Geometria Algebraiczna, Seria 7

Ex. 1.

Let X be an integral noetherian scheme and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. For $x \in X$ we write $\mathcal{M}(x) := \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ for the *fiber* of $\mathcal{M}$ (it is a vector space over $k(x) = \mathcal{O}_{X,x}/m_x$). Let η be the generic point of X. A *rank* of $\mathcal{M}$ is the dimension of $\mathcal{M}(\eta)$ over $k(\eta)$. Show that:

1. If for some non-empty open $U \subset X$, the module $\mathcal{M}(U)$ is generated over $\mathcal{O}_X(U)$ by r elements, then the rank of $\mathcal{M}$ is $\leq r$.
2. There exists non-empty open $U \subset X$, such that $\mathcal{M}|_U$ is a free $\mathcal{O}_U$-module of rank equal to the rank of $\mathcal{M}$.
3. For every $x \in X$ the dimension of $\mathcal{M}(x)$ over $k(x)$ is at least the rank of $\mathcal{M}$.

Ex. 2.

Let A be a domain and M a finitely generated A-module. Let us set $X = \operatorname{Spec} A$, $\mathcal{M} = \tilde{M}$. Find the rank of $\mathcal{M}$ and dimensions of fibers in the following cases:

1. $A = k[x]$, $M = k[x]/(x)$.
2. $A = k[x, y]$, $M = (x, y) \subset k[x, y]$.

Ex. 3.

Let X be a scheme and $Y \subset X$ a closed subset. Let us define a presheaf $\mathcal{J}_Y$ by

$$\mathcal{J}_Y(U) := \{f \in \mathcal{O}_X(U) : f(x) = 0 \text{ for } x \in U \cap Y\},$$

where $f(x)$ is the image of f in $k(x)$. Show that:

1. $\mathcal{J}_Y$ is quasi-coherent and it is a subsheaf of $\mathcal{O}_X$.
2. The support of $\mathcal{O}_X/\mathcal{J}_Y$ is equal to Y.

Ex. 4.

Let X be a noetherian scheme and let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Show that if two of $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ are coherent, then the third is also coherent. Does this hold if we don't assume that the modules are quasi-coherent?

Ex. 5.

Let X be a variety over an algebraically closed field k. For $f \in \mathcal{O}_X(X)$ we write

$$X_f = \{x \in X : f_x \text{ is invertible in } \mathcal{O}_{X,x}\}.$$

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that there exists a natural isomorphism of $\mathcal{O}_X(X)$-modules

$$\mathcal{M}(X)_f \to \mathcal{M}(X_f).$$