Conjugacy classes of left ideals of an associative algebra

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Notation

1. R – an associative ring (with identity!),
2. $U(R)$ – the unit group of R,
3. $L(R)$ – the set of left ideals of R,
4. $J(R)$ – the Jacobson radical of a ring R,
5. A – an algebra over a field $\mathbb{K}$
   (in most cases – algebraically closed).

Conjugacy classes of left ideals

Definition

Let $U(R)$ be the group of units of R. Consider an action $U(R) \times R \to R$ of $U(R)$ on R such that

$$(u, r) \mapsto uru^{-1}, \quad \text{for } u \in U(R), r \in R.$$

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The orbits of this action are called **conjugacy classes**.
By $[L]$ we denote the conjugacy class of a left ideal L in R.
By $C(R)$ we denote the set of conjugacy classes of left ideals on R.

The semigroup $C(R)$

Definition

If $L_1, L_2 \in L(R)$ and $g, h \in U(R)$, then $L_1 g L_2 h = L_1 L_2 h$. So we can equip the set $C(R)$ with a binary operation:

$$[L_1][L_2] := [L_1 L_2].$$

This operation is well defined and associative, so there is a **natural structure of a semigroup** on $C(R)$.

What information on R can be deduced from $C(R)$?

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Examples (1)

$R = \mathrm{M}_n(D)$

If R is a simple ring of matrices $\mathrm{M}_n(D)$ over a division ring D, then $C(\mathrm{M}_n(D))$ consists of exactly $n+1$ elements. Every nonzero left ideal L of $\mathrm{M}_n(D)$ is a conjugate of one of the ideals:

$$\mathrm{M}_n(D)(e_{11} + \ldots + e_{jj}), \text{ for } 1 \leq j \leq n,$$

e_{ij} are matrix units in $M_n(D)$.

Corollary

If R is an artinian ring with identity and $J(R) = 0$ then $C(R)$ is finite.

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1. (J. Han) the conjugate action on R:

$$(g, r) \mapsto grg^{-1}, \text{ for } g \in U(R), r \in R,$$

2. (J. Han, Y. Hirano) the regular action on R:

$$(g, r) \mapsto gr, \text{ for } g \in U(R), r \in R,$$

3. (J. Okniński & L. Renner, J. Krempa & M. Hryniewicka) $U(R)$-orbits:

$$(g, h, r) \mapsto grh^{-1}, \text{ for } g, h \in U(R), r \in R,$$

Some general lemmas

Lemma

Let R be a left perfect ring with identity. Then the $U(R)$-orbits on R are precisely the conjugacy classes of principal left ideals.

Lemma

Let R be a semilocal ring with identity. Assume that L, L' are left ideals of R. Then $R/L \simeq R/L'$ are isomorphic, as left R-modules, if and only if $L = L'g$, for some $g \in U(R)$.

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Examples (2)

Definition

Let A be a finite dimensional algebra over a field $\mathbb{K}$. We say that A is of **finite representation type** if A has finitely many isomorphism classes of finite dimensional indecomposable modules.

Theorem (J. Okniński, L. Renner)

Let A be a finite dimensional algebra over a field $\mathbb{K}$.

- if A is of finite representation type, then $C(A)$ is finite,
- if the field $\mathbb{K}$ is algebraically closed and if $C(\mathrm{M}_n(A))$ is finite for all $n > 1$,then A is of finite representation type.

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The conjugacy classes in the Jacobson radical

Theorem

Let A be a finite dimensional algebra with identity over an arbitrary field $\mathbb{K}$. The following conditions are equivalent:

- $C(A)$ is finite,
- the number of conjugacy classes of nilpotent left ideals in A is finite.

When the radical is nonzero

Theorem

Let A be a finite dimensional algebra over an algebraically closed field $\mathbb{K}$ and let $J(A)^2 = 0$. We know when $C(A)$ is finite in the following cases:

- $A/J(A)$ is a direct sum of finitely many copies of $\mathbb{K}$
- $A/J(A) \simeq \mathrm{M}_{n_1}(\mathbb{K}) \oplus \ldots \oplus \mathrm{M}_{n_k}(\mathbb{K})$, for $n_i \leq 2$
- $A/J(A) \simeq \mathrm{M}_{n_1}(\mathbb{K}) \oplus \ldots \oplus \mathrm{M}_{n_k}(\mathbb{K})$, for $n_i \geq 6$

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$C(A)$ as a finite invariant of an algebra A

Theorem

- Let A, B be finite dimensional algebras over an algebraically closed field $\mathbb{K}$. Assume that $J(A)^2 = 0$ and $C(A)$ is finite. If the semigroups $C(A)$ and $C(B)$ are isomorphic then the algebras A and B are isomorphic.
- Let A, B be finite dimensional algebras over an algebraically closed field $\mathbb{K}$. If the semigroups $C(A)$ and $C(B)$ are finite and isomorphic, then the algebras $A/J(A)$ and $B/J(B)$ are isomorphic.

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Isomorpshim problem

Question

Take two algebras A, B, finite dimensional algebras over an algebraically closed field $\mathbb{K}$ such that $C(A) \simeq C(B)$ as **finite semigroups**. Is $A \simeq B$?

Probably not, so maybe consider the case when:

- $A/J(A)$ is a direct product of $\mathbb{K}$,
- $A/J(A) \simeq \mathrm{M}_n(\mathbb{K})$, for some n,
- A is of finite representation type?

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