On Hecke-Kiselman monoids

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Simple digraphs and anti-reflexive relations

Definition

By a **simple digraph** we mean a directed graph $\Theta = (V, E)$ without loops and multiple arrows, where $V = \{1, \dots, n\}$ and

- if $(i,j) \in E$ and $(j,i) \in E$ we say that (i,j) is an **edge** between i and j, denoted as $i - j$,
- if $(i,j) \in E$ and $(j,i) \notin E$ we say that (i,j) is an **arrow** from i to j, denoted as $i \to j$.

There is a bijection between the set $\mathcal{M}_n$ of all simple digraphs (V, E), where $V = \{1, \dots, n\}$ and the set of all anti-reflexive binary relations on V.

Hecke-Kiselman monoid

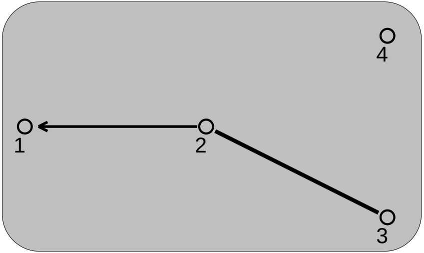
Definition (Ganyushkin, Mazorchuk, 2011)

For any simple digraph $\Theta \in \mathcal{M}_n$ the corresponding monoid HK_Θ generated by idempotents a_i, $i \in \{1, \dots, n\}$ is defined by the following relations, for any $i \neq j$:

1. $a_i a_j = a_j a_i$, when there is no edge/arrow between i, j in Θ,
2. $a_i a_j a_i = a_j a_i a_j$, when we have an edge $i - j$ in Θ,
3. $a_i a_j a_i = a_j a_i a_j = a_i a_j$, when we have an arrow $i \to j$ in Θ,
4. $a_i a_j a_i = a_j a_i a_j = a_j a_i$, when we have an arrow $j \to i$ in Θ.

An example

If we take the following digraph $\Theta \in \mathcal{M}_4$



then the corresponding monoid HK_Θ is generated by idempotents a_1, a_2, a_3, a_4 and by the following relations:

$$a_4 a_i = a_i a_4, \text{ for } i = 1, 2, 3, \quad a_1 a_3 = a_3 a_1,$$

$$a_1 a_2 a_1 = a_2 a_1 a_2 = a_2 a_1, \quad a_2 a_3 a_2 = a_3 a_2 a_3.$$

Questions about Hecke-Kiselman monoids

Open problems:

- For which Θ is HK_Θ finite?
- Is there a faithful representation of HK_Θ in $M_{n\times n}(\mathbb{N})$?
- Is HK_Θ always a $\mathcal{J}$-trivial monoid?

Motivations (1): 0-Hecke monoids

If we assume that $\Theta \in \mathcal{M}_n$ has no oriented arrows then HK_Θ is generated by n idempotents $a_1, \ldots, a_n$ and satisfies relations:

$$a_i a_j = a_j a_i, \quad \text{if there is no edge i-j in } \Theta,$$

$$a_i a_j a_i = a_j a_i a_j, \quad \text{if there is an edge i-j in } \Theta.$$

This is a special case of the so-called **Coxeter monoid** $M(W)$ (also called a 0**-Hecke monoid**) of a Coxeter group W. Every such monoid has n idempotent generators $a_1, \ldots, a_n$ and satisfies the following relations:

$$\underbrace{a_i a_j a_i \ldots}_{m_{ij}} = \underbrace{a_j a_i a_j \ldots}_{m_{ij}},$$

where $M = (m_{ij}) \in \mathrm{M}_{n\times n}(\mathbb{N}_+)$ is a Coxeter matrix of W.

Motivations (1): 0-Hecke monoids

Let $M = (m_{ij}) \in \mathrm{M}_{n \times n}(\mathbb{N}_+ \cup \{\infty\})$ such that:

- M is symmetric
- $m_{ij} = 1 \Leftrightarrow i = j$.

The Coxeter group of M is defined as:

$$W = \langle s_1, \dots, s_n \,|\, (s_i s_j)^{m_{ij}} = 1, \text{ for } |m_{ij}| < \infty \rangle$$

Examples:

- the symmetric groups Σ_n,
- symmetry groups for regular polytopes,
- Weyl groups.

Motivations (1): 0-Hecke monoids

	the Coxeter group W	the Coxeter monoid $M(W)$
generators	$s_1, \dots, s_n$	$a_1, \dots, a_n$
relations	$s_i^2 = 1$	$a_i^2 = a_i$
for $m_{ij} > 1$:	$\underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}}$	$\underbrace{a_i a_j a_i \dots}_{m_{ij}} = \underbrace{a_j a_i a_j \dots}_{m_{ij}}$

An important fact (use exchange property!):

$$|W| = |M(W)|.$$

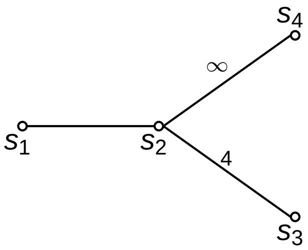
Motivations (1): 0-Hecke monoids

To any Coxeter matrix $M = (m_{ij}) \in \mathrm{M}_{n\times n}(\mathbb{N}_+ \cup \{\infty\})$ we associate a graph (V, E), where

- $V = \{1, 2, \ldots, n\}$
- $\{i, j\} \in E \Leftrightarrow m_{ij} \geq 3$
- and edge has a label m_{ij} if $m_{ij} > 3$.

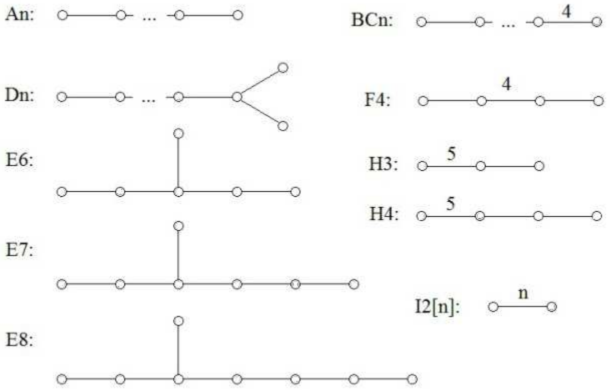
For example:

$$M = (m_{ij}) = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & \infty \\ 2 & 4 & 1 & 2 \\ 2 & \infty & 2 & 1 \end{bmatrix} \longleftrightarrow$$



Motivations (1): 0-Hecke monoids

The graphs of finite connected Coxeter groups:



Motivations (1): 0-Hecke monoids

Corollary

If Θ is a simple unoriented graph then the Hecke-Kiselman monoid HK_{Θ} is finite if and only if Θ is a disjoint union of Dynkin diagrams.

Motivations (1): 0-Hecke algebras

Definition

If W is a Coxeter group of n generators and K is a field then the Iwahori-Hecke algebra $\mathcal{H}_q(W)$ is defined, for every $q \in$ K, by generators $S_1, \ldots, S_n$ and relations:

$$S_i^2 = q + (q-1)S_i, \quad \underbrace{S_iS_jS_i\ldots}_{m_{ij}} = \underbrace{S_jS_iS_j\ldots}_{m_{ij}}.$$

- For $q = 1$ this is just the group algebra K$[W]$,
- for $q = 0$ this is a semigroup algebra of the Coxeter monoid $M(W)$ – the 0-Hecke algebra,
- the representation theory of $\mathcal{H}_q(W)$ is quite well understood, for $q \neq 0$,
- there are still a lot of questions about 0-Hecke algebras.

Interlude: tropicalization and duality theorems

„Tropicalization" means: „replacing" a sum or an integral by a supremum. An example:

the l^p-norm:

$$||x||_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \quad x \in \mathbb{R}^n, \quad 1 \leq p < +\infty$$

becomes the sup-norm:

$$||x||_\infty = \left(\sup_{j=1,\dots,n} |x_j|^p \right)^{1/p} = \sup_{j=1,\dots,n} |x_j|. \quad x \in \mathbb{R}^n$$

Interlude: tropicalization and duality theorems

Fenchel conjugate

Let X be a real normed space. Take any function $f : X \to \overline{\mathbb{R}}$. Then the Fenchel conjugate $f^* : X^* \to \overline{\mathbb{R}}$ is defined as

$$f^*(y) = \sup_{x \in X}(y(x) - f(x)).$$

Examples and motivations:

- If $X = \mathbb{R}^n$ then $f^*(y) = \sup_{x \in \mathbb{R}^n}(y \circ x - f(x))$, for $y \in \mathbb{R}^n$.
- Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be convex and $y(z) = f(z_0) + f'(z_0)(z - z_0)$ be the tangent to the graph of y in $(z_0, f(z_0))$. Then

$$f^*(y) = -y(0) = f'(z_0)z_0 - f(z_0).$$

- interpretations in economy, computer modelling, optimization theory...

Interlude: tropicalization and duality theorems

Theorem (Fenchel-Moreau (1949?))

Let X be a normed space and $f : X \to \overline{\mathbb{R}}$. Then we have an equality $f^{**} = f$ if and only if one of the conditions is satisfied:

- f is convex, lower semicontinuous and proper,
- $f \equiv +\infty$,
- $f \equiv -\infty$.

Remarks:

- If $f^{**} = f$ then f can be represented as a supremum of affine functions,
- If $X = \mathbb{R}$ then you can think about the Fenchel transform as a "tropicalization" of a Laplace transform:

$$(\mathcal{L}g)(\zeta) = \int_0^\infty g(x)e^{-\zeta x}dx, \quad \zeta \in \mathbb{R}$$

Motivations (2): the Kiselman monoids

Consider a monoid $G(E)$ generated by all compositions of three closure operators c, l, m defined on functions from $E \to \overline{\mathbb{R}}$:

- the **convex hull** of f:

$$c(f)(x) = \inf \left\{ \sum_{i=1}^{N} \lambda_i f(x_i) \,|\, \lambda_i > 0, f(x_i) < +\infty, \sum_{i=1}^{N} \lambda_i x_i = x \right\},$$

- the **largest lower semicontinuous minorant** of f:

$$l(f)(x) = \liminf_{y \to x} f(y),$$

- the **„proper function checking" operator**:

$$m(f)(x) = \begin{cases} f(x), & \text{if } f(y) > -\infty \text{, for all } y \in E, \\ -\infty, & \text{otherwise.} \end{cases}.$$

Motivations (2): the Kiselman monoids

Theorem (Kiselman, 2002)

If E is a normed space of infinite dimension over $\mathbb{R}$ then the monoid $G(E)$ consists of 18 elements. It is generated by c, l, m and the following relations give a presentation of $G(E)$:

$$c^2 = c, \quad l^2 = l, \quad m^2 = m$$

$$clc = lcl = lc, \quad cmc = mcm = mc, \quad lml = mlm = ml.$$

Moreover, there exists a faithful representation of $G(E)$ by 3x3 matrices with non-negative integer coefficients. Namely, we can represent c, l, m by the following matrices C, L, M:

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Motivations (2): the Kiselman monoids

Definition (Ganyushkin, Mazorchuk 2002)

By a Kiselman semigroup K_n we denote the monoid generated by n elements $a_1, \ldots, a_n$ with the following relations:

$$a_1^2 = a_1, \ldots, a_n^2 = a_n$$

$$a_i a_j a_i = a_j a_i a_j = a_j a_i, \quad 1 \leq i < j \leq n.$$

Theorem (Kudryavtseva, Mazorchuk, 2009)

The monoid K_n:

- is finite for all n,
- has a faithful representation by $n \times n$ matrices over $\mathbb{N}$
- is $\mathcal{J}$-trivial, namely $K_n a K_n = K_n b K_n \Rightarrow a = b$.

The finiteness of K_n (1)

Proof. Let e be the unit element in K_n. For a finite alphabet $\mathcal{A}$, we denote by $W(\mathcal{A})$ the set of all finite words over this alphabet, including the empty word. Let $l : W(\mathcal{A}) \to \mathbb{N} \cup \{0\}$ be the lenght function.

First, observe that:

(i) if $i \in \{1, \ldots, n\}$ and $w \in W(\{a_1, \ldots, a_{i-1}\})$. Then we have $a_i w a_i = a_i w$ in K_n,

(ii) if $i \in \{1, \ldots, n\}$ and $w \in W(\{a_{i+1}, \ldots, a_n\})$. Then we have $a_i w a_i = w a_i$ in K_n.

We (only) prove (i) by induction on $l(w)$. For $l(w) = 0$ and $l(w) = 1$ this is just the definition of K_n. Let $l(w) > 1$ and write $w = w' a_j$, for some $j < i$. Then $l(w') = l(w) - 1$. Thus:

$$a_i w a_i = \mathbf{a_i w'} a_j a_i \stackrel{i}{=} \mathbf{a_i w' a_i} a_j a_i \stackrel{d}{=} a_i w' \mathbf{a_i a_j} \stackrel{i}{=} \mathbf{a_i w'} a_j = a_i w.$$

Proof (continued). Second, observe that if $\alpha \in K_n$, $\alpha \neq e$ and if $w \in W(\{a_1, \ldots, a_n\})$ is a word of the shortest possible lenght such that $\alpha = w$ in K_n, then:

(iii) for $i \leq \lceil \frac{n}{2} \rceil$ the letter a_i occurs in w at most 2^{i-1} times,
(iv) for $i \geq \lceil \frac{n+1}{2} \rceil$ the letter a_i occurs in w at most 2^{n-i} times.

We (only) prove (iii) by induction on i. If the letter a_1 occurs in w more than once, the word w can be reduced (shortened) using (ii). Let $1 < i \leq \lceil \frac{n}{2} \rceil$. By the inductive hypothesis the total number of occurencies of $a_1, \ldots, a_{i-1}$ in w does not exceed $2^{i-1} - 1$. Hence we can write

$$w = w_1 b_1 w_2 b_2 w_3 \ldots w_{2^{i-1}-1} b_{2^{i-1}-1} w_{2^{i-1}},$$

where $b_j \in \{a_1, \ldots, a_{i-1}\}$, and $w_j \in W(\{a_i, \ldots, a_n\})$. If a_i occurs in some w_j more than once, the word w_j can be reduced by (ii). Thus a_i may occur no more than 2^{i-1} times in w.

Proof (continued). From (iii) and (iv) it follows that the length of any reduced word $w \in W(\{a_1, \dots, a_n\})$ is less than, or equal to:

$$L(n) = \begin{cases} \sum_{i=1}^{k} 2^{i-1} + \sum_{i=k+1}^{n} 2^{n-i} = 2^{k+1} - 2, & n = 2k \\ \sum_{i=1}^{k+1} 2^{i-1} + \sum_{i=k+2}^{n} 2^{n-i} = 3 \cdot 2^{k} - 2, & n = 2k+1 \end{cases}.$$

Since K_n is generated by n elements and every element of K_n, different from the unit element e, can be written as a product of at most $L(n)$ generators, we can see that:

$$|K_n| \leq 1 + n^{L(n)}.$$

Thus K_n is finite.

Motivations (3): The $\mathcal{J}$-trivial monoids

Definition

The monoid M is $\mathcal{J}$-trivial if and only if for all $a, b \in M$ we have:

$$MaM = MbM \Rightarrow a = b.$$

Connections:

- **finite automata theory** (Simon 72'∷ a language is piecewise testable iff its syntactic monoid is $\mathcal{J}$-trivial),
- **theory of partially ordered monoids** (Straubing-Therien 85': every finite $\mathcal{J}$-trivial monoid is a quotient of a finite partially ordered monoid - satisfying the identity $x \leq 1$)
- **theory of matrix semigroups** (every $\mathcal{J}$-trivial monoid is a quotient of a monoid of unitriangular matrices),
- **representation theory of** 0**-Hecke algebras** (every 0-Hecke algebra is a semigroup algebra of a $\mathcal{J}$-trivial monoid).

Motivations (3): The $\mathcal{J}$-trivial monoids

Definition

We say that the monoid M is partially ordered if there exists a partial order $\leq$ on M such that:

- 1 is the maximum element,
- $\leq$ is compatible with mutliplication on M, namely for all m_1, m_1', m_2, m_2' in M we have:

$$m_1 \leq m_1', m_2 \leq m_2' \Rightarrow m_1 m_2 \leq m_1' m_2'.$$

Example. $M = \{1, x, y, z, 0\}$ with relations $x^2 = x, y^2 = y$, $xz = zy = z$, and all other products $= 0$. Then:

$$MxM = \{x, z, 0\}, MyM = \{y, z, 0\}, MzM = \{z, 0\}.$$

Thus M is $\mathcal{J}$-trivial. But no partial order $\leq$ is compatible with multiplication in M. Otherwise we would have:

$$0 = z^2 \leq z = xzy \leq xy = 0 \Rightarrow z = 0.$$

Fact. A partially ordered finite monoid M is $\mathcal{J}$-trivial.

Proof. If $xM = yM$, then $x = ya$ and $y = xb$, for some $a, b \in M$. Since $a \leq 1$ this implies $x = ya \leq y$ and $y = xb \leq x$ so that $x = y$. Analogously $Mx = My \Rightarrow x = y$. But a finite monoid that is both $\mathcal{R}$-trivial and $\mathcal{L}$-trivial is $\mathcal{J}$-trivial.

Corollary. A finite Coxeter (0-Hecke) monoid $M(W)$ is $\mathcal{J}$-trivial.

An idea for the proof. We define the so-called Bruhat order $\leq_B$ on the Coxeter group W. Let $w = s_{i_1} \ldots s_{i_l}$ be a reduced expression for $w \in W$. Then $u \leq_B w$ if and only if there exists a reduced expression $u = s_{j_1} \ldots s_{j_k}$, where $j_1 \ldots j_k$ is a subword of $i_1 \ldots i_l$. Then $\leq_B$ is a partial order (classical fact) on both W and $M(W)$. Moreover $\leq_B$ is compatible with multiplication on $M(W)$ and 1 is the minimal element...

Motivations (3): The $\mathcal{J}$-trivial monoids

Another interesting example of a $\mathcal{J}$-trivial monoid.

Monoid of order preserving regression (contraction) functions

Let $(P, \leq_P)$ be a poset. The set $\mathcal{OR}(P)$ of functions $f : P \to P$ which are:

- order preserving: for all $x, y \in P$
$$x \leq_P y \Rightarrow f(x) \leq f(y),$$
- regressive (alt. contractions): for all $x \in P$ one has $f(x) \leq x$.

In fact all finite $\mathcal{J}$-trivial monoids „divide” one of these kind (it a theorem of J.E. Pin):

Certain results on HK_Θ

Basic facts:

- Let $\Theta, \Psi \in \mathcal{M}_n$ and assume that Ψ is obtained from Θ by deleting some edges. Then mapping between vertices of Θ, Ψ extends uniquely to an epimorphism $\mathrm{HK}_\Theta \to \mathrm{HK}_\Psi$.

  Proof. For any two idempotents x, y of any semigroup from $xy = yx$ it follows that: $xyx = xxy = xy = xyy = yxy$. Thus any relations satisfied by canonical generators of HK_Θ are satisfied by the corresponding generators in HK_Ψ .

- Let $m, n \in \mathbb{N}$ and $\Theta \in \mathcal{M}_m, \Psi \in \mathcal{M}_n$. Assume that $f : \Theta \to \Psi$ os a full embedding of graphs. Then the mapping $e_i \to e_{f(i)}$ induces a monomorphism $\overline{f} : \mathrm{HK}_\Theta \to \mathrm{HK}_\Psi$.

Certain results on HK_Θ

Theorem (Mazorchuk, Ganyushkin)

Let $m, n \in \mathbb{N}$ and $\Theta \in \mathcal{M}_m, \Psi \in \mathcal{M}_n$. Then the semigroups HK_Θ and HK_Ψ are isomorphic if and only if the graphs Θ and Ψ are isomorphic.

Finiteness problem for HK_Θ

Basic facts on the finiteness problem for HK_Θ.

- Let $\Theta, \Psi \in \mathcal{M}_n$. Assume that HK_Θ is finite and that Ψ is obtained from Θ by either:
  - orienting an edge or,
  - removing an arrow or,
  - removing all arrows connected to a sink or source vertex.

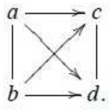
  Then HK_Ψ is finite.
- If HK_Θ is finite then there are no oriented cycles in Θ.
- Let $\Theta, \Psi \in \mathcal{M}_n$. Assume that HK_Θ is finite and that Ψ is obtained from Θ by removing every arrow. Then Ψ is a disjoint union of Dynkin diagrams. These are called the Coxeter componets of Θ.

Coxeter components and the finiteness problem

- **Fact.** Let C be a Dynkin diagram and Θ – an oriented graph obtained from C by choosing an orientation of every edge. Then HK_Θ is finite.

- **The general approach to the finiteness problem.** Take a simple digraph Θ with n elements and try to understand the case when removing all arrows from Θ leads to a small number of Coxeter components.

- **Theorem (Aragona, D'Andrea 2012).** Let $\Theta \in \mathcal{M}_n$ be an acyclic simple digraph. If $n = 3$ then HK_Θ is finite. If $n = 4$ then HK_Θ is finite except for the case when Θ is of form:



An order on simple digraphs

Consider two digraphs Θ and Ψ in $\mathcal{M}_n$. We say that $\Theta \subseteq \Psi$ if and only if Θ is a subgraph of Ψ. In the language of relations we have an inclusion $\Theta \subseteq \Psi$.

We also define two members of $\mathcal{M}_n$:

- $\Theta_C = \{i+1, i) \,|\, 1, 2, \ldots, n-1\}$
- $\Theta_K = \{(j, i) \,|\, 1 \leq i < j \leq n\}$.

The Hecke-Kiselman monoids of these relations are:

- HK_{Θ_C} - the Catalan monoid C_n (it is a monoid $\mathcal{OR}(P)$ on a chain $1 < \ldots < n$)
- HK_{Θ_K} - and the Kiselman monoid K_n.

An order on simple digraphs

Ashikhmin, Volkov, Zhang (2015)

Let $n \geq 2$. Then for every relation $\Theta \in \mathcal{M}_n$ such that $\Theta_C \subseteq \Theta \subseteq \Theta_K$, the set of identities of the Hecke-Kiselman monoid HK_Θ is the following:

$$\{w \equiv w' \Leftrightarrow w \sim_n w'\},$$

where $w \sim_n w'$ means that the words w and w' have the same scattered subwords of lenght $\leq n$. In particular:

- for $n = 2, 3$ the monoid HK_Θ is finitely based,
- for $n \geq 4$ the monoid HK_Θ is nonfinitely based.

Corollary: The Hecke-Kiselman monoids HK_Θ are $\mathcal{J}$-trivial for all $\Theta \subseteq \Theta_K$.

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