On Hecke-Kiselman algebras

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This talk concerns a certain class of semigroup algebras that are being under research in the Algebra Group at the University of Warsaw for more than two years now by Jan Okniński, Łukasz Kubat (before leaving), Magdalena Wiertel and myself. First, I will introduce those monoids and give some background, as their definition covers a number of well known objects and there are natural yet non-trivial questions to be considered. Then I will talk about the problem of determining the Gröbner basis of the semigroup algebras of Hecke-Kiselman monoids, a procedure allowing us to work in the combinatorial setting of the language of normal words, reduced polynomials and automata. Lastly, I will recall some results that were obtained on these algebras and state some question that are yet to be answered.

1 Hecke-Kiselman monoids

Consider a finite graph Θ with n vertices $\{1, \dots, n\}$. We assume that these vertices can be connected by unoriented edges and also by oriented arrows, but for each pair of vertices there is at most one edge or arrow that connects them. This is called a finite simple digraph. For such graph Θ we define a monoid HK_Θ generated by n elements $x_1, \dots, x_n$ that are related by the following conditions.

(i) $x_i^2 = x_i$, where $1 \leqslant i \leqslant n$,

(ii) if the vertices i, j are not connected in Θ, then $x_i x_j = x_j x_i$,

(iii) if i, j are connected by an arrow $i \to j$ in Θ, then $x_i x_j x_i = x_j x_i x_j = x_i x_j$,

(iv) if i, j are connected by an (unoriented) edge in Θ, then $x_i x_j x_i = x_j x_i x_j$.

This presentation yields the Hecke-Kiselman monoid HK_Θ of a graph Θ. The monoid algebra of $k[\mathrm{HK}_\Theta]$ will be denoted as A_Θ. The ground field is irrelevant in our considerations.

The notion of HK_Θ was introduced by Ganyushkin and Mazorchuk in 2011 as an attempt to generalize a class of Kiselman monoids and their quotients, which corresponds to the case when the graph Θ is oriented. I will go into details in a moment. To put it briefly, this new object was considered with a combinatorial motiviations and approach. One aim of that first paper was to decide if two non-isomorphic graphs can yield two isomorphic Hecke-Kiselman monoids, which proved impossible. Certain important examples were also considered. Some very natural questions were stated, that are still unanswered in full generality:

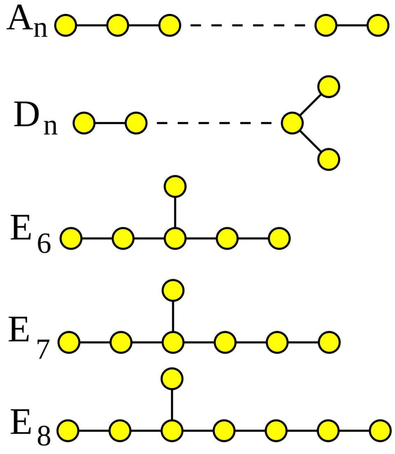
- For which graphs Θ is HK_Θ finite?
- Are there any faithful representations of HK_Θ into matrices (preferably over $\mathbb{Z}$)?
- What is the ideal structure of HK_Θ, for instance, can two principal two-sided ideals generated by two different elements can be equal?

If Θ is an unoriented graph, then the most bizarre-looking relation of type (iii) is omitted, and we obtain a monoid, whose presentation resembles, in many ways, a presentation of a symmetric group. If we actually assumed Θ to be a chain of $n-1$ vertices and if we generated a group of $n-1$ generators that satisfy (ii), (iv), along with a modified relation (i) of form $x_i^2 = 1$, then we would obtain exactly a presentation of the symmetric group S_n. This is no coincidence, as for every simple unoriented graph Θ of vertices $\{1, 2, \dots, n\}$ there exists a corresponding finitely generated Coxeter group W_Θ of presentation:

$$(1)\, x_i^2 = 1, \quad (2)\, (x_i x_j)^2 = 1, \quad (3)\, (x_i x_j)^3 = 1,$$

where the condition (1) is satisfied by all generators, (2) is satisfied by those generators which correspond to non-connected vertices Θ and (3) is satisfied for generators corresponding to connected vertices $i - j$. It is obvious, that via group manipulations of equalities (1)-(3) one can obtain a presentation that resembles that of a Hecke-Kiselman monoid in a way described above. These are called simply laced Coxeter groups, as such groups can be formed also for graphs with multiple connections between vertices of an unoriented graph. These groups describe symmetries in the real space and appear in many other algebraic settings.

To make the resemblance between the Coxeter group W_Θ and a Hecke-Kiselman monoid HK_Θ clear we should mention that the latter in the unoriented case is called a 0-Hecke monoid, or a Coxeter monoid. This name comes from the representation theory, and originates from the modular representation theory of Coxeter groups, in particular a symmetric group. In essence, this is a specialization of the so called Hecke-Iwahori algebra $H_0(W_\Theta, v)$, a unital algebra generated by generators T_i such that are related by braid relations (they correspond to the ones for W_Θ) and the relation of kind (i) is modified to the form $(T_i - q)(T_i + 1) = 0$, where q is an indeterminate. Roughly speaking, when $q = 1$ this is just a group algebra of a Coxeter group, and when $q = 0$ this is exactly a semigroup algebra of our Hecke-Kiselman monoid. The last important bit of information is that the dimension of the Iwahori-Hecke algebra is the same as the number of elements in the Coxeter group, no matter the specialization of indeterminate q. Therefore, it can be proved, that there is a bijection between the Coxeter group, and the associated Coxeter monoid. It is a very strong one as the reduced forms of words in these two objects are completely the same. This yields, for instance, that the finiteness problem for Hecke-Kiselman monoids in the unoriented case is in fact equivalent to the finiteness problem for Coxeter groups, and it is thus solved years ago. The graph, for which this monoid is finite are exactly the simply laced Dynkin diagrams.



In the oriented case, when we do not allow edges in the graph Θ, one obtains a completely different object, of a more combinatorial flavour. The name of Kiselman appears here, and I would assume that it honors Christer Kiselman, a professor emeritus in Uppsala. Kiselman, a specialist in analysis and convexity theory, wrote a paper in 2002 called "A semigroup of operators in convexity theory", concerning family of convex functions in a real normed space. He proved that certain family of closure operators called the convex hull, largest lower semicontinous minorant and the proper function checking operator, represented as three elements c, l, m and forms a monoid that has a presentation

$$c^2 = c, l^2 = l, m^2 = m, clc = lcl = lc, cmc = mcm = mc, lml = mlm = ml$$

so it involves relations (i)-(iii). Later, other authors from Uppsala took this monoid and started to investigate it in the case of n generators. Mazorchuk, who works in Uppsala since 2001, was one of them. He found out the graph setting in which he could realize it as a semigroup defined in the beginning for a graph Θ of n vertices such that for every $i > j$ there is an arrow $i \to j$. This way the Kiselman monoid was put into one family with one famous example of monoid is obtained when we consider an oriented chain in $n-1$ vertices and obtain the so-called Catalan C_n monoid of all order-preserving, weakly increasing self-maps f of $\{1, \ldots, n+1\}$. It is well known that the cardinality of C_n is the n-th Catalan number. The representation theory of this monoid is very interesting, especially since it is know, that its semigroup algebra is an incidence algebra of a certain natural poset. Both Kiselman and Catalan monoids are examples of finite Hecke-Kiselman monoids in the oriented case.

It is, by no means completely trivial to say when in the oriented Θ case, is such semigroup HK_Θ finite. Even if we take an oriented cycle X_n of n-elements, which can be called an affine Catalan monoid, it does not seem immediately obvious from the defining relations how to prove that it is infinite. If this cycle is of form $x_1 \to x_2 \to \ldots \to x_n \to x_1$ then an obvious candidate for an infinite-order element is $x_1 \ldots x_n$, but how to prove this. In general, a method of the Gröbner bases will prove appropriate for this. For over a year, however, we did not realize that such general method can be applied and proved (in quite a forceful way) a number of results on the growth and the Gelfand-Kirillov dimension of such algebras. Before I mention these results and recall the Grobner bases setting, let me give you an example of some other approaches. I will show you one tricky method that involves some kind of representation of the oriented cycle monoid. It was used to show that the finite oriented Hecke-Kiselman monoids are exactly the ones that follow from acyclic graphs.

Remark 1. *Let Θ be an oriented simple graph. Then HK_Θ is finite if and only if Θ is acyclic.*

Proof Denote the oriented cycle of n-elements by X_n. Let $M(\mathbb{Z}^n)$ stand for the collection of all maps $f : \mathbb{Z}^n \to \mathbb{Z}^n$, which is a semigroup under composition. We construct a semigroup homomorphism

$$\rho : \mathrm{HK}_{X_n} \to M(\mathbb{Z}^n)$$

and show that the image is infinite. Notice that due to the presentation of HK_{X_n}, a map ρ is given as soon as we choose images $u_i = \rho(x_i)$, when x_i are consecutive elements of the cycle (see above), that are bound to satisfy some relations. Let u_i be defined as follows:

$$u_i(m_1, \ldots, m_n) = (m_1, \ldots, m_{i-1}, m_{i+1}, m_{i+1}, \ldots, m_n), \text{ for } i = 1, \ldots, n-1,$$
$$u_n(m_1, \ldots, m_n) = (m_1, \ldots, m_{n-1}, m_1 + 1).$$

A straightforward check shows that u_i satisfy the defining relations (i)-(iii). However,

$$(u_1 \dots u_n)(m_1, \dots, m_n) = (m_1 + 1, m_1 + 1, \dots, m_1 + 1).$$

All powers of $u_1 \dots u_n$ are therefore distinct. So the image of ρ is infinite and hence HK_{X_n} too.

We proceed to the actual proof. If Θ contains a cycle Θ', then there is an obvious monoid monomorphism $\mathrm{HK}_{\Theta'} \to \mathrm{HK}_{\Theta}$ and HK_{Θ} is therefore infinite. The other way around we argue by an inductive argument on the number of vertices of Θ. If there is only one vertex, the semigroup HK_{Θ} is clearly finite. Otherwise, since Θ is acyclic there exists a terminal vertex x in Θ ie. such that there are only incoming or only outcoming arrows that are incicent to x. Assume the first case. It is easy to show that every word of form xwx, where w is a word in the free monoid generated by the vertices of Θ, can be reduced, via the relations in HK_{Θ} to xw. Therefore every element of HK_{Θ} can be represented by a word which allows only one appearance of x. Thus from the induction hypothesis there are finitely many such words. So HK_{Θ} is infinite. The second case is resolved instantaneously.

2 Gröbner bases

The use of Gröbner bases is a standard method of checking if a certain polynomial belongs to an ideal generated by a finite numbers of polynomials. The method is to transform the polynomial generators to a set of „nicer generators" by introducing some kind of ordering on the monomials and by some kind of division algorithm. As a result: if the leading elements of a probed polynomial divides a leading polynomial of those "nice generators" this means that our candidate indeed belongs to the ideal. This form of approach is especially fruitful in the commutative case, as shown by Buchberger in 1965, there is an algorithm of obtaining such a nice set of generators for an ideal I in commutative polynomials in finitely many variables in finitely many steps. In the noncommutative settings things do not work that smooth as the procedure for obtaining the so-called Gröbner basis of an ideal may not terminate after finitely many steps. But if we obtain a basis somehow (finite or not), there is a way to retrieve a particularly elegant basis of a quotient algebra. The method of obtaining such basis requires the use of the so-called diamond lemma. Let us recall some details.

Let F denote the free monoid on the set X of $n \geqslant 3$ free generators $x_1, \dots, x_n$. Let k be a field and let $k[F] = k\langle x_1, \dots, x_n \rangle$ denote the corresponding free algebra over k. Assume that a well order $<$ is fixed on X and consider the induced degree-lexicographical order on F (also denoted by $<$). Let A be a finitely generated algebra over k with a set of generators $r_1, \dots, r_n$ and let $\pi : k[F] \to A$ be the natural homomorphism of k-algebras with $\pi(x_i) = r_i$. We will assume that $\ker(\pi)$ is spanned by elements of the form $w - v$, where $w, v \in F$ (in other words, A is a semigroup algebra). Let I be the ideal of F consisting of all leading monomials of $\ker(\pi)$. The set of normal words corresponding to the chosen presentation for A and to the chosen order on F is defined by $N(A) = F \setminus I$.

A subset G of an ideal $J = \ker(\pi)$ is called a Gröbner basis of J (or of $A = k[F]/J$) if $0 \notin G$, J is generated by G as an ideal and for every nonzero $f \in J$ there exists $g \in G$ such that the leading monomial $\overline{g} \in F$ of g is a factor of the leading monomial $\overline{f}$ of f. A word $w \in F$ is normal if and only if w has no factors that are leading monomials in $g \in G$.

The so-called diamond lemma, or as others say: a composition-lemma is often used in this context. By a reduction in $k[F]$ determined by a pair $(w, w') \in F^2$, where $w' < w$ (the deg-lex order of F), we mean any operation of replacing a factor w in a word $f \in F$ by the factor w'. For a set $T \subseteq F^2$ of such pairs (these pairs will be called reductions as well) we say that the word $f \in F$ is T-reduced if no factor of f is the leading term w of a reduction (w, w') from the set T. The deg-lex order on F satisfies the descending chain condition, which means there is no infinite decreasing chain of elements in F. This means that a T-reduced form of a word $w \in F$ can always be obtained in a finite series of steps. The linear space spanned by T-reduced monomials in $k[F]$ is denoted by $R(T)$.

The diamond lemma gives necessary and sufficient conditions for the set $N(A)$ of normal words to coincide with the set of T-reduced words in F. The key tool is the notion of ambiguity. Let $\sigma = (w_\sigma, v_\sigma)$, $\tau = (w_\tau, v_\tau)$ be reductions in T. By an overlap ambiguity we mean a quintuple (σ, τ, l, w, r), where $1 \neq l, w, r \in F$ are such that $w_\sigma = wr$ and $w_\tau = lw$. A quintuple (σ, τ, l, w, r) is called an inclusive ambiguity if $w_\sigma = w$ and $w_\tau = lwr$. For brevity we will denote these ambiguities as $l(wr) = (lw)r$ and $l(w)r = (lwr)$, respectively. We will also say that they are of type σ-τ. We say that the overlap (inclusive, respectively) ambiguity is resolvable if $v_\tau r$ and lv_σ (v_τ and $lv_\sigma r$, respectively) have equal T-reduced forms. Recall the following simplified version of Bergman's diamond lemma.

Lemma 2. *Let T be a reduction set in the free algebra $k[F]$ over a field k, with a fixed deg-lex order in the free monoid F over X. Then the following conditions are equivalent:*

- *all ambiguities on T are resolvable,*
- *each monomial $f \in F$ can be uniquely T-reduced,*
- *if $I(T)$ denotes the ideal of $k[F]$ generated by $\{w - v : (w, v) \in T\}$ then $k[F] = I(T) \oplus R(T)$ as vector spaces.*

Moreover if the conditions above are satisfied then the k-algebra $A = k[F]/I(T)$ can be identified with $R(T)$ equipped with a k-algebra structure with $f \cdot g$ defined as the T-reduced form of fg, for $f, g \in R(T)$. In this case, $\{w - v : (w, v) \in T\}$ is a Gröbner basis of A.

3 Some results and open questions

The use of diamond-lemma involves somehow-guessing the appropriate reduction set for an algebra, even if it can be derived algorithmically (indeed, we use some packages of GAP to do certain computations) This was successful in the case of Hecke-Kiselman algebras of oriented graphs, as we were able to prove:

Theorem 3 (Okniński, M. (2018)). *Let Θ be a finite simple oriented graph with vertices $V(\Theta) = \{x_1, x_2, \ldots, x_n\}$. Extend the natural ordering $x_1 < x_2 < \cdots < x_n$ on the set $V(\Theta)$ to the deg-lex order on the free monoid $F = \langle V(\Theta)\rangle$. Consider the following set T of reductions on the algebra $k[F]$:*

(i) (twt, tw), *for any $t \in V(\Theta)$ and $w \in F$ such that $w \not\twoheadrightarrow t$,*

(ii) (twt, wt), *for any $t \in V(\Theta)$ and $w \in F$ such that $t \not\twoheadrightarrow w$,*

(iii) (t_1wt_2, t_2t_1w), *for any $t_1, t_2 \in V(\Theta)$ and $w \in F$ such that $t_1 > t_2$ and $t_2 \not\twoheadleftarrow\!\!\!\twoheadrightarrow t_1w$.*

where for $t \in V(\Theta)$ and $w \in F = \langle V(\Theta) \rangle$ we write $w \nrightarrow t$ if $t \notin \operatorname{supp}(w)$ and there are no $x \in \operatorname{supp}(w)$ such that $x \to t$ in Θ. Similarly, we define $t \nrightarrow w$: again we assume that $t \notin \operatorname{supp}(w)$ and there is no arrow $t \to y$, where $y \in \operatorname{supp}(w)$. Moreover, when $t \nrightarrow w$ and $w \nrightarrow t$, we write $t \nleftrightarrow w$. Then the set $\{w - v,\ \text{where } (w, v) \in T\}$ forms a Gröbner basis of the algebra $k[\mathrm{HK}_\Theta]$.

In other words, if a word in F does not contain a factor (a block of letters) of form twt, as in (i) or (ii) or of form $t_1 w t_2$, then it is reduced. And all elements of HK_Θ can be obtained in that way, as T-reduced elements.

This result, allows a number of corollaries, some of which were obtained in the earlier works. Here are some of them:

Theorem 4 (Okninski, M. (2017))**.** *Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.*

(1) Θ does not contain two different cycles connected by an oriented path of length $\geqslant 0$,

(2) A_Θ is a PI-algebra,

(3) $\mathrm{GKdim}(A_\Theta) < \infty$,

(4) the monoid HK_Θ does not contain a free submonoid of rank 2.

Theorem 5 (Okninski, Wiertel (2018))**.** *Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.*

(1) A_Θ is right Noetherian,

(2) A_Θ is left Noetherian,

(3) each of the connected components of Θ is either an oriented cycle or an acyclic graph.

Corollary 6. *The Hecke-Kiselman algebra of an oriented cycle embedds into the matrix algebra over a field.*

These results involved a careful considerations on the oriented cycle graph monoid HK_{X_n}, mentioned before. Its semigroup algebra has a lot of interesting properties. It can be proved that its growth is linear, that is, its Gelfand-Kirillov dimension is 1. Moreover, its Gröbner basis is finite, which is a non-trivial corollary from the description of the Gröbner basis in the general oriented case. With great computational effort of Okniński and Wiertel it was proved, that the cycle monoid allows a chain of ideals whose factors are co-finite with the generalized semigroups of matrix types, known in the classical theory of representations of semigroups. There is no place to comment on how are the ideals of the Hecke-Kilesman algebras studied, but some methods require representation ideas, as the one used before in the proof of the finiteness of the acyclic graph monoid, some require the use of graph automorphisms that yield some word-invariant operations in the monoid.

Another result that followed from those main theorem concerned the automaton property of Hecke-Kiselman monoids - an especially elegant property for a finitely generated algebra.

Theorem 7 (Okninski, M. (2018))**.** *Assume that Θ is a finite oriented simple graph. Then the algebra A_Θ is automaton, that is: the set $N(A)$ of normal words of A is obtained from a finite subset of F by applying a finite sequence of operations of union, multiplication and operation $*$ defined by $T^* = \bigcup_{i \geqslant 1} T^i$, for $T \subseteq F$.*

In other words, and this is famous Kleene theorem, the set of normal words of A is determined by a finite deterministic automaton.

The class of automaton algebras was introduced by Ufnarovskii in the late 80'. The main motivation was to study a class of finitely generated algebras that generalizes the class of algebras that admit a finite Gröbner basis with respect to some choice of generators and an ordering on monomials. The difficulty here lies in the fact that there are infinitely many generating sets as well as infinitely many admissible orderings on monomials to deal with. There are examples of algebras with finite Gröbner bases with respect to one ordering, and infinite bases with respect to the other. Up until recently it was not known whether for any of known examples of automaton algebras with infinite Gröbner bases with respect to certain orderings one could find a better ordering that would yield a finite Gröbner basis. First counterexamples were found by Iyudu and Shkarin.

There are many results indicating that the class of automaton algebras not only has better computational properties but also several structural properties that are better than in the class of arbitrary finitely generated algebras. For example, in this context one can refer to results on the Gelfand-Kirillov dimension, results on the radical in the case of monomial automaton algebras by Ufnarovskii, results on prime algebras of this type by Jason Bell, and also structural results concerned with the special case of finitely presented monomial algebras (Okniński). In particular, finitely generated algebras of the following types are automaton: commutative algebras, algebras defined by not more than two quadratic relations, algebras for which all the defining relations have the form $[x_i x_j] = 0$, for some pairs of generators. Moreover, algebras that are finite modules over commutative finitely generated subalgebras are also of this type (Cedo, Okniński).

In our setting there are many examples of algebras with infinite Gröbner basis. Attach even one outgoing arrow to a cycle, and you will obtain an algebra of Gelfand-Kirillov dimension 2, with an infinite Gröbner basis. As there are still many open question on the structure of Hecke-Kiselman algebras, even the ones obtained from the oriented cycle, for instance: what are the prime algebras, semiprime, semiprimitive, are the Hecke-Kiselman algebras cellular?

There is also a context of mixed graphs Θ and the corresponding Hecke-Kiselman algebras. Here all the questions raised in the beginning are still unanswered. What are finite Hecke-Kiselman monoids? Are they J-trivial? Are there any representations that would prove useful?

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