979), 152–155.
21. V.D. Zurkowski, *Locally finite derivations*, To appear in Rocky Mount. J. of Math.