On Hecke-Kiselman algebras

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Hecke-Kiselman monoid

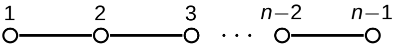
Definition (Ganyushkin, Mazorchuk, 2011)

For any simple graph Θ of vertices $\{1, \ldots, n\}$ the monoid HK_Θ generated by $a_1, \ldots, a_n$ is defined by the following relations:

- $a_i^2 = a_i$, for all $i = 1, \ldots, n$,
- $a_i a_j = a_j a_i$, when i, j are not connected in Θ,
- $a_i a_j a_i = a_j a_i a_j$, when $i - j$ in Θ,
- $a_i a_j a_i = a_j a_i a_j = a_i a_j$, when $i \to j$ in Θ,

The oriented graphs case

Consider an unoriented chain Θ of $n-1$ vertices



HK_Θ is **the Coxeter monoid of the symmetric group S_n.**
It is generated by s_i, $1 \leq i \leq n-1$ such that:

$$\begin{aligned} s_i^2 &= s_i, && \text{for all } 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

An important fact (non-trivial!): $|S_n| = |\,\mathrm{HK}_\Theta\,|$.

Hecke algebras

Remainder

Let Θ be an unoriented graph with n vertices, K - a field, $q \in$ K. The **Iwahori-Hecke algebra** is generated by $S_1, \ldots, S_n$ with:

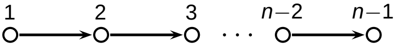
$$S_i^2 = q + (q-1)S_i, \quad \underbrace{S_iS_jS_i\ldots}_{m_{ij}} = \underbrace{S_jS_iS_j\ldots}_{m_{ij}}.$$

where $m_{ij} - 2$ denotes the number of arrows between i and j.

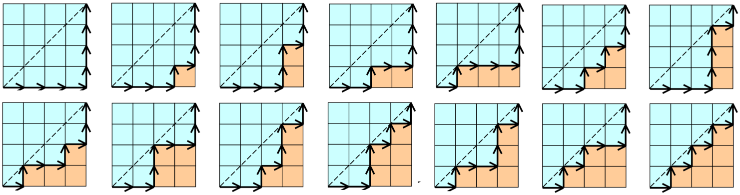
- For $q = 1$ this is just the group algebra K$[W]$ of a Coxeter group W_Θ, constructed from the graph Θ,
- for $q = 0$ this is a semigroup algebra of HK_Θ, if Θ is simple.

The oriented graphs case: the chain

Consider an oriented chain Θ of $n-1$ vertices.



HK_Θ is **the Catalan monoid** C_n of the order-preserving, weakly-increasing self maps of $\{1, \ldots, n\}$ represented by monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. For n=4:



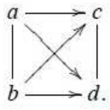
Questions about Hecke-Kiselman monoids

Open problems:

- The word problem.
- For which Θ is HK_Θ finite?
- Is there a faithful representation of HK_Θ in $M_{n \times n}(\mathbb{N})$?
- The ideal structure in the monoids. Are they $\mathcal{J}$-trivial?

The finiteness problem for HK_Θ

- **Tsaranov (1990).** When Θ is unoriented, then HK_Θ is finite if and only if Θ is a disjoint union of Dynkin diagrams.
- **Aragona, D'Andrea (2012)** When Θ is oriented, then HK_Θ is finite if and only if Θ is acyclic.
- **Theorem (Aragona, D'Andrea 2012).** Let Θ be acyclic mixed graph. If $|V(\Theta)| = 3$ then HK_Θ is finite. If $V(\Theta)| = 4$ then HK_Θ is finite, except for the case when Θ is of form:



Questions about Hecke-Kiselman algebras

Open problems:

- Is the Gröbner basis finite? Is $K[HK_\Theta]$ automaton?
- What is the Gelfand-Kirillov dimension of $K[HK_\Theta]$?
- Is there a faithful representation of $K[HK_\Theta]$ in $M_{n\times n}(\mathbb{K})$?
- The ideal structure in the algebra. Is it a cellular algebra?

The key problem - how to reduce?

The word problem.

Let S be a finitely generated semigroup $\langle X \rangle / I$. Let $v, w \in \langle X \rangle$. Find an algorithm to decide if $v = w$ in S.

- In general - this is impossible (Tsejtin, 1958).
- Do we reduce only with the defining relations? No!
- Problem: reduce to obtain smaller deg-lex words.
- How many reductions do we need to add? Finitely many?
- What if we change the presentation and the monomial order? Can the problem become decidable?

Normal words, reduced words

Let $\pi : \mathrm{K}\langle X \rangle \longrightarrow \mathrm{K}\langle X \rangle / I$ be a natural map of finitely generated semigroup algebras. Let $<$ be a monomial order on $\langle X \rangle$.

- an element $w \in \langle X \rangle$ is **normal**, if w is not a leading term in $\ker(\pi)$,
- a set G is a **Gröbner basis** of I, if no normal word has a factor being a leading term in G.

If T is a set of reductions of form (w, w') where $w > w'$ on $\langle X \rangle$, then w is T-reduced if no reduction from T can be applied.

How to choose T such that T-reduced words $\equiv$ normal words? When is $\{w - w' \mid, (w, w') \in T\}$ a Gröbner basis?

Use: **Bergman's Diamond Lemma**.

The Gröbner basis for an oriented graph

Theorem (Okniński, M. (2018))

Let Θ be a finite simple oriented graph with the vertex set $V(\Theta)$. Consider a deg-lex order on the free monoid $F = \langle V(\Theta) \rangle$.

Consider the following set T of reductions on the algebra $k[F]$:

(i) (twt, tw), *for any* $t \in V(\Theta)$ *and* $w \in F$ *such that* $w \nrightarrow t$,

(ii) (twt, wt), *for any* $t \in V(\Theta)$ *and* $w \in F$ *such that* $t \nrightarrow w$,

(iii) $(t_1 w t_2, t_2 t_1 w)$, *for any* $t_1, t_2 \in V(\Theta)$ *and* $w \in F$ *such that* $t_1 > t_2$ *and* $t_2 \nleftrightarrow t_1 w$.

Then the set $T(F)$ of T-reduced words and the set of normal words of $\mathrm{K}[\mathrm{HK}_\Theta]$ are equal. Moreover, as linear spaces:

$$\mathrm{K}[F] = \mathrm{K}[I] \oplus \mathrm{lin}_{\mathrm{K}}(T(F)).$$

A few remarks

- If we choose a „better presentation” and a „better order"’, perhaps a finite Gröbner basis can be obtained?
- If Θ is simple oriented, then the set of normal words of HK_Θ forms a regular language: it is obtained from a finite subset of F by applying a finite sequence of operations of union, multiplication and operation $*$ defined by

$$T^* = \bigcup_{i \geq 1} T^i, \text{ for } T \subseteq F.$$

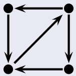
- Corollary: The Hecke-Kiselman algebra is automaton!

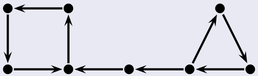
The growth of HK_Θ

Theorem (M. Okniński, 2017)

Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.

(1) The graph Θ does not contain two different cycles connected by an oriented path of length ≥ 0, for instance





(2) the monoid algebra $K[HK_\Theta]$ satisfies a polynomial identity,

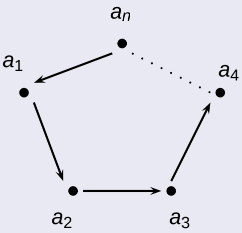
(3) $\mathrm{GKdim}(K[HK_\Theta]) < \infty$,

(4) HK_Θ does not contain a free submonoid of rank 2.

The oriented cycle case

Theorem (Okniński, M., 2018)

Consider an oriented cycle Θ_n with n vertices, where $n \geq 4$:



Then:

- *the Gröbner basis of* HK_{Θ_n} *is finite,*
- $\mathrm{GKdim}(K[\mathrm{HK}_{\Theta_n}]) = 1$,

Open problem: Is $K[\mathrm{HK}_{\Theta_n}]$ semiprimitive? Is it semiprime?

The Noetherianity criterion

Theorem (Okniński, Wiertel, 2018)

Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.

(1) $K[HK_\Theta]$ *is right Noetherian,*

(2) $K[HK_\Theta]$ *is left Noetherian,*

(3) *each of the connected components of Θ is either an oriented cycle or an acyclic graph.*

References

(1) Ganyushkin O., Mazorchuk V.: *On Kiselman quotients of* 0*-Hecke Monoids*. Int. Electron. J. Algebra 10 (2) (2011), 174-191.

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(3) A. Męcel, J. Okniński: *Gröbner basis and the automaton property of Hecke-Kiselman algebras*, to appear in Semigroup Forum (2019).

(4) J. Okniński, M. Wiertel: *Combinatorics and structure of Hecke-Kiselman algebras*, preprint, arXiv: 1904.12202.