THE UNIVERSAL EULER CHARACTERISTIC FOR VARIETIES OF CHARACTERISTIC ZERO

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ABSTRACT. Using the weak factorization theorem we give a simple presentation for the value group of the universal Euler characteristic with compact support for varieties of characteristic zero and describe the value group of the universal Euler characteristic of pairs. This gives a new proof for the existence of natural Euler characteristics with values in the Grothendieck group of Chow motives. A generalization of the presentation to the relative setting allows us to define duality and the six operations.

1. INTRODUCTION

Let k be a field of characteristic zero. For an abelian group A, an *A-valued Euler characteristic with compact support* for varieties over k assigns to every such variety X an element $e(X) \in A$ only depending on the isomorphism class of X such that $e(X) = e(X - Y) + e(Y)$ for $Y \subset X$ a closed subvariety. There is an evident universal Euler characteristic for varieties. It takes values in the *naive Grothendieck group of varieties over k*, denoted by $\mathrm{K}_0(Var_k)$, which is the free abelian group on isomorphism classes $[X]$ of varieties X over k modulo the relations $[X] = [X-Y]+[Y]$ for $Y \subset X$ a closed subvariety. It can be given a ring structure by taking products of varieties. This group and its relative and equivariant analogues (to be more precise, certain localizations and completions) are the natural value groups for motivic integrals, as explained e.g. by Looijenga in [6] and by Denef and Loeser in [3].

In this note we give a simple presentation of this group in terms of smooth projective varieties. In fact we show that $\mathrm{K}_0(Var_k)$ is the free abelian group on isomorphism classes of smooth projective varieties modulo the relations $[\emptyset] = 0$ and $[X] - [Y] = [\mathrm{Bl}_Y X] - [E]$, where $Y \subset X$ is a smooth closed subvariety, $\mathrm{Bl}_Y X$ denotes the blow-up of X along Y and E denotes the exceptional divisor of this blow-up. The key ingredient is the factorization theorem proven by Włodarczyk in [8] and Abramovich, Karu, Matsuki and Włodarczyk in [1]: Any birational map between smooth complete irreducible varieties over k can be factored into a sequence of blow-ups and blow-downs with smooth centers. It follows that there is a unique Euler characteristic with compact support for k-varieties with values in the Grothendieck group of Chow motives over k which assigns to a smooth projective variety the class of its Chow motive. This has already been proven by Gillet and Soulé in [4] and by Guillen and Navarro Aznar in [5].

There is a similar, though somewhat more involved definition of an Euler characteristic of pairs of varieties. We show that the universal Euler characteristic for

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pairs has the same value ring as the universal Euler characteristic with compact support, hence for every pair (X,Y) we get an element $\langle X,Y\rangle$ in $\mathrm{K}_0(\mathit{Var}_k)$. In these terms $[X]$ corresponds to $\langle \overline{X}, \overline{X}-X\rangle$, where $X \subset \overline{X}$ is a completion of X.

We are then in a position to show that there is a unique Euler characteristic for pairs of k-varieties with values in the Grothendieck group of Chow motives which for a smooth projective variety X assigns to the pair $(X,\emptyset)$ the Chow motive of X. This gives a new proof for a result by Guillen and Navarro Aznar in [5].

We also develop simple presentations for the Grothendieck group of varieties over a base variety since this will make the theory much more effective. For instance, we are then able to bring in the operations $f_*, f_!, f^*, f^!$, tensor product, internal homomorphisms and duality familiar from Grothendieck's duality theory and to derive some expected formulae.

Furthermore we generalize the presentations to an equivariant setting.

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Conventions: In the sequel k denotes a field of characteristic zero. By a *variety* over k we mean a reduced scheme of finite type over k, not necessarily irreducible.

2. The weak factorization theorem

The weak factorization theorem for varieties over a (not necessarily algebraically closed) field k of characteristic zero is proven by Włodarczyk in [8] and Abramovich, Karu, Matsuki and Włodarczyk in [1]. For convenience we recall parts of Theorem 0.3.1 in [1] and the subsequent Remark 2.

Theorem 2.1. *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete smooth connected varieties over k, let $U \subset X_1$ be an open set where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from U: There exists a sequence of birational maps*

$$X_1 = V_0 \overset{\phi_1}{\dashrightarrow} V_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_i}{\dashrightarrow} V_i \overset{\phi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\phi_{i+2}}{\dashrightarrow} \cdots \overset{\phi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\phi_l}{\dashrightarrow} V_l = X_2$$

where $\phi = \phi_l \circ \phi_{l-1} \circ \cdots \phi_2 \circ \phi_1$, such that each factor ϕ_i is an isomorphism over U, and $\phi_i : V_i \dashrightarrow V_{i+1}$ or $\phi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth center disjoint from U (here U is identified with an open subset of V_i).

Moreover, there is an index i_0 such that for all $i \leq i_0$ the map $V_i \dashrightarrow X_1$ is defined everywhere and projective, and for all $i \geq i_0$ the map $V_i \dashrightarrow X_2$ is defined everywhere and projective.

If $X_1 - U$ (respectively, $X_2 - U$) is a simple normal crossings divisor, then the factorization can be chosen such that the inverse images of this divisor under $V_i \to X_1$ (respectively, $V_i \to X_2$) are also simple normal crossings divisors, and the centers of blowing up have normal crossings with these divisors.

If ϕ is equivariant under the action of a finite group, then the factorization can be chosen equivariantly.

We make some remarks on this theorem.

Remark 2.2. It is not explicitly stated in theorem 0.3.1 of [1] that in the case of $X_1 - U$ (respectively, $X_2 - U$) a simple normal crossings divisor the inverse images of this divisor under $V_i \to X_1$ (respectively, $V_i \to X_2$) are also simple normal crossings divisors, but it can be read off from the proof (see 5.9 and 5.10).

Remark 2.3. The completeness of the varieties X_1 and X_2 is not necessary as has already been pointed out by Bonavero in [2]. It suffices that ϕ is a *proper* birational map $X_1 \dashrightarrow X_2$ between smooth connected varieties. This means that the projections to X_1 and X_2 from the graph of ϕ are proper, which gives back usual properness for an everywhere defined ϕ.

Remark 2.4. We note that the theorem implies that if X_1 and X_2 are varieties over a base variety S and ϕ is a map of S-varieties then the factorization is a factorization over S. If X_1 and X_2 are projective over S then so are the V_i's.

Definition 2.5. An action of a finite group on a variety is said to be *good* if every orbit is contained in an affine open set.

Remark 2.6. Note that on a quasi-projective variety every action of a finite group is good. More generally, if $X \longrightarrow S$ is equivariant and X is quasi-projective over S, then the action on X is good if the action on S is. In particular, in case of an equivariant factorization with respect to a good action of a finite group on X_1 and X_2 the action of on the V_i's is automatically good as they are projective over X_1, or X_2 respectively.

Remark 2.7. If X is a not necessarily connected variety with a (good) G-action such that G acts transitively on the connected components of X one can identify X with the induced variety $G \times^{G_0} X_0 = (G \times X_0)/G_0$, where X_0 is a connected component of X and G_0 is the stabilizer of X_0. Here the action of G_0 is given by $h(g, x) = (gh^{-1}, hx)$.

Using this and the equivariant version of the factorization theorem for connected smooth varieties also yields an equivariant factorization for smooth, not necessarily connected varieties where G acts transitively on the connected components.

3. The universal Euler characteristic with compact support

Let $\mathrm{K}_0(\mathit{Var}_k)$ be the free abelian group on isomorphism classes $[X]$ of varieties X over k where we impose the relation $[X] = [X - Y] + [Y]$ for $Y \subset X$ a closed subvariety (so in particular $[\emptyset] = 0$). This group is also called *the (naive) Grothendieck group of k-varieties*. It is the value group of the universal Euler characteristic with compact support. It is a commutative ring for the multiplication induced by the product of k-varieties, because the subgroup divided out actually is an ideal with respect to this multiplication.

Theorem 3.1. *The Grothendieck group of k-varieties has the following alternative presentations:*

(sm) *As the abelian group generated by the isomorphism classes of smooth varieties over k subject to the relations $[X] = [X - Y] + [Y]$, where X is smooth and $Y \subset X$ is a smooth closed subvariety,*

(bl) *as the abelian group generated by the isomorphism classes of smooth complete k-varieties subject to the relations $[\emptyset] = 0$ and $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$, where X is smooth and complete, $Y \subset X$ is a closed smooth subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Moreover, we get the same group if in case (sm) we restrict to quasi-projective varieties or if in case (bl) we restrict to projective varieties. We can also restrict to connected varieties in both presentations.

Remark 3.2. The subgroups divided out are again ideals with respect to the product of k-varieties.

Definition 3.3. Let $\mathbb{L} = [\mathbb{A}^1]$ denote the class of the affine line. Define the *naive motivic ring of k-varieties*, $\mathcal{M}_k$, as the localization $\mathrm{K}_0(Var_k)[\mathbb{L}^{-1}]$.

Corollary 3.4. *We have a ring involution $\mathcal{D}_k$ of $\mathcal{M}_k$ that sends $\mathbb{L}$ to $\mathbb{L}^{-1}$ and is characterized by the property that it sends the class of a complete connected smooth variety X to $\mathbb{L}^{-\dim X}[X]$.*

We call this involution the *duality map*.

We first deduce Corollary 3.4 from Theorem 3.1.

A vector bundle $V \longrightarrow X$ of rank n on a variety X is locally trivial by definition and hence its class $[V]$ in $\mathrm{K}_0(Var_k)$ is equal to $\mathbb{L}^n[X]$. Similarly the class of its projectivization $[\mathbb{P}(V)]$ equals $[\mathbb{P}^{n-1}][X] = (1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1})[X]$.

Lemma 3.5. *Let X be a smooth connected variety and $Y \subset X$ a smooth connected subvariety, let d denote the codimension of Y in X. Let E be the exceptional divisor of the blow-up $\mathrm{Bl}_Y X$ of X along Y. Then $[\mathrm{Bl}_Y X] - \mathbb{L}[E] = [X] - \mathbb{L}^d[Y]$ in $\mathrm{K}_0(Var_k)$.*

Proof. In $\mathrm{K}_0(Var_k)$ we have the fundamental relation $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$. Furthermore $[E] = (1 + \mathbb{L} + \cdots + \mathbb{L}^{d-1})[Y]$, thus $(1 - \mathbb{L})[E] = (1 - \mathbb{L}^d)[Y]$. Adding this to the fundamental relation finishes the proof. □

Proof of Corollary 3.4. Using the presentation (bl) and Lemma 3.5 we can define a group homomorphism $\mathrm{K}_0(Var_k) \longrightarrow \mathcal{M}_k$ by sending the class of a smooth connected complete variety X to $\mathbb{L}^{-\dim X}[X]$. This morphism is multiplicative and maps $\mathbb{L}$ to $\mathbb{L}^{-1}$, hence it can be extended uniquely to a ring endomorphism $\mathcal{D}_k$ of $\mathcal{M}_k$. Obviously $\mathcal{D}_k\mathcal{D}_k = \mathrm{id}_{\mathcal{M}_k}$. □

We now deduce the presentation (sm) of Theorem 3.1. For this purpose let us provisionally introduce the group $\mathrm{K}_0^{\mathrm{sm}}(Var_k)$, defined as the free abelian group on isomorphism classes $[X]_{\mathrm{sm}}$ of smooth varieties modulo the relations for smooth closed subvarieties Y of smooth varieties X. It is a commutative ring with respect to the product of varieties.

Step 1. The ring homomorphism $\mathrm{K}_0^{\mathrm{sm}}(Var_k) \longrightarrow \mathrm{K}_0(Var_k)$, $[X]_{\mathrm{sm}} \mapsto [X]$ is an isomorphism.

Proof. To construct an inverse for any variety X we stratify $X = \coprod_{N \in \mathcal{N}} N$ such that N smooth and equidimensional and $\overline{N}$ a union of strata for all $N \in \mathcal{N}$. Consider the expression $\sum_{N \in \mathcal{N}}[N]_{\mathrm{sm}}$ in $\mathrm{K}_0^{\mathrm{sm}}(Var_k)$. If X is smooth itself $\sum_{N \in \mathcal{N}}[N]_{\mathrm{sm}}$ equals $[X]_{\mathrm{sm}}$ as can be seen by induction on the number of elements of $\mathcal{N}$: Let $N \in \mathcal{N}$ be an element of minimal dimension, then $[X]_{\mathrm{sm}} = [X - N]_{\mathrm{sm}} + [N]_{\mathrm{sm}}$, and by the induction hypothesis $[X - N]_{\mathrm{sm}} = \sum_{N' \in \mathcal{N} - \{N\}}[N']_{\mathrm{sm}}$.

For two stratifications $\mathcal{N}$ and $\mathcal{N}'$ of X we can always find a common refinement $\mathcal{L}$. The above argument shows that for $N \in \mathcal{N}$ we get $\sum_{N \supset L \in \mathcal{L}}[L]_{\mathrm{sm}} = [N]_{\mathrm{sm}}$. Hence $\sum_{L \in \mathcal{L}}[L]_{\mathrm{sm}}$ is equal to $\sum_{N \in \mathcal{N}}[N]_{\mathrm{sm}}$ and analogously it equals $\sum_{N \in \mathcal{N}'}[N]_{\mathrm{sm}}$, therefore $\sum_{N\in\mathcal{N}}[N]_{\mathrm{sm}}$ is independent of the choice of the stratification. Thus we can set $e(X) := \sum_{N\in\mathcal{N}}[N]_{\mathrm{sm}}$.

If $Y \subset X$ is a closed subvariety we can find a stratification for which Y is a union of strata which yields $e(X) = e(X-Y)+e(Y)$, hence e is an Euler characteristic with compact support and factors through $\mathrm{K}_0(\mathit{Var}_k)$. The induced map on $\mathrm{K}_0(\mathit{Var}_k)$ obviously is an inverse for $\mathrm{K}_0^{\mathrm{sm}}(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathit{Var}_k)$. □

Decomposing into connected components and noting that instead of cutting a smooth closed subvariety Y out of a smooth connected variety X we can also take out the connected components of Y one by one shows that we can restrict to smooth connected varieties in the presentation (sm).

Stratifying by smooth quasi-projective varieties shows that we can restrict to smooth (connected) quasi-projective varieties.

In the rest of this section we establish the presentation (bl) of Theorem 3.1.

Again we introduce an auxiliary group: Let $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k)$ be the free abelian group on isomorphism classes $[X]_{\mathrm{bl}}$ of smooth complete varieties X over k modulo the relations for blow-ups of smooth complete varieties X along smooth closed subvarieties Y and the relation $[\emptyset]_{\mathrm{bl}} = 0$ (then $[X \sqcup Y]_{\mathrm{bl}} = [X]_{\mathrm{bl}} + [Y]_{\mathrm{bl}}$, which can be seen by blowing up along Y).

Decomposing into connected components and noting that the blow-up along a disjoint union is the successive blow-up along the connected components one sees that this can also be described as the free abelian group on isomorphism classes $[X]_{\mathrm{bl}}$ of connected smooth complete varieties with imposed relations $[\emptyset]_{\mathrm{bl}} = 0$ and $[\mathrm{Bl}_Y X]_{\mathrm{bl}} - [E]_{\mathrm{bl}} = [X]_{\mathrm{bl}} - [Y]_{\mathrm{bl}}$, where $Y \subset X$ is a connected closed smooth subvariety.

Also $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k)$ carries a commutative ring structure induced by the product of varieties.

Step 2. The ring homomorphism $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathit{Var}_k)$ which sends $[X]_{\mathrm{bl}}$ to $[X]$ is an isomorphism.

Proof. Again we construct an inverse. Using the presentation (sm) in Theorem 3.1 we see that defining an Euler characteristic with compact support is equivalent to defining an Euler characteristic with compact support for smooth connected varieties.

Let X be a smooth connected variety, let $X \subset \overline{X}$ be a smooth completion with $D = \overline{X} - X$ a simple normal crossings divisor. Let $D^{(l)}$ be the normalization of the l-fold intersections of D, where $D^{(0)}$ is understood to be $\overline{X}$ (so $D^{(l)}$ is the disjoint union of the l-fold intersections of the irreducible components of D). Consider the expression $\sum(-1)^l[D^{(l)}]_{\mathrm{bl}}$ in $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k)$.

We first show that this expression is independent of the choice of the completion:

Let $X \subset \overline{X}'$ and $X \subset \overline{X}$ be two smooth completions of X with $\overline{X} - X = D$ and $\overline{X}' - X = D'$ simple normal crossings divisors. Due to the weak factorization theorem the birational map $\overline{X}' \dashrightarrow \overline{X}$ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from X which have normal crossings with the complement of X. Hence we may assume that $\overline{X}' = \mathrm{Bl}_Z \overline{X}$ with $Z \subset D$ smooth and connected such that Z has normal crossings with D.

Let D_0 be the irreducible component of D containing Z and let $\{D_i\}_{i\in I}$ be the remaining irreducible components. Then the irreducible components of D' are $D'_i = \mathrm{Bl}_{Z\cap D_i} D_i$ (where $i \in \{0\} \cup I$) and the exceptional divisor E of the blow-up.

For $K \subset \{0\} \cup I$ we put $D_K := \bigcap_{j\in K} D_j$ (where $D_\emptyset$ is understood to be $\overline{X}$), $D'_K := \bigcap_{j\in K} D'_j$, $Z_K := Z \cap D_K$ and $E_K := E \cap D'_K$. As Z has normal crossings with D we get $D'_K = \mathrm{Bl}_{Z_K} D_K$ with exceptional divisor E_K, hence we have

$$[D'_K]_{\mathrm{bl}} - [E_K]_{\mathrm{bl}} = [D_K]_{\mathrm{bl}} - [Z_K]_{\mathrm{bl}}.$$

Denote by $E^{(l)}$ the preimage of E in $D'^{(l)}$ and by $Z^{(l)}$ the preimage of Z in $D^{(l)}$. Then for $l = 0, \ldots, n$ the preceding identity yields

$$\begin{aligned}
[D'^{(l)}]_{\mathrm{bl}} &= \sum_{|K|=l} [D'_K]_{\mathrm{bl}} + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= \sum_{|K|=l} ([D_K]_{\mathrm{bl}} + [E_K]_{\mathrm{bl}} - [Z_K]_{\mathrm{bl}}) + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= [D^{(l)}]_{\mathrm{bl}} + [E^{(l)}]_{\mathrm{bl}} - [Z^{(l)}]_{\mathrm{bl}} + [E^{(l-1)}]_{\mathrm{bl}},
\end{aligned}$$

(for $l = 0$ the last term is zero). As $Z \subset D_0$ we get $Z_{\{0\}\cup K} = Z_K$ for $K \subset I$, thus $\sum(-1)^l [Z^{(l)}]_{\mathrm{bl}} = 0$. Taking the alternating sum hence yields

$$\sum(-1)^l [D'^{(l)}]_{\mathrm{bl}} = \sum(-1)^l [D^{(l)}]_{\mathrm{bl}}.$$

Therefore we can set $e(X) := \sum(-1)^l [D^{(l)}]_{\mathrm{bl}}$.

We have to check that $e(X) = e(X - Y) + e(Y)$ for $Y \subset X$ a connected closed smooth subvariety of a connected smooth variety X. We choose $\overline{X} \supset X$ smooth and complete such that $D = \overline{X} - X$ is a simple normal crossings divisor and such that the closure $\overline{Y}$ of Y in $\overline{X}$ is also smooth and has normal crossings with D (we can take first a smooth completion of X with boundary a simple normal crossings divisor and then an embedded resolution of the closure of Y compatible with this divisor – compare e.g. Section 1.2 of [1]). In particular $D \cap \overline{Y}$ is a simple normal crossings divisor in $\overline{Y}$. Denote the irreducible components of D by $\{D_i\}_{i\in I}$, for $K \subset I$ let D_K be defined as above, let $Y_K := \overline{Y} \cap D_K$ and $Y^{(l)} = \bigsqcup_{|K|=l} Y_K$. Then $e(Y) = \sum(-1)^l [Y^{(l)}]_{\mathrm{bl}}$.

Let $\widetilde{X} := \mathrm{Bl}_{\overline{Y}} \overline{X}$ and denote the exceptional divisor by E. Denote the proper transform of D_i by $\widetilde{D}_i$. The complement $\widetilde{D}$ of $X - Y$ in $\widetilde{X}$ is the simple normal crossings divisor $\bigcup \widetilde{D}_i \cup E$. If $E_K := E \cap \widetilde{D}_K$ then as above $\widetilde{D}_K$ is the blow-up of D_K along Y_K with exceptional divisor E_K and hence

$$[\widetilde{D}_K]_{\mathrm{bl}} - [E_K]_{\mathrm{bl}} = [D_K]_{\mathrm{bl}} - [Y_K]_{\mathrm{bl}}.$$

Thus

$$\begin{aligned}
[\widetilde{D}^{(l)}]_{\mathrm{bl}} &= \sum_{|K|=l} [\widetilde{D}_K]_{\mathrm{bl}} + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= \sum_{|K|=l} ([D_K]_{\mathrm{bl}} - [Y_K]_{\mathrm{bl}} + [E_K]_{\mathrm{bl}}) + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= [D^{(l)}]_{\mathrm{bl}} - [Y^{(l)}]_{\mathrm{bl}} + [E^{(l)}]_{\mathrm{bl}} + [E^{(l-1)}]_{\mathrm{bl}},
\end{aligned}$$

where $E^{(l)}$ denotes the preimage of E in $\widetilde{D}^{(l)}$. Taking the alternating sum yields $e(X - Y) = e(X) - e(Y)$. Hence e induces a morphism $\mathrm{K}_0(\mathit{Var}_k) \longrightarrow \mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k)$ which clearly is an inverse for the mapping $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathit{Var}_k)$. $\square$

Using the fact that we can restrict to quasi-projective generators in the presentation (sm) of Theorem 3.1 and that a connected smooth quasi-projective variety has a smooth projective simple normal crossings completion we see that we can restrict to projective generators in the description (bl) of Theorem 3.1.

4. The universal Euler characteristic of pairs

Let A be an abelian group. Then an *A-valued Euler characteristic* is a mapping which associates to each pair (X, Y) of varieties over k with $Y \subset X$ a closed subvariety an element $\chi(X, Y) \in A$ (where we denote $\chi(X, \emptyset)$ by $\chi(X)$) such that the following properties hold:

- *Excision*: If $f : X' \longrightarrow X$ is proper and $Y \subset X$ is a closed subvariety such that f induces an isomorphism $X' - f^{-1}Y \cong X - Y$, then $\chi(X', f^{-1}Y) = \chi(X, Y)$.
- *Gysin*: If X is smooth and connected and $D \subset X$ is a smooth divisor, then $\chi(X - D) = \chi(X) - \chi(\mathbb{P}^1 \times D, \{\infty\} \times D)$.
- *Exactness*: If $X \supset_{\text{closed}} Y \supset_{\text{closed}} X$, then $\chi(X, Z) = \chi(X, Y) + \chi(Y, Z)$.

Excision implies that $\chi(X, Y)$ only depends on the isomorphism class of the pair (X, Y). Using exactness we get $\chi(X, Y) = \chi(X) - \chi(Y)$. Exactness and excision yield $\chi(X \sqcup Y) = \chi(X \sqcup Y, Y) + \chi(Y) = \chi(X) + \chi(Y)$ and $\chi(\emptyset) = 0$. If $X \subset W$ is an open embedding with W complete, then the the excision property implies that $\chi(W, W - X)$ is independent of the choice of the open embedding. We denote $\chi(W, W - X)$ by $\chi_c(X)$. For $Y \subset X$ closed we have

$$\begin{aligned}\chi_c(X) &= \chi(W, W - X) \\ &= \chi(W, \overline{Y} \cup (W - X)) + \chi(\overline{Y} \cup (W - X), W - X) = \chi_c(X - Y) + \chi_c(Y),\end{aligned}$$

where by $\overline{Y}$ we denote the closure of Y in W. Hence χ_c factors through $\mathrm{K}_0(\mathit{Var}_k)$.

Definition 4.1. The *value ring of the universal Euler characteristic for pairs of k-varieties* is defined to be the free abelian group on isomorphism classes of pairs (X, Y) with $Y \subset X$ closed modulo the relations for an Euler characteristic. We denote it by $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$. The class of a pair (X, Y) in $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ is denoted by $\langle X, Y\rangle$, where we write $\langle X\rangle$ shorthand for $\langle X, \emptyset\rangle$.

The above construction yields a group homomorphism

$$\chi_c : \mathrm{K}_0(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0(\mathit{Var}_k)$$

Indeed $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ carries a ring structure by defining the product of $\langle X, Y\rangle$ and $\langle X', Y'\rangle$ to be $\langle X \times X', X \times Y' \cup Y \times X'\rangle$:

To see that this is compatible with the Gysin relation we introduce an auxiliary group $\widehat{\mathrm{K}_0}(\mathit{Var}_k)$, defined as the free abelian group on isomorphism classes of pairs of varieties over k modulo excision and exactness. We denote the class of a pair (X, Y) in $\widehat{\mathrm{K}_0}(\mathit{Var}_k)$ by $\{X, Y\}$, where we write $\{X\}$ for $\{X, \emptyset\}$. We get a ring structure on $\widehat{\mathrm{K}_0}(\mathit{Var}_k)$ by setting $\{X, Y\}\{X', Y'\} = \{X \times X', X \times Y' \cup Y \times X'\}$. Noting that $\{X, Y\} = \{X\} - \{Y\}$ and $\{X, S\} = \{\widetilde{X}, \widetilde{S}\}$ for an arbitrary X where S is the singular locus of X and $\widetilde{X}$ a resolution of singularities with $\widetilde{S}$ the inverse image of S we get $\{X\} = \{\widetilde{X}\} - \{\widetilde{S}\} + \{S\}$, where the dimension of $\widetilde{S}$ and S is strictly less than the dimension of X. Proceeding inductively on the dimension of X we can write every class $\{X, Y\}$ as a linear combination of classes of smooth varieties. Hence the subgroup generated by expressions of the form $\{X - D\} - \{X\} + \{\mathbb{P}^1 \times D, \{\infty\} \times D\}$ with X smooth and D a smooth divisor on X actually is an ideal.

Using the multiplicative structure of $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ the Gysin relation can be rewritten as
$$\langle X - D\rangle = \langle X\rangle - \langle \mathbb{P}^1, \{\infty\}\rangle\langle D\rangle.$$
The group homomorphism χ_c defined above actually is a ring homomorphism.

Theorem 4.2. *The map $\chi_c : \mathrm{K}_0(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ is a ring isomorphism. If X is a smooth connected k-variety and $(\overline{X}, D)$ is a simple normal crossings completion of X over k, then the inverse of χ_c assigns to $\langle X\rangle$ the element $\sum(-\mathbb{L})^l[D^{(l)}]$. Its image in $\mathcal{M}_k$ is also equal to $\mathcal{D}_k(\mathbb{L}^{-\dim X}[X]) = \mathbb{L}^{\dim X}\mathcal{D}_k([X])$.*

Before we prove this we give an application of the theorems 3.1 and 4.2.

Let $\mathfrak{M}_k$ denote the category of Chow k-motives (see e.g. the introduction [7] by Scholl for the definition of $\mathfrak{M}_k$). For a smooth connected projective variety X over k, denote by $h(X)$ the motive of X and its class in $\mathrm{K}_0(\mathfrak{M}_k)$ by $[h(X)]$. Denote the class $[h(\mathbb{P}^1)]-[h(\operatorname{Spec} k)]$ of the Tate motive by $\mathbb{L}_{\mathrm{mot}}$, and by $[h(X)]^\vee = \mathbb{L}_{\mathrm{mot}}^{-\dim X}\otimes[h(X)]$ the class of the dual motive of $h(X)$.

Corollary 4.3. *There is a ring homomorphism (in fact the unique group homomorphism) $\chi_{c,\mathrm{mot}} : \mathrm{K}_0(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathfrak{M}_k)$ which sends the class of a smooth projective variety X to $[h(X)]$. The class of the affine line is mapped to the class of the Tate motive by this morphism. Likewise, there is a ring homomorphism (in fact the unique group homomorphism) $\chi_{\mathrm{mot}} : \widetilde{\mathrm{K}}_0(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathfrak{M}_k)$ which sends the class of a smooth projective variety X to $[h(X)]$. The two are related by the property that for X smooth and connected we have $(\chi_{\mathrm{mot}}\langle X\rangle)^\vee = \mathbb{L}_{\mathrm{mot}}^{-\dim X}\otimes\chi_{c,\mathrm{mot}}[X]$.*

Proof of Corollary 4.3. For $Y \subset X$ a smooth closed subvariety of a smooth projective variety we have $[h(\mathrm{Bl}_Y X)] - [h(E)] = [h(X)] - [h(Y)]$, where E denotes the exceptional divisor. We now use the presentation (bl) of Theorem 3.1 to see that there is a unique group homomorphism $\widetilde{\mathrm{K}}_0(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathfrak{M}_k)$ which sends the class of a smooth projective variety X to $[h(X)]$ and therefore is a ring homomorphism. As $\mathbb{L}$ is mapped to $\mathbb{L}_{\mathrm{mot}}$ it can be extended to $\mathcal{M}_k$.

Furthermore due to Theorem 4.2 there is a unique group homomorphism χ_{mot} (in fact a ring homomorphism) which makes

$$\begin{tikzcd} \mathrm{K}_0(\mathit{Var}_k) \arrow[d, "\chi_c"'] \arrow[dr, "\chi_{c,\mathrm{mot}}"] & \\ \widetilde{\mathrm{K}}_0(\mathit{Var}_k) \arrow[r, "\chi_{\mathrm{mot}}"] & \mathrm{K}_0(\mathfrak{M}_k) \end{tikzcd}$$

commutative.

Now let X be a smooth connected variety. As the image of the inverse of $\langle X\rangle$ under χ_c in $\mathcal{M}_k$ is $\mathcal{D}_k(\mathbb{L}^{-\dim X}[X])$ and as

$$\begin{tikzcd} \mathcal{M}_k \arrow[r, "\chi_{c,\mathrm{mot}}"] \arrow[d, "\mathcal{D}_k"'] & \mathrm{K}_0(\mathfrak{M}_k) \arrow[d, "\vee"] \\ \mathcal{M}_k \arrow[r, "\chi_{c,\mathrm{mot}}"] & \mathrm{K}_0(\mathfrak{M}_k) \end{tikzcd}$$

is commutative we conclude that $(\chi_{\mathrm{mot}}\langle X\rangle)^\vee = \mathbb{L}_{\mathrm{mot}}^{-\dim X}\otimes\chi_{c,\mathrm{mot}}[X]$. ☐

The rest of this section is devoted to the proof of Theorem 4.2.

For this purpose we give a more convenient presentation of $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$.

For the moment let $\widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k)$ be the free abelian group on isomorphism classes of pairs of *smooth* varieties (denoted by $\langle X, Y\rangle_{\mathrm{bl}}$, where $\langle X\rangle_{\mathrm{bl}}$ stands for $\langle X, \emptyset\rangle_{\mathrm{bl}}$), with the same relations as for $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ except that we impose the excision relation only for blow-ups at smooth centers (as $\langle X, \emptyset\rangle_{\mathrm{bl}} = \langle X \sqcup Y, Y\rangle_{\mathrm{bl}}$ by blowing up $X \sqcup Y$ along Y we get by exactness $\langle X \sqcup Y\rangle_{\mathrm{bl}} = \langle X\rangle_{\mathrm{bl}} + \langle Y\rangle_{\mathrm{bl}}$ and $\langle\emptyset\rangle_{\mathrm{bl}} = 0$).

Proposition 4.4. *The obvious group homomorphism $\widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ is an isomorphism. In other words, the value group $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$ of the universal Euler characteristic for pairs of k-varieties has a presentation as the free group on isomorphism classes of pairs of smooth k-varieties, modulo exactness, Gysin and excision for blow-ups at smooth centers.*

The strategy for the proof of the proposition is to construct an inverse by defining a map $\Psi : \widehat{\mathrm{K}_0}(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k)$ that is the obvious map on a smooth pair $\{X, Y\}$ and factors through the Gysin relation. It is enough to define $\Psi\{X\}$ for all varieties $\{X\}$ and then to set $\Psi\{X, Y\} = \Psi\{X\} - \Psi\{Y\}$. Then Ψ will automatically fulfill exactness. For a smooth variety X we want $\Psi\{X\}$ to be equal to $\langle X\rangle_{\mathrm{bl}}$.

We proceed by induction on the dimension. To be more precise we use that $\varinjlim \widehat{\mathrm{K}_0}(\mathit{Var}_k)_n \cong \widehat{\mathrm{K}_0}(\mathit{Var}_k)$, where $\widehat{\mathrm{K}_0}(\mathit{Var}_k)_n$ denotes the free abelian group on isomorphism classes of pairs of varieties of dimension at most n modulo excision and exactness. In dimension zero we just set $\Psi\{X\} = \langle X\rangle_{\mathrm{bl}}$. Suppose now that Ψ has already been defined for pairs $\{X, Y\}$ with $\dim X < n$, that it factors through the excision relation for such pairs (in particular it is additive on disjoint unions in dimensions smaller than n) and that $\Psi\{X\} = \langle X\rangle_{\mathrm{bl}}$ for X smooth and $\dim X < n$. In the following five steps we extend Ψ to $\widehat{\mathrm{K}_0}(\mathit{Var}_k)_n$.

Step 1. Suppose $f : X' \longrightarrow X$ is proper with $Y' = f^{-1}Y$, suppose that f induces an isomorphism $X' - Y' \cong X - Y$. Suppose $\dim X, \dim X' \leq n$ and $\dim Y, \dim Y' < n$, X and X' smooth. Then $\langle X'\rangle_{\mathrm{bl}} - \Psi\{Y'\} = \langle X\rangle_{\mathrm{bl}} - \Psi\{Y\}$.

Proof. Suppose first that X and X' are connected. Due to the factorization theorem f can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from $X - Y$. So we can assume that $X' \longrightarrow X$ is a blow-up along $Z \subset X$ a smooth closed subvariety with $Z \subset Y$. Let $E \subset Y'$ be the exceptional divisor of this blow-up. We have $\langle X'\rangle_{\mathrm{bl}} - \langle E\rangle_{\mathrm{bl}} = \langle X\rangle_{\mathrm{bl}} - \langle Z\rangle_{\mathrm{bl}}$ and furthermore $\Psi\{Y'\} - \langle E\rangle_{\mathrm{bl}} = \Psi\{Y\} - \langle Z\rangle_{\mathrm{bl}}$ by induction hypothesis. Hence $\langle X'\rangle_{\mathrm{bl}} - \Psi\{Y'\} = \langle X\rangle_{\mathrm{bl}} - \Psi\{Y\}$.

Now in general X can be written as the disjoint union of connected components $X_1 \sqcup \cdots \sqcup X_l \sqcup \cdots \sqcup X_n$ such that $Y_i := X_i \cap Y$ is a proper subset of X_i for $i \leq l$ and $Y_i = X_i$ for $i > l$. In particular Y_i is smooth for $i > l$, and as $\dim Y_i < n$ the induction hypothesis yields $\Psi\{Y_i\} = \langle Y_i\rangle_{\mathrm{bl}} = \langle X_i\rangle_{\mathrm{bl}}$ and hence

$$\langle X\rangle_{\mathrm{bl}} - \psi\{Y\} = \sum_{i=1}^{l} \langle X_i\rangle_{\mathrm{bl}} - \psi\{Y_i\}.$$

Decomposing $X' = X'_1 \sqcup \cdots \sqcup X'_{l'} \sqcup \cdots \sqcup X'_{n'}$ in the same way and setting $Y'_i = X'_i \cap Y$ yields

$$\langle X'\rangle_{\mathrm{bl}} - \psi\{Y'\} = \sum_{i=1}^{l'} \langle X'_i\rangle_{\mathrm{bl}} - \psi\{Y'_i\}.$$

As f induces an isomorphism $\bigsqcup_{i=1}^{l'} X'_i - Y'_i \cong \bigsqcup_{i=1}^{l} X_i - Y_i$ it follows that $l = l'$ and we may assume that f induces $(X'_i, Y'_i) \longrightarrow (X_i, Y_i)$ for $i \leq l$, hence the two sums are equal. □

Now let X be an arbitrary n-dimensional variety. Choose $\pi : \widetilde{X} \longrightarrow X$ proper with $\widetilde{X}$ smooth such that π induces an isomorphism over the smooth locus of X. Let $S \subset X$ be the singular locus of X, let $\widetilde{S} \subset \widetilde{X}$ be the inverse image of S. As we want $\Psi\{\widetilde{X}, \widetilde{S}\}$ to be equal to $\Psi\{X, S\}$ we set $\Psi\{X\} := \langle \widetilde{X} \rangle_{\mathrm{bl}} - \Psi\{\widetilde{S}\} + \Psi\{S\}$ (the right hand side has already been defined as $\widetilde{X}$ is smooth and $\dim S, \dim \widetilde{S} < n$).

Since any two choices of $\widetilde{X} \longrightarrow X$ are dominated by a third one (take for example the closure of $X - S$ which is diagonally embedded in the product and resolve the singularities), Step 1 implies that $\langle \widetilde{X} \rangle_{\mathrm{bl}} - \Psi\{\widetilde{S}\} + \Psi\{S\}$ is independent of this choice. It is clear that Ψ is additive with respect to disjoint unions and that for smooth X we get $\Psi\{X\} = \langle X \rangle_{\mathrm{bl}}$.

We have to prove that Ψ factors through excision for pairs of varieties of dimension $\leq n$.

Step 2. The map Ψ factors through excision for $f : (X', Y') \longrightarrow (X, Y)$ with X and X' both smooth of dimension $\leq n$:

Proof. Like in the proof of Step 1 we can decompose X and X' into connected components. Now use $\Psi\{X_i\} = \langle X_i \rangle_{\mathrm{bl}}$ and $\Psi\{X'_i\} = \langle X'_i \rangle_{\mathrm{bl}}$ and apply Step 1 to the X_i and X'_i with $i \leq l$. □

Step 3. The map Ψ factors through excision for $f : (X', Y') \longrightarrow (X, Y)$ with $\dim X, \dim X' \leq n$, X' smooth and $\dim Y < n$.

Proof. Let $S \subset Y$ be the singular locus of X, let $\widetilde{X} \longrightarrow X$ be a resolution of singularities, let $\widetilde{Y}$ be the inverse image of Y and $\widetilde{S}$ the inverse image of S. Then by definition $\Psi\{X, S\} = \Psi\{\widetilde{X}, \widetilde{S}\}$ and by induction $\Psi\{Y, S\} = \Psi\{\widetilde{Y}, \widetilde{S}\}$. Hence $\Psi\{X, Y\} = \Psi\{\widetilde{X}, \widetilde{Y}\}$.

Let $\widehat{X}$ be a resolution of singularities of the closure of $X - Y$ in $\widetilde{X} \times X'$ (we embed $X - Y$ diagonally). We have a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d] & X' \arrow[d] \\ \widetilde{X} \arrow[r] & X \end{tikzcd}$$

of proper birational maps which yield isomorphisms over $X - Y$. Let $\widehat{Y}$ be the inverse image of Y in $\widehat{X}$. Then $\Psi\{\widehat{X}, \widehat{Y}\} = \Psi\{X', Y'\}$ and $\Psi\{\widehat{X}, \widehat{Y}\} = \Psi\{\widetilde{X}, \widetilde{Y}\}$ as $\widehat{X}$, $\widetilde{X}$ and X' are smooth. □

Step 4. The map Ψ factors through excision for $f : (X_1, Y_1) \longrightarrow (X_2, Y_2)$ with $\dim X_1, \dim X_2 \leq n$ and $\dim Y_1, \dim Y_2 < n$.

Proof. For $i = 1, 2$ let S_i be the singular locus of X_i, let $Z_i = Y_i \cup S_i$. Then $f^{-1}(Z_2) = Z_1$ and f induces an isomorphism of smooth varieties $X_1 - Z_1 \cong X_2 - Z_2$ (as the singular locus of $X_i - Y_i$ is $S_i - Y_i$). Let $\widetilde{X_i}$ be a resolution of singularities of X_i that is an isomorphism over $X_i - Z_i$: so if $\widetilde{Z_i}$ is the preimage of Z_i in $\widetilde{X_i}$ then $\widetilde{X_i} - \widetilde{Z_i} \cong X_i - Z_i$. Let X_3 be the closure of $X_1 - Z_1$ (embedded diagonally) in

$\widetilde{X_1} \times \widetilde{X_2}$. Let $\widetilde{X_3}$ a resolution of singularities of X_3 (that is an isomorphism over $X_1 - Z_1$). We thus have a commutative diagram

$$\begin{tikzcd} & \widetilde{X_3} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \widetilde{X_1} \arrow[d] & & \widetilde{X_2} \arrow[d] \\ X_1 \arrow[rr] & & X_2 \end{tikzcd}$$

where $\pi_1^{-1}(\widetilde{Z_1}) = \pi_2^{-1}(\widetilde{Z_2}) =: \widetilde{Z_3}$ and π_1, π_2 induce isomorphisms outside $\widetilde{Z_3}$. As $\widetilde{X_i}$ are smooth for $1 \leq i \leq 3$ and $\dim Z_1, \dim Z_2 < n$ we have

$$\Psi\{X_1, Z_1\} = \Psi\{\widetilde{X_1}, \widetilde{Z_1}\} = \Psi\{\widetilde{X_3}, \widetilde{Z_3}\} = \Psi\{\widetilde{X_2}, \widetilde{Z_2}\} = \Psi\{X_2, Z_2\}.$$

The induction hypothesis gives

$$\Psi\{Z_1, Y_1\} = \Psi\{Z_2, Y_2\}.$$

Adding these two equations up yields $\Psi\{X_1, Y_1\} = \Psi\{X_2, Y_2\}$. □

Step 5. The map Ψ factors through excision for $f : (X', Y') \longrightarrow (X, Y)$ with $\dim X, \dim X' \leq n$.

Proof. Let $X_1, \ldots, X_l$ be the irreducible components of X, let $Y_i := X_i \cap Y$. Let $B := \bigcup_{Y_i \neq X_i} X_i$, let $N := Y \cap B$. Let $A = \bigcup_{Y_i = X_i} X_i$. Hence $X = A \cup B$ and $Y = A \cup N$.

Consider the proper map $A \sqcup B \longrightarrow X$ which induces an isomorphism $A \sqcup B - (A \cap B \sqcup A \cap B) \cong X - (A \cap B)$. As $\dim A \cap B < n$ we get $\Psi\{A \sqcup B, A \cap B \sqcup A \cap B\} = \Psi\{X, A \cap B\}$, hence $\Psi\{A \sqcup B\} = \Psi\{X\} + \Psi\{A \cap B\}$.

Analogously we get $\Psi\{A \sqcup N\} = \Psi\{Y\} + \Psi\{A \cap N\}$.

Subtracting yields $\Psi\{A \sqcup B, A \sqcup N\} = \Psi\{X, Y\}$ (because $A \cap N = A \cap B \cap Y = A \cap B$).

On the other hand $\Psi\{A \sqcup B, A \sqcup N\} = \Psi\{B, N\}$, hence $\Psi\{B, N\} = \Psi\{X, Y\}$.

The same reasoning yields $\Psi\{B', N'\} = \Psi\{X', Y'\}$ (with analogous notations).

Restricting f yields a proper map $(B', N') \longrightarrow (B, N)$ which induces an isomorphism $B' - N' \cong B - N$, and as $\dim N, \dim N' < n$ we are done. □

This finishes the induction step.

As the Gysin relation holds in $\widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k)$, we get an induced homomorphism $\overline{\Psi} : \widetilde{\mathrm{K}}_0(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k)$ which can inductively be seen to be an inverse of $\widetilde{\mathrm{K}}_0^{\mathrm{bl}}(\mathit{Var}_k) \longrightarrow \widetilde{\mathrm{K}}_0(\mathit{Var}_k)$. This establishes Proposition 4.4. □

We now prove Theorem 4.2 by defining an inverse $\chi : \widetilde{\mathrm{K}}_0(\mathit{Var}_k) \longrightarrow \mathrm{K}_0(\mathit{Var}_k)$ on classes of smooth varieties and checking compatibility with excision for blow-ups and the Gysin relation. Let X be a smooth connected variety. Choose a smooth completion $\overline{X} \supset X$ of X with $D = \overline{X} - X$ a divisor with simple normal crossings.

Step 1. The element $\sum(-\mathbb{L})^l[D^{(l)}]$ of $\mathrm{K}_0(\mathit{Var}_k)$ is independent of the choice of the completion of X (we denote it by $\chi(X)$).

Proof. In view of the weak factorization theorem we only have to compare the result for a completion $X \subset \overline{X}$ and the completion $X \subset \mathrm{Bl}_Z \overline{X} =: \overline{X}'$, with Z a smooth and irreducible closed subvariety of $\overline{X}$ disjoint from X which has normal crossings with $D = \overline{X} - X$.

Let D_0 be the irreducible component of D containing Z and let $\{D_i\}_{i \in I}$ be the remaining irreducible components. Then the irreducible components of D' are $D'_i = \mathrm{Bl}_{Z \cap D_i} D_i$ (where $i \in \{0\} \cup I$) and the exceptional divisor E of the blow-up. For $K \subset \{0\} \cup I$ we put $D_K := \bigcap_{j \in K} D_j$ (where $D_\emptyset$ is understood to be $\overline{X}$), $D'_K := \bigcap_{j \in K} D'_j$, $Z_K := Z \cap D_K$ and $E_K := E \cap D'_K$. Denote the codimension of Z_K in D_K by d_K. As Z has normal crossings with D we get $D'_K = \mathrm{Bl}_{Z_K} D_K$ with exceptional divisor E_K, hence by Lemma 3.5 we have

$$[D'_K] - \mathbb{L}[E_K] = [D_K] - \mathbb{L}^{d_K}[Z_K].$$

Denote by $E^{(l)}$ the inverse image of E in $D'^{(l)}$. For $l = 0, \dots, n$ the preceding equality yields

$$\begin{aligned}
\mathbb{L}^l[D'^{(l)}] &= \mathbb{L}^l \sum_{|K|=l} [D'_K] + \mathbb{L}^l \sum_{|K|=l-1} [E_K] \\
&= \mathbb{L}^l \sum_{|K|=l} ([D_K] + \mathbb{L}[E_K] - \mathbb{L}^{d_K}[Z_K]) + \mathbb{L}^l \sum_{|K|=l-1} [E_K] \\
&= \mathbb{L}^l[D^{(l)}] + \mathbb{L}^{l+1}[E^{(l)}] - \sum_{|K|=l} \mathbb{L}^{l+d_K}[Z_K] + \mathbb{L}^l[E^{(l-1)}],
\end{aligned}$$

(for $l = 0$ the last term is zero).

For $K \subset I$ we get $Z_{K \cup \{0\}} = Z_K$ and hence $d_{K \cup \{0\}} = d_K - 1$, which yields $\mathbb{L}^{|K \cup \{0\}| + d_{K \cup \{0\}}}[Z_{K \cup \{0\}}] = \mathbb{L}^{|K| + d_K}[Z_K]$ and $\sum_{K \subset I \cup \{0\}} (-1)^{|K|} \mathbb{L}^{|K| + d_K}[Z_K] = 0$. Taking the alternating sum we conclude

$$\sum (-\mathbb{L})^l [D'^{(l)}] = \sum (-\mathbb{L})^l [D^{(l)}]$$

as claimed. □

Step 2. For a smooth X with connected components X_i we set $\chi(X) := \sum \chi(X_i)$.

Step 3. Let X be a smooth variety, let $Z \subset X$ be a smooth closed subvariety. Let E be the exceptional divisor of $\mathrm{Bl}_Z X \longrightarrow X$. Then $\chi(\mathrm{Bl}_Z X, E) = \chi(X, Z)$.

Proof. By decomposing X and Z into connected components we see that it suffices to prove the claim for connected X and Z. Denote the dimension of X by n. Choose a smooth completion $\overline{X}$ of X such that $D = \overline{X} - X$ is a divisor with simple normal crossings and $\overline{Z}$ is smooth and has normal crossings with D. So in particular $D \cap \overline{Z} \subset \overline{Z}$ is a simple normal crossings divisor. Then $\mathrm{Bl}_{\overline{Z}} \overline{X}$ is a smooth completion of $\mathrm{Bl}_Z X$ with $D' := \mathrm{Bl}_{\overline{Z}} \overline{X} - \mathrm{Bl}_Z X$ a normal crossings divisor. The exceptional divisor $\overline{E}$ of this blow-up is a smooth completion of E with $\overline{E} - E = \overline{E} \cap D'$ a simple normal crossings divisor. Denote by $E^{(l)}$ the preimage of $\overline{E}$ in $D'^{(l)}$ and by

$Z^{(l)}$ the preimage of $\overline{Z}$ in $D^{(l)}$. Now

$$\begin{aligned}\chi(\mathrm{Bl}_Z X) &= [D'^{(0)}] - \mathbb{L}[D'^{(1)}] + \cdots + (-\mathbb{L})^n [D'^{(n)}],\\ \chi(E) &= [E^{(0)}] - \mathbb{L}[E^{(1)}] + \cdots + (-\mathbb{L})^n [E^{(n)}],\\ \chi(X) &= [D^{(0)}] - \mathbb{L}[D^{(1)}] + \cdots + (-\mathbb{L})^n [D^{(n)}],\\ \chi(Z) &= [Z^{(0)}] - \mathbb{L}[Z^{(1)}] + \cdots + (-\mathbb{L})^n [Z^{(n)}].\end{aligned}$$

As $D'^{(l)} - E^{(l)} \cong D^{(l)} - Z^{(l)}$ we get $[D'^{(l)}] - [E^{(l)}] = [D^{(l)}] - [Z^{(l)}]$ in $\mathrm{K}_0(\mathit{Var}_k)$ and consequently

$$\chi(\mathrm{Bl}_Z X, E) = \chi(\mathrm{Bl}_Z X) - \chi(E) = \chi(X) - \chi(Z) = \chi(X, Z).$$

□

Step 4. For a smooth connected variety X and $D \subset X$ a smooth divisor we have $\chi(X - D) = \chi(X) - \mathbb{L}\chi(D)$.

Proof. We choose a smooth completion $\overline{X} \supset X$ such that $D_1 := (\overline{X} - X) \cup \overline{D}$ is a simple normal crossings divisor, where by $\overline{D}$ we denote the closure of D in $\overline{X}$. Let $D_2 := \overline{X} - X$. Denote by $\Delta^{(l)}$ the inverse image of $\overline{D}$ in $D_2^{(l)}$. Then $\chi(D) = \sum(-\mathbb{L})^l[\Delta^{(l)}]$ and on the other hand

$$\chi(X - D) = \sum(-\mathbb{L})^l[D_1^{(l)}] = \sum(-\mathbb{L})^l[D_2^{(l)}] - \mathbb{L}\sum(-\mathbb{L})^l[\Delta^{(l)}].$$

□

Step 5. For X smooth and connected $\chi(\mathbb{P}^1 \times X, \{\infty\} \times X) = \mathbb{L}\chi(X)$.

Proof. Let $\overline{X}$ be a smooth completion of X with $D = \overline{X} - X$ a simple normal crossings divisor. Then $\mathbb{P}^1 \times \overline{X}$ is a smooth completion of $\mathbb{P}^1 \times X$ such that the complement of $\mathbb{P}^1 \times X$ is $\mathbb{P}^1 \times D$, a simple normal crossings divisor. This yields

$$\chi(\mathbb{P}^1 \times X, \{\infty\} \times X) = \chi(\mathbb{P}^1 \times X) - \chi(X) = \sum(-\mathbb{L})^l([(\mathbb{P}^1 \times D)^{(l)}] - [D^{(l)}]) = \mathbb{L}\chi(X).$$

□

Hence χ factors through $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$. We will now denote the induced mapping by χ.

Step 6. Let X be a smooth connected variety. Then $\chi_c\chi\langle X\rangle = \langle X\rangle$ in $\widetilde{\mathrm{K}}_0(\mathit{Var}_k)$.

Proof. Let $\overline{X} \supset X$ be a smooth completion with $D := \overline{X} - X$ a simple normal crossings divisor. Let $D_1, \ldots, D_n$ be the irreducible components of D. We proceed by induction on n.

If X is complete then obviously $\chi_c\chi\langle X\rangle = \langle X\rangle$. This establishes the claim for $n = 0$.

Let $X = X_0 \subset X_1 \subset \cdots \subset X_n = \overline{X}$ be given by $X_i := X_{i+1} - (D_{i+1} \cap X_{i+1})$. Then

$$\begin{aligned}\chi_c\chi\langle X\rangle &= \chi_c\chi\langle X_1 - (D_1 \cap X_1)\rangle\\ &= \chi_c\chi\langle X_1\rangle - \chi_c\chi\langle \mathbb{P}^1 \times (D_1 \cap X_1), \{\infty\} \times (D_1 \cap X_1)\rangle\end{aligned}$$

by the Gysin relation. Now $X_1 \subset \overline{X}$ is a smooth completion with $\overline{X} - X_1$ a simple normal crossings divisor with less then n irreducible components. The same holds for $D_1 \cap X_1 \subset D_1$ and for $\mathbb{P}^1 \times (D_1 \cap X_1) \subset \mathbb{P}^1 \times D_1$. Hence by induction we get

$$\begin{aligned}
&\chi_c\chi\langle X_1\rangle - \chi_c\chi\langle \mathbb{P}^1 \times (D_1 \cap X_1), \{\infty\} \times (D_1 \cap X_1)\rangle \\
&\qquad = \chi_c\chi\langle X_1\rangle - \chi_c\chi\langle \mathbb{P}^1 \times (D_1 \cap X_1)\rangle + \chi_c\chi\langle \{\infty\} \times (D_1 \cap X_1)\rangle \\
&\qquad = \langle X_1\rangle - \langle \mathbb{P}^1 \times (D_1 \cap X_1)\rangle + \langle \{\infty\} \times (D_1 \cap X_1)\rangle = \langle X\rangle \text{ again by Gysin.}
\end{aligned}$$

□

Step 7. The mappings χ_c and χ are mutually inverse.

Proof. Obviously $\chi\chi_c[X] = [X]$ for X smooth and complete, and we already know that $\chi_c\chi\langle X\rangle = \langle X\rangle$ for X smooth. □

Step 8. For X a smooth and connected variety the image of $\chi\langle X\rangle$ in $\mathcal{M}_k$ is equal to $\mathcal{D}_k(\mathbb{L}^{-\dim X}[X]) = \mathbb{L}^{\dim X}\mathcal{D}_k([X])$.

Proof. Let $(\overline{X}, D)$ be a smooth simple normal crossings completion. Then

$$\mathcal{D}_k([X]) = \sum (-1)^l \mathcal{D}_k([D^{(l)}]) = \sum (-1)^l \mathbb{L}^{l-\dim X}[D^{(l)}].$$

□

This finishes the proof of Theorem 4.2. □

5. Relative Grothendieck groups of varieties

Let S be a (not necessarily irreducible) variety over k. Let $\mathrm{K}_0(\mathit{Var}_S)$ be the free abelian group on isomorphism classes $[X]_S$ of varieties X over S where we impose the relations $[X]_S = [(X - Y)]_S - [Y]_S$ for $Y \subset X$ a closed subvariety (hence again $[\emptyset]_S = 0$). We call it the *Grothendieck group of S-varieties.*

This group, too, has simple presentations.

Theorem 5.1. *The Grothendieck group of S-varieties has the following alternative presentations:*

(sm) *As the abelian group generated by the isomorphism classes of S-varieties which are smooth over k subject to the relations $[X]_S = [X - Y]_S + [Y]_S$, where X is smooth and $Y \subset X$ is a smooth closed subvariety,*

(bl) *as the abelian group generated by the isomorphism classes of S-varieties which are smooth over k and proper over S subject to the relations $[\emptyset]_S = 0$ and $[\mathrm{Bl}_Y X]_S - [E]_S = [X]_S - [Y]_S$, where X is smooth over k and proper over S, $Y \subset X$ is a closed smooth subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Moreover, we get the same group if in case (sm) we restrict to varieties which are quasi-projective over S or if in case (bl) we restrict to varieties which are projective over S. We can also restrict to connected varieties in both presentations.

We proceed in the same way as in the absolute case. For the moment denote by $\mathrm{K}_0^{\mathrm{sm}}(\mathit{Var}_S)$ the free abelian group on isomorphism classes $[X]_{S,\mathrm{sm}}$ of varieties over S which are smooth over k, modulo $[X]_{S,\mathrm{sm}} = [X - Y]_{S,\mathrm{sm}} + [Y]_{S,\mathrm{sm}}$ for $X \subset Y$ a smooth closed subvariety of X.

Step 1. The group homomorphism $\mathrm{K}_0^{\mathrm{sm}}(\mathit{Var}_S) \longrightarrow \mathrm{K}_0(\mathit{Var}_S)$ is an isomorphism.

Proof. We use the same stratification argument as in the absolute case. □

Again we can restrict to smooth connected varieties in the presentation (sm) by decomposing into connected components.

We can also stratify by smooth varieties which are quasi-projective over S.

Now we establish the presentation (bl). Denote by $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_S)$ the free abelian group on isomorphism classes of varieties $[X]_{S,\mathrm{bl}}$, where X is a variety which is proper over S and smooth over k, modulo $[X]_{S,\mathrm{bl}} - [Y]_{S,\mathrm{bl}} = [\mathrm{Bl}_Y X]_{S,\mathrm{bl}} - [E]_{S,\mathrm{bl}}$ for Y a closed smooth subvariety of X and E the exceptional divisor of the blowup of X along Y and $[\emptyset]_{S,\mathrm{bl}} = 0$ (blowing up along Y yields then $[X \sqcup Y]_{S,\mathrm{bl}} = [X]_{S,\mathrm{bl}} + [Y]_{S,\mathrm{bl}}$).

Decomposing into connected components and noting that the blow-up along a disjoint union is the successive blow-up along the connected components one sees that this can also be described as the free abelian group on isomorphism classes $[X]_{S,\mathrm{bl}}$ of smooth complete connected varieties with imposed relations $[\emptyset]_{S,\mathrm{bl}} = 0$ and $[\mathrm{Bl}_Y X]_{S,\mathrm{bl}} - [E]_{S,\mathrm{bl}} = [X]_{S,\mathrm{bl}} - [Y]_{S,\mathrm{bl}}$, where $Y \subset X$ is a closed connected smooth subvariety.

Step 2. The group homomorphism $\mathrm{K}_0^{\mathrm{bl}}(\mathit{Var}_S) \longrightarrow \mathrm{K}_0(\mathit{Var}_S)$ which maps $[X]_{S,\mathrm{bl}}$ to $[X]_S$ is an isomorphism.

Proof. For an arbitrary variety X over S there exist a proper $\overline{X} \to S$ and an open dense immersion $X \hookrightarrow \overline{X}$ over S. If X is smooth over k and connected we can even find $\overline{X}$ smooth with $\overline{X} - X$ a simple normal crossings divisor (by resolution of singularities and principalization). As the weak factorization theorem also works over a base variety (note that a birational map between two irreducible varieties which are proper over S is automatically proper) we can construct an inverse in the same way as in the absolute case. □

Using the fact that we can restrict to generators which are quasi-projective over S in the presentation (sm) and that a connected smooth variety which is quasi-projective over S has a smooth simple normal crossings completion which is projective over S we see that we can restrict to generators projective over S in the description (bl).

6. The six operations

The product of varieties makes $\mathrm{K}_0(\mathit{Var}_S)$ a $\mathrm{K}_0(\mathit{Var}_k)$-module. Let $\mathcal{M}_k$ denote $\mathrm{K}_0(\mathit{Var}_k)[\mathbb{L}^{-1}]$, let $\mathcal{M}_S$ denote the localization $\mathrm{K}_0(\mathit{Var}_S)[\mathbb{L}^{-1}]$. Denote the class $[S]_S$ by $\mathbb{1}_S$.

Definition 6.1. If $f : S \longrightarrow S'$ is a morphism of k-varieties, composition with f yields a $\mathrm{K}_0(\mathit{Var}_k)$-linear mapping $f_! : \mathrm{K}_0(\mathit{Var}_S) \longrightarrow \mathrm{K}_0(\mathit{Var}_{S'})$, hence we get an $\mathcal{M}_k$-linear mapping $f_! : \mathcal{M}_S \longrightarrow \mathcal{M}_{S'}$. Pulling back along f yields a $\mathrm{K}_0(\mathit{Var}_k)$-linear mapping $f^* : \mathrm{K}_0(\mathit{Var}_{S'}) \longrightarrow \mathrm{K}_0(\mathit{Var}_S)$ and hence an $\mathcal{M}_k$-linear mapping $\mathcal{M}_{S'} \longrightarrow \mathcal{M}_S$.

Taking products yields a $\mathrm{K}_0(\mathit{Var}_k)$-bilinear associative exterior product

$$\boxtimes : \mathrm{K}_0(\mathit{Var}_S) \times \mathrm{K}_0(\mathit{Var}_T) \longrightarrow \mathrm{K}_0(\mathit{Var}_{S\times T})$$

and hence an $\mathcal{M}_k$-bilinear associative map

$$\boxtimes : \mathcal{M}_S \times \mathcal{M}_T \longrightarrow \mathcal{M}_{S\times T}.$$

For morphisms $f : S \longrightarrow S'$ and $g : T \longrightarrow T'$ of varieties we get the identities $(f \times g)_!(A \boxtimes B) = f_!(A) \boxtimes g_!(B)$ and $(f \times g)^*(C \boxtimes D) = f^*(C) \boxtimes g^*(D)$. If $p : S \times S' \longrightarrow S$ is the projection to the first factor we get $p^*(A) = A \boxtimes \mathbb{1}_{S'}$ for $A \in M_S$.

Definition 6.2. Taking fiber products yields an internal $\mathrm{K}_0(\mathit{Var}_k)$-bilinear symmetric associative product $\otimes : \mathrm{K}_0(\mathit{Var}_S) \times \mathrm{K}_0(\mathit{Var}_S) \longrightarrow \mathrm{K}_0(\mathit{Var}_S)$ and hence an $\mathcal{M}_k$-bilinear associative map $\otimes : \mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M}_S$

This satisfies $f^*(C \otimes D) = f^*(C) \otimes f^*(D)$ and $\mathbb{1}_S \otimes A = A \otimes \mathbb{1}_S = A$.

For $A \in \mathcal{M}_{S'}$ and $B \in \mathcal{M}_S$ we have $f_!(f^*A \otimes B) = A \otimes f_!B$. In other words, if we regard $\mathcal{M}_S$ as a $\mathcal{M}_{S'}$-module via f^*, then $f_!$ is a $\mathcal{M}_{S'}$-module homomorphism.

For $A, B \in \mathcal{M}_S$ and $C, D \in \mathcal{M}_T$ we get

$$(A \boxtimes C) \otimes (B \boxtimes D) = (A \otimes B) \boxtimes (C \otimes D),$$

hence the exterior product provides $\mathcal{M}_{S\times T}$ with the structure of an $\mathcal{M}_S \otimes \mathcal{M}_T$-algebra.

Definition 6.3. The *duality involution relative to* S is the map defined as follows:

There is a morphism $\mathcal{D}_S : \mathrm{K}_0(\mathit{Var}_S) \longrightarrow \mathcal{M}_S$ which sends a generator $[X]_S$ with X connected and smooth over k, proper over S, to $\mathbb{L}^{-\dim X}[X]_S$. A relative version of lemma 3.5 shows that this is indeed compatible with blow-up relations. For $A \in \mathrm{K}_0(\mathit{Var}_k)$ and $B \in \mathrm{K}_0(\mathit{Var}_S)$ we get $\mathcal{D}_S(AB) = \mathcal{D}_k(A)\mathcal{D}_S(B)$ (this is easily checked on smooth proper generators), so $\mathcal{D}_S$ can be extended to a $\mathcal{D}_k$-linear morphism $\mathcal{D}_S : \mathcal{M}_S \longrightarrow \mathcal{M}_S$.

Indeed $\mathcal{D}_S$ is an involution. We have already noted in section 4 that $\mathcal{D}_k(\mathbb{L}) = \mathbb{L}^{-1}$ and that $\mathcal{D}_k$ is a lift of the duality morphism on the Grothendieck ring of Chow motives. For $A \in \mathcal{M}_S$ and $B \in \mathcal{M}_T$ we get

$$\mathcal{D}_{S\times T}(A \boxtimes B) = \mathcal{D}_S(A) \boxtimes \mathcal{D}_T(B).$$

Definition 6.4. For $f : S \longrightarrow S'$ define $f^! := \mathcal{D}_S f^* \mathcal{D}_{S'}$ and $f_* = \mathcal{D}_{S'} f_! \mathcal{D}_S$.

These are both $\mathcal{M}_k$-linear mappings.

Remark 6.5. If f is a proper mapping then f_* coincides with $f_!$. If f is an open embedding then $f^!$ coincides with f^*. More generally, if f is smooth of relative dimension m, then for $A \in \mathcal{M}_{S'}$ we have $f^*A = \mathbb{L}^m f^! A$.

Proposition 6.6. *Given a cartesian diagram as follows*

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "\pi"'] & T' \arrow[d, "\pi'"] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

we have $g_!\pi^* = \pi'^* f_!$ *and* $g_*\pi^! = \pi'^! f_*$. *If* f *is proper or* π' *is smooth, then* $g_*\pi^* = \pi'^* f_*$.

Proof. The only thing to prove is $g_*\pi^* = \pi'^* f_*$ for π' smooth. So suppose π' is smooth of relative dimension m, hence the same holds for π.

Let $A \in \mathcal{M}_S$. Then due to Remark 6.5

$$\pi'^* f_* A = \mathbb{L}^m \pi'^! f_* A = \mathbb{L}^m g_* \pi^! A g_*(\mathbb{L}^m \pi^! A) = g_* \pi^* A.$$

□

Definition 6.7. Define the *dualizing element* of $\mathcal{M}_S$ by $\mathbb{D}_S := \mathcal{D}_S(\mathbb{1}_S)$. For $A, B \in \mathcal{M}_S$ define

$$\mathrm{Hom}(A,B) := \mathcal{D}_S(A \otimes \mathcal{D}_S B).$$

This yields a mapping $\mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M}_S$ which is $\mathcal{D}_k$-linear in the first and $\mathcal{M}_k$-linear in the second component. For $A \in \mathcal{M}_S$ we get $\mathrm{Hom}(A, \mathbb{D}_S) = \mathcal{D}_S(A)$.

List of properties 6.8. *Let $f : S \longrightarrow S'$ be a morphism of k-varieties. Then the following identities hold:*

$$\mathrm{Hom}(A, f_* B) = f_* \mathrm{Hom}(f^* A, B) \text{ for } A \in \mathcal{M}_{S'} \text{ and } B \in \mathcal{M}_S,$$
$$\mathrm{Hom}(f^* A, f^! B) = f^! \mathrm{Hom}(A, B) \text{ for } A, B \in \mathcal{M}_{S'}.$$
$$f_* \mathrm{Hom}(A, f^! B) = \mathrm{Hom}(f_! A, B) \text{ for } A \in \mathcal{M}_S \text{ and } B \in \mathcal{M}_{S'}.$$

For $A, B, C \in \mathcal{M}_S$ we get

$$\mathrm{Hom}(A, \mathrm{Hom}(B, C)) = \mathrm{Hom}(A \otimes B, C).$$

For $p : S \times S' \longrightarrow S$ the projection to the first factor and for $A, B \in \mathcal{M}_S$ we have

$$\mathrm{Hom}(p^* A, p^* B) = p^* \mathrm{Hom}(A, B).$$

Proof. For we have

$$\begin{aligned}
f_* \mathrm{Hom}(f^* A, B) &= f_* \mathcal{D}_S(f^* A \otimes \mathcal{D}_S B) = \mathcal{D}_{S'} f_!(f^* A \otimes \mathcal{D}_S B) = \mathcal{D}_{S'}(A \otimes f_! \mathcal{D}_S B) \\
&= \mathcal{D}_{S'}(A \otimes \mathcal{D}_S f_* B) = \mathrm{Hom}(A, f_* B), \\
\mathrm{Hom}(f^* A, f^! B) &= \mathcal{D}_S(f^* A \otimes \mathcal{D}_S f^! B) = \mathcal{D}_S(f^* A \otimes f^* \mathcal{D}_{S'} B) = \mathcal{D}_S f^*(A \otimes \mathcal{D}_{S'} B) \\
&= f^! \mathcal{D}_{S'}(A \otimes \mathcal{D}_{S'} B) = f^! \mathrm{Hom}(A, B), \\
f_* \mathrm{Hom}(A, f^! B) &= f_* \mathcal{D}_S(A \otimes \mathcal{D}_S f^! B) = f_* \mathcal{D}_S(A \otimes f^* \mathcal{D}_{S'} B) \\
&= \mathcal{D}_{S'} f_!(A \otimes f^* \mathcal{D}_{S'} B) = \mathcal{D}_{S'}(f_! A \otimes \mathcal{D}_{S'} B) = \mathrm{Hom}(f_! A, B).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Hom}(A, \mathrm{Hom}(B, C)) &= \mathrm{Hom}(A, \mathcal{D}_S(B \otimes \mathcal{D}_S C)) \\
&= \mathcal{D}_S(A \otimes (B \otimes \mathcal{D}_S C)) = \mathrm{Hom}(A \otimes B, C).
\end{aligned}$$

Finally, using $\mathcal{D}_{S \times S'}(C \boxtimes D) = \mathcal{D}_S(C) \boxtimes \mathcal{D}_{S'}(D)$ for $C \in \mathcal{M}_S$ and $D \in \mathcal{M}_{S'}$ we get

$$\begin{aligned}
\mathrm{Hom}(p^* A, p^* B) &= \mathcal{D}_{S \times S'}((A \boxtimes \mathbb{1}_{S'}) \otimes \mathcal{D}_{S \times S'}(B \boxtimes \mathbb{1}_{S'})) \\
&= \mathcal{D}_{S \times S'}((A \boxtimes \mathbb{1}_{S'}) \otimes (\mathcal{D}_S B \boxtimes \mathbb{D}_{S'}) \\
&= \mathcal{D}_{S \times S'}((A \otimes \mathcal{D}_S B) \boxtimes \mathbb{D}_{S'}) = \mathcal{D}_S(A \otimes \mathcal{D}_S B) \boxtimes \mathcal{D}_{S'}(\mathbb{D}_{S'}) \\
&= \mathrm{Hom}(A, B) \boxtimes \mathbb{1}_{S'} = p^* \mathrm{Hom}(A, B).
\end{aligned}$$

$\square$

7. Equivariant Grothendieck groups of varieties

Let G be a finite group. There are also G-equivariant versions of Grothendieck groups of varieties:

Let $\mathrm{K'}_0^G(\mathit{Var}_k)$ be the free abelian group on G-isomorphism classes $[G \circlearrowright X]$ (or shorthand $[X]$) of varieties X with a good G-action, divided out by the relations $[X] = [X - Y] + [Y]$ for $Y \subset X$ a closed G-invariant subvariety. The product of varieties (with diagonal G-action) induces a commutative ring structure on $\mathrm{K'}_0^G(\mathit{Var}_k)$.

Now define $\mathrm{K}_0^G(Var_k)$ to be $\mathrm{K'}_0^G(Var_k)$ modulo the ideal I^G generated by expressions of the form $[G \circlearrowright \mathbb{P}(V)] - [G \text{ trivial} \circlearrowright \mathbb{P}(V)]$, where V is a k-vector space on which G acts linearly, $\mathbb{P}(V)$ denotes its projectivization and $[G \text{ trivial} \circlearrowright \mathbb{P}(V)]$ denotes the class of $\mathbb{P}(V)$ with the *trivial* G-action (this is motivated by the fact that on cohomology such an operation is not visible). Note that as the class $[G \circlearrowright V]$ is equal to $[G \circlearrowright \mathbb{P}(V \oplus k)] - [G \circlearrowright \mathbb{P}(V)]$ this implies that $[G \circlearrowright V] - [G \text{ trivial} \circlearrowright V]$ in $\mathrm{K}_0^G(Var_k)$.

Lemma 7.1. *The group $\mathrm{K'}_0^G(Var_k)$ has the following alternative presentations:*

(sm) *As the abelian group generated by the isomorphism classes of smooth varieties with good G-action subject to the relations $[X] = [X - Y] + [Y]$, where X is smooth with good G-action and $Y \subset X$ is a smooth G-invariant closed subvariety,*

(bl) *as the abelian group generated by the isomorphism classes of smooth complete varieties with good G-action subject to the relations $[\emptyset] = 0$ and $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$, where X is a smooth complete variety with good G-action, $Y \subset X$ is a closed smooth G-invariant subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Moreover, we get the same group if in case (sm) we restrict to quasi-projective varieties or if in case (bl) we restrict to projective varieties. We can also restrict to varieties such that G acts transitively on the connected components in both presentations.

Proof. Stratifying by smooth G-invariant equidimensional varieties establishes the presentation (sm). We can also stratify by quasi-projective G-invariant varieties. We can decompose further into smooth varieties such that G acts transitively on the connected components.

Now note that a smooth connected variety X with a good G-action has a good smooth completion with as complement a G-invariant simple normal crossings divisor: Take a completion $\overline{X/G}$ of X/G and take the normalization $\widetilde{X}$ of $\overline{X/G}$ in X. This is finite over $\overline{X/G}$, so the induced G-operation is automatically good. Furthermore $\widetilde{X}$ is complete and contains X as an open dense subvariety. An equivariant resolution of singularities of $\widetilde{X}$ making the complement of X a simple normal crossings divisor then gives a completion $\overline{X}$ with the desired properties. Actually as in remark 2.7 it suffices that G acts transitively on the connected components of X. The same reasoning as before then establishes the presentation (bl). Note that in the equivariant version of the factorization theorem the centers of the blow-ups need not be connected, but we can assume that G acts transitively on the connected components. Restricting to quasi-projective varieties in the presentation (sm) shows that we can restrict to projective varieties in (bl) (here we can take any smooth projective equivariant simple normal crossings completion, as on a projective variety a G-action is automatically good). □

Summarizing we get:

Theorem 7.2. *The ring $\mathrm{K}_0^G(Var_k)$ is the free abelian group on smooth projective (respectively, complete) varieties with good G-action (transitive on the connected components), with the product described above, modulo blow-up relations and the ideal I^G generated by expressions of the form $[G \circlearrowright \mathbb{P}(V)] - [G \text{ trivial} \circlearrowright \mathbb{P}(V)]$.*

We can also consider varieties over a base variety S with good G-action over S (by this we mean that $X \to S$ is G-equivariant where we give S the trivial G-action). Denote by $\mathrm{K}'^G_0(\mathit{Var}_S)$ the free group on isomorphism classes $[X]_S$ of those modulo relations for closed subvarieties. It has the structure of a $\mathrm{K}'^G_0(\mathit{Var}_k)$-module provided by the product of varieties with diagonal G-action. We define $\mathrm{K}^G_0(\mathit{Var}_S)$ to be $\mathrm{K}'^G_0(\mathit{Var}_S)$ modulo the submodule $I^G\,\mathrm{K}'^G_0(\mathit{Var}_S)$ (so in particular it is a $\mathrm{K}^G_0(\mathit{Var}_k)$-module). By the same reasoning as above we get:

Lemma 7.3. *The group $\mathrm{K}'^G_0(\mathit{Var}_S)$ has the following alternative presentations:*

(sm) *As the abelian group generated by the isomorphism classes of S-varieties, smooth over k, with a good G-action over S, subject to the relations $[X]_S = [X-Y]_S + [Y]_S$, where X is smooth with a good G-action over S and $Y \subset X$ is a smooth closed G-invariant subvariety,*

(bl) *as the abelian group generated by the isomorphism classes of S-varieties with good G-action over S which are smooth over k and proper over S subject to the relations $[\emptyset]_S = 0$ and $[\mathrm{Bl}_Y X]_S - [E]_S = [X]_S - [Y]_S$, where X is smooth over k and proper over S and carries a good G-action over S, $Y \subset X$ is a closed smooth G-invariant subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Moreover, we get the same group if in case (sm) we restrict to varieties which are quasi-projective over S or if in case (bl) we restrict to varieties which are projective over S. We can also restrict to varieties such that G acts transitively on the connected components in both presentations.

Theorem 7.4. *The group $\mathrm{K}^G_0(\mathit{Var}_S)$ is the free abelian group on smooth varieties, projective (respectively, proper) over S with (good) G-action over S (transitive on the connected components), modulo blow-up relations, divided out by the $\mathrm{K}'^G_0(\mathit{Var}_k)$-submodule $I^G\,\mathrm{K}'^G_0(\mathit{Var}_S)$.*

References

[1] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk. Torification and factorization of birational maps. arXiv:math.AG/9904135.

[2] L. Bonavero. Factorisation faible des applications birationelles [d'après Abramovich, Karu, Matsuki, Włodarczyk et Morelli]. *Séminaire Bourbaki*, 2000–2001(880), 2000. arXiv:math.AG/0011043.

[3] J. Denef and F. Loeser. Geometry on arc spaces of algebraic varieties. To appear in Proceedings of 3rd ECM, Barcelona, July 10–14, 2000. arXiv:math.AG/0006050.

[4] H. Gillet and C. Soulé. Descent, motives and K-theory. *J. Reine Angew. Mathematik*, 478:127–176, 1996.

[5] F. Guillen and V. Navarro Aznar. Un critère d'extension d'un foncteur défini sur les schémas lisses. arXiv:math.AG/9505008.

[6] E. Looijenga. Motivic measures. *Séminaire Bourbaki*, 1999–2000(874), 2000. arXiv:math.AG/0006220.

[7] A. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55, I of *Proc. Sympos. Pure Math.*, pages 163–187, Providence, RI, 1994. Amer. Math. Soc.

[8] J. Włodarczyk. Toroidal varieties and the weak factorization theorem. arXiv:math.AG/9904076.

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