Variational methods in PDEs

Bartosz Bieganowski

Institute of Applied Mathematics and Mechanics

09.05.2024

Unconstrained problems

Variational PDEs

Consider the following second order elliptic problem

$$-\Delta u + V(x)u = f(u), \quad u : \mathbb{R}^N \to \mathbb{R},\ u(x) \to 0 \text{ as } |x| \to \infty.$$

We say that u is a *weak solution*, if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv\,dx = \int_{\mathbb{R}^N} f(u)v\,dx$$

for all v.
All weak solutions are *critical points* of the energy (Euler-Lagrange) functional

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx$$

with $F(u) := \int_0^u f(s)\,ds$.

Variational PDEs

Consider the following second order elliptic problem

$$-\Delta u + V(x)u = f(u), \quad u : \mathbb{R}^N \to \mathbb{R}, \;\; u(x) \to 0 \text{ as } |x| \to \infty.$$

We say that u is a *weak solution*, if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv \, dx = \int_{\mathbb{R}^N} f(u)v \, dx$$

for all v.

All weak solutions are *critical points* of the energy (Euler-Lagrange) functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx$$

with $F(u) := \int_0^u f(s) \, ds$.

Variational PDEs

Consider the following second order elliptic problem

$$-\Delta u + V(x)u = f(u), \quad u : \mathbb{R}^N \to \mathbb{R},\ u(x) \to 0 \text{ as } |x| \to \infty.$$

We say that u is a *weak solution*, if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv\, dx = \int_{\mathbb{R}^N} f(u)v\, dx$$

for all v.
All weak solutions are *critical points* of the energy (Euler-Lagrange) functional

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\, dx - \int_{\mathbb{R}^N} F(u)\, dx$$

with $F(u) := \int_0^u f(s)\, ds$.

Variational functionals

Hence, we look for critical points of some nonlinear functional

$$\mathcal{J} : X \to \mathbb{R}$$

defined on some function space X (in applications: some Sobolev-type space).

First idea: **look for minimizers of $\mathcal{J}$**! Minimizers are critical points, so we will find solutions....

But, if e.g. $F(u) = \frac{1}{p}|u|^p$ with some $p > 2$, we have

$$\mathcal{J}(tu) = t^2 \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - t^p \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx \to -\infty$$

as $t \to \infty$. The functional is not bounded from below!

Variational functionals

Hence, we look for critical points of some nonlinear functional

$$\mathcal{J} : X \to \mathbb{R}$$

defined on some function space X (in applications: some Sobolev-type space).

First idea: **look for minimizers of $\mathcal{J}$**! Minimizers are critical points, so we will find solutions....

But, if e.g. $F(u) = \frac{1}{p}|u|^p$ with some $p > 2$, we have

$$\mathcal{J}(tu) = t^2 \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - t^p \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx \to -\infty$$

as $t \to \infty$. The functional is not bounded from below!

Variational functionals

Hence, we look for critical points of some nonlinear functional

$$\mathcal{J} : X \to \mathbb{R}$$

defined on some function space X (in applications: some Sobolev-type space).

First idea: **look for minimizers of $\mathcal{J}$**! Minimizers are critical points, so we will find solutions....

But, if e.g. $F(u) = \frac{1}{p}|u|^p$ with some $p > 2$, we have

$$\mathcal{J}(tu) = t^2 \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2 \, dx - t^p \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx \to -\infty$$

as $t \to \infty$. The functional is not bounded from below!

Variational functionals

Hence, we look for critical points of some nonlinear functional

$$\mathcal{J} : X \to \mathbb{R}$$

defined on some function space X (in applications: some Sobolev-type space).
First idea: **look for minimizers of $\mathcal{J}$!** Minimizers are critical points, so we will find solutions....
But, if e.g. $F(u) = \frac{1}{p}|u|^p$ with some $p > 2$, we have

$$\mathcal{J}(tu) = t^2 \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2 \, dx - t^p \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx \to -\infty$$

as $t \to \infty$. The functional is not bounded from below!

Possible approaches

- Look for other type of solutions: Mountain Pass Theorem, Palais-Smale sequences, ...
- Constrained minimization: look for minimizers on appropriate subsets of X on which the functional is bounded from below. *Are such minimizers critical points, and therefore - solutions?*

Possible approaches

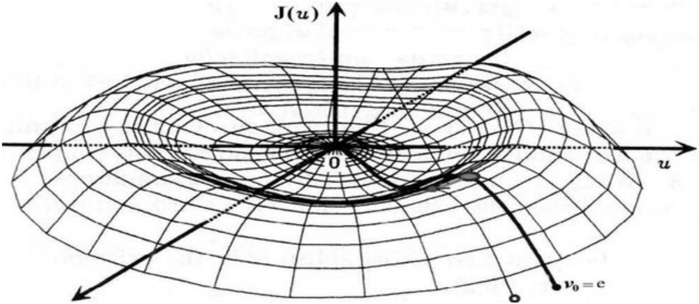
- Look for other type of solutions: Mountain Pass Theorem, Palais-Smale sequences, ...
- Constrained minimization: look for minimizers on appropriate subsets of X on which the functional is bounded from below. *Are such minimizers critical points, and therefore - solutions?*

Possible approaches

- Look for other type of solutions: Mountain Pass Theorem, Palais-Smale sequences, ...
- Constrained minimization: look for minimizers on appropriate subsets of X on which the functional is bounded from below. *Are such minimizers critical points, and therefore - solutions?*

Mountain Pass Approach

One can check whether the functional has a *mountain pass geometry*.
Amrosetti, Rabinowitz, 1973



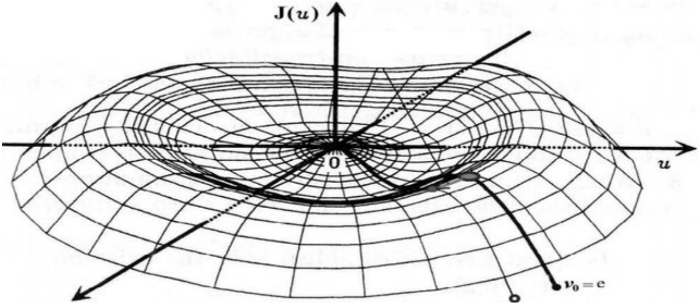
Then one can expect the existence of a Palais-Smale sequence:

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0,$$

where $c > 0$ is some number. Is such a sequence convergent...? Usually not.

Mountain Pass Approach

One can check whether the functional has a *mountain pass geometry*.
Amrosetti, Rabinowitz, 1973



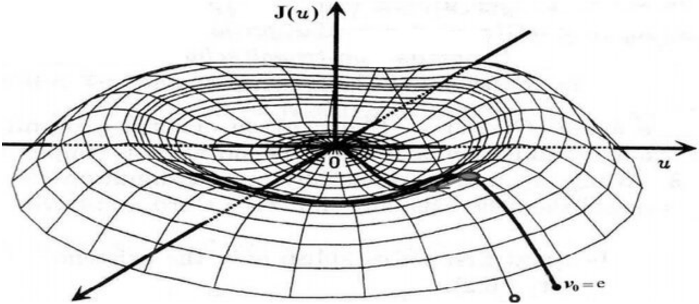
Then one can expect the existence of a Palais-Smale sequence:

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0,$$

where $c > 0$ is some number. Is such a sequence convergent...? Usually not.

Mountain Pass Approach

One can check whether the functional has a *mountain pass geometry*.
Amrosetti, Rabinowitz, 1973



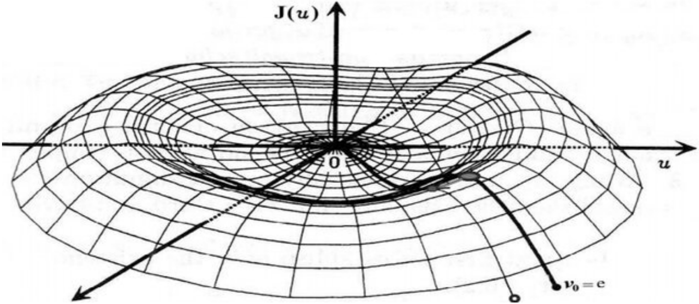
Then one can expect the existence of a Palais-Smale sequence:

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0,$$

where $c > 0$ is some number. Is such a sequence convergent...? Usually not.

Mountain Pass Approach

One can check whether the functional has a *mountain pass geometry*.
Amrosetti, Rabinowitz, 1973



Then one can expect the existence of a Palais-Smale sequence:

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0,$$

where $c > 0$ is some number. Is such a sequence convergent...? Usually not.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \ : \ \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960

$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.

Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \ : \ \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960
$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.
Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \ : \ \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960
$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.
Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \;:\; \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960

$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.

Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \ : \ \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960
$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.
Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Nehari manifold approach

We look for critical points on the following constraint

$$\mathcal{N} := \{u \in X \setminus \{0\} \ : \ \mathcal{J}'(u)(u) = 0\}.$$

Nehari, 1960
$\mathcal{N}$ contains all nontrivial critical points of $\mathcal{J}$.
Properties (under reasonable assumptions, if f is sufficiently regular):

- $\mathcal{J}$ is bounded from below on $\mathcal{N}$!
- $\mathcal{N}$ is a $\mathcal{C}^{1,1}$ manifold.
- $\mathcal{N}$ is a *natural* constraint to $\mathcal{J}$. Namely - if $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$, then $\mathcal{J}'(u) = 0$.

Corollary: it is enough to look for minimizers of $\mathcal{J}$ on $\mathcal{N}$.

Uniwersytet Warszawski
MIM
Wydział
Matematyki, Informatyki i Mechaniki

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $\left(\mathcal{J}|_{\mathcal{N}}\right)'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

Does it work when f is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{J}|_{\mathcal{N}})'(u) = 0$.

Szulkin, Weth, 2009 There is a homeomorphism $m : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the unit sphere in X.

- Although m is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $\mathcal{C}^1$ class;
- $\mathcal{S}$ is a manifold of $\mathcal{C}^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : \mathcal{S} \to \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
- Transform the minimizer (in fact, the minimizing sequence) back to $\mathcal{N}$ through m.
- It appears that this function is a critical point of $\mathcal{J}$ (and is a minimizer on $\mathcal{N}$).

Nehari manifold approach

When we find a minimizer of $\mathcal{J}$ on $\mathcal{N}$, we gain the additional information: the solution we find is the **ground state solution** (the least energy solution) – it minimizes $\mathcal{J}$ among all critical points.

Nehari manifold approach

When we find a minimizer of $\mathcal{J}$ on $\mathcal{N}$, we gain the additional information: the solution we find is the **ground state solution** (the least energy solution) – it minimizes $\mathcal{J}$ among all critical points.

Strongly indefinite problems

For strongly indefinite problems the mentioned methods have their counterparts:

- Mountain Pass Theorem ↔ Linking Theorem (**Kryszewski, Szulkin, 1998**)
- Nehari manifold ↔ Nehari-Pankov manifold (**Pankov, 2005**)

Strongly indefinite problems

For strongly indefinite problems the mentioned methods have their counterparts:

- Mountain Pass Theorem ↔ Linking Theorem (**Kryszewski, Szulkin, 1998**)
- Nehari manifold ↔ Nehari-Pankov manifold (**Pankov, 2005**)

Strongly indefinite problems

For strongly indefinite problems the mentioned methods have their counterparts:

- Mountain Pass Theorem ↔ Linking Theorem (**Kryszewski, Szulkin, 1998**)
- Nehari manifold ↔ Nehari-Pankov manifold (**Pankov, 2005**)

Constrained problems

The Schrödinger equation

We consider the following nonlinear Schrödinger wave equation

$$i\frac{\partial \Psi}{\partial t} = -\Delta_x \Psi - f(|\Psi|)\Psi, \ (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\Psi = \Psi(t,x)$ is the state (wave) function. Looking for solutions of the form (so-called *standing waves*)

$$\Psi(t,x) = e^{-i\lambda t}u(x),$$

where the so-called *soliton* u vanishes at infinity, leads to the equation

$$-\Delta u + \lambda u = g(u), \quad x \in \mathbb{R}^N.$$

The Schrödinger equation

We consider the following nonlinear Schrödinger wave equation

$$i\frac{\partial\Psi}{\partial t} = -\Delta_x\Psi - f(|\Psi|)\Psi,\ (t,x)\in\mathbb{R}\times\mathbb{R}^N,$$

where $\Psi = \Psi(t,x)$ is the state (wave) function. Looking for solutions of the form (so-called *standing waves*)

$$\Psi(t,x) = e^{-i\lambda t}u(x),$$

where the so-called *soliton* u vanishes at infinity, leads to the equation

$$-\Delta u + \lambda u = g(u), \quad x\in\mathbb{R}^N.$$

The normalized problem

We are looking for solutions to the following problem

$$\begin{cases} -\Delta u + \lambda u = g(u) & \text{in } \mathbb{R}^N,\ N \geqslant 3, \\ \int_{\mathbb{R}^N} |u|^2\,dx = \rho > 0, \end{cases}$$

where ρ is prescribed and $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ has to be determined.

In the time-dependent equation, the mass

$$\int_{\mathbb{R}^N} |\Psi(t,x)|^2\,dx \quad \text{is independent of } t$$

thus

it makes sense to prescribe $\int_{\mathbb{R}^N} |u|^2\,dx$ instead of λ.

The normalized problem

We are looking for solutions to the following problem

$$\begin{cases} -\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}$$

where ρ is prescribed and $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ has to be determined. In the time-dependent equation, the mass

$$\int_{\mathbb{R}^N} |\Psi(t,x)|^2 \, dx \quad \text{is independent of } t$$

thus

it makes sense to prescribe $\int_{\mathbb{R}^N} |u|^2 \, dx$ instead of λ.

The normalized problem

We are looking for solutions to the following problem

$$\begin{cases} -\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}$$

where ρ is prescribed and $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ has to be determined.
In the time-dependent equation, the mass

$$\int_{\mathbb{R}^N} |\Psi(t, x)|^2 \, dx \quad \text{is independent of } t$$

thus

it makes sense to prescribe $\int_{\mathbb{R}^N} |u|^2 \, dx$ instead of λ.

The normalized problem

We are looking for solutions to the following problem

$$\begin{cases} -\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}$$

where ρ is prescribed and $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ has to be determined.
In the time-dependent equation, the mass

$$\int_{\mathbb{R}^N} |\Psi(t,x)|^2 \, dx \quad \text{is independent of } t$$

thus

it makes sense to prescribe $\int_{\mathbb{R}^N} |u|^2 \, dx$ instead of λ.

Variational methods

Let us denote

$$\mathcal{S} = \left\{ u \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho \right\}.$$

Under suitable assumptions, solutions are critical points of the energy functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx,$$

where $G(u) := \int_0^u g(s) \, ds$, on the constraint $\mathcal{S}$ with a Lagrange multiplier $\lambda \in \mathbb{R}$.

Variational methods

Let us denote

$$\mathcal{S} = \left\{ u \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho \right\}.$$

Under suitable assumptions, solutions are critical points of the energy functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx,$$

where $G(u) := \int_0^u g(s)\, ds$, on the constraint $\mathcal{S}$ with a Lagrange multiplier $\lambda \in \mathbb{R}$.

Variational methods

If $\mathcal{J}$ is bounded from below on $\mathcal{S}$, one can just minimize it there. What to do if $\mathcal{J}$ is not bounded from below on $\mathcal{S}$?

- Restrict the problem to look for radial solutions (**Jeanjean, 1997**; **Bartsch, de Valeriola, 2013**);
- Find another constraint like "Nehari manifold"?

Variational methods

If $\mathcal{J}$ is bounded from below on $\mathcal{S}$, one can just minimize it there. What to do if $\mathcal{J}$ is not bounded from below on $\mathcal{S}$?

- Restrict the problem to look for radial solutions (**Jeanjean, 1997**; **Bartsch, de Valeriola, 2013**);
- Find another constraint like "Nehari manifold"?

Variational methods

If $\mathcal{J}$ is bounded from below on $\mathcal{S}$, one can just minimize it there. What to do if $\mathcal{J}$ is not bounded from below on $\mathcal{S}$?

- Restrict the problem to look for radial solutions (**Jeanjean, 1997**; **Bartsch, de Valeriola, 2013**);
- Find another constraint like "Nehari manifold"?

$$-\Delta u + \lambda u = g(u)$$

Nehari manifold:

$$\mathcal{J}'(u)(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} g(u) u \, dx = 0.$$

Pohožaev manifold (**Pohožaev, 1965**):

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u) - \frac{\lambda}{2} u^2 \, dx.$$

Idea: take the linear combination of them to rule out λ!

$$-\Delta u + \lambda u = g(u)$$

Nehari manifold:

$$\mathcal{J}'(u)(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} g(u) u \, dx = 0.$$

Pohožaev manifold (**Pohožaev, 1965**):

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u) - \frac{\lambda}{2} u^2 \, dx.$$

Idea: take the linear combination of them to rule out λ!

$$-\Delta u + \lambda u = g(u)$$

Nehari manifold:

$$\mathcal{J}'(u)(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} g(u) u \, dx = 0.$$

Pohožaev manifold (**Pohožaev, 1965**):

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u) - \frac{\lambda}{2} u^2 \, dx.$$

Idea: take the linear combination of them to rule out λ!

Nehari-Pohožaev manifold

$$\mathcal{M} = \{u \neq 0 \;:\; M(u) = 0\},$$

where

$$M(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \int_{\mathbb{R}^N} H(u) \, dx = 0,$$

where $H(u) := g(u)u - 2G(u)$.
Idea: look for solutions in $\mathcal{M} \cap \mathcal{S}$.

- $\mathcal{M}$ is a $\mathcal{C}^1$-manifold,
- $\mathcal{J}$ is bounded from below on $\mathcal{M} \cap \mathcal{S}$.

One can use variational techniques to find a kind of Palais-Smale sequence on $\mathcal{M} \cap \mathcal{S}$. Is such a sequence bounded? Convergent? Is the limit still in $\mathcal{S}$? ...?

It can be done:

- a mini-max approach in $\mathcal{M}$ based on the σ-homotopy stable family of compact subsets of $\mathcal{M}$ and some minimax principles (**Bartsch, Soave, 2018**)
- mountain-pass-type approach connected with the analysis of the *ground state energy map* (**Lu, Jeanjean, 2020**)

- $\mathcal{M}$ is a $\mathcal{C}^1$-manifold,
- $\mathcal{J}$ is bounded from below on $\mathcal{M} \cap \mathcal{S}$.

One can use variational techniques to find a kind of Palais-Smale sequence on $\mathcal{M} \cap \mathcal{S}$. Is such a sequence bounded? Convergent? Is the limit still in $\mathcal{S}$? ...?

It can be done:

- a mini-max approach in $\mathcal{M}$ based on the σ-homotopy stable family of compact subsets of $\mathcal{M}$ and some minimax principles (**Bartsch, Soave, 2018**)
- mountain-pass-type approach connected with the analysis of the *ground state energy map* (**Lu, Jeanjean, 2020)**

- $\mathcal{M}$ is a $\mathcal{C}^1$-manifold,
- $\mathcal{J}$ is bounded from below on $\mathcal{M} \cap \mathcal{S}$.

One can use variational techniques to find a kind of Palais-Smale sequence on $\mathcal{M} \cap \mathcal{S}$. Is such a sequence bounded? Convergent? Is the limit still in $\mathcal{S}$? ...?
It can be done:

- a mini-max approach in $\mathcal{M}$ based on the σ-homotopy stable family of compact subsets of $\mathcal{M}$ and some minimax principles (**Bartsch, Soave, 2018**)
- mountain-pass-type approach connected with the analysis of the *ground state energy map* (**Lu, Jeanjean, 2020)**

The new approach

Assumptions:

- don't work with radial functions;
- don't work with Palais-Smale sequences, and avoid the mini-max approach in $\mathcal{M}$ involving strong topological arguments.

The new idea (**B., Mederski, 2021**):
work in $\mathcal{D} \cap \mathcal{M}$ instead of $\mathcal{S} \cap \mathcal{M}$, where

$$\mathcal{D} := \left\{ u \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx \leqslant \rho \right\}.$$

Obviously $\mathcal{S} \cap \mathcal{M} \subset \mathcal{D} \cap \mathcal{M}$.

The new approach

Assumptions:

- don't work with radial functions;
- don't work with Palais-Smale sequences, and avoid the mini-max approach in $\mathcal{M}$ involving strong topological arguments.

The new idea (**B., Mederski, 2021**):
work in $\mathcal{D} \cap \mathcal{M}$ instead of $\mathcal{S} \cap \mathcal{M}$, where

$$\mathcal{D} := \left\{ u \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx \leqslant \rho \right\}.$$

Obviously $\mathcal{S} \cap \mathcal{M} \subset \mathcal{D} \cap \mathcal{M}$.

The new approach

Assumptions:

- don't work with radial functions;
- don't work with Palais-Smale sequences, and avoid the mini-max approach in $\mathcal{M}$ involving strong topological arguments.

The new idea (**B., Mederski, 2021**):
work in $\mathcal{D} \cap \mathcal{M}$ instead of $\mathcal{S} \cap \mathcal{M}$, where

$$\mathcal{D} := \left\{ u \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx \leqslant \rho \right\}.$$

Obviously $\mathcal{S} \cap \mathcal{M} \subset \mathcal{D} \cap \mathcal{M}$.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D} \cap \mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S} \cap \mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D}\cap\mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S}\cap\mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D}\cap\mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S}\cap\mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D}\cap\mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S}\cap\mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D}\cap\mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S}\cap\mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

The new approach

- $\mathcal{J}$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$;
- minimizing sequences $\mathcal{J}(u_n) \to \inf_{\mathcal{D}\cap\mathcal{M}} \mathcal{J}$ are bounded!;
- one can pass to the weak limit and show that the limit is non-zero;
- $\mathcal{D}$ is weakly closed! Hence the limit point still lies in $\mathcal{D}$ and is a minimizer of $\mathcal{J}$ on $\mathcal{D} \cap \mathcal{M}$.
- one can show the crucial inequality $\inf_{\mathcal{S}\cap\mathcal{M}} \mathcal{J} < \mathcal{J}(v)$ for $v \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}$ – the minimizer lies in $\mathcal{S}$!
- it seems that $\mathcal{M}$ is a "natural" constaint - the Lagrange multiplier for $\mathcal{M}$ equals to 0.

Thank you for your attention!