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University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

June 29, 2023

Solving time: 150 minutes

*In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **YES** or **NO** in the box close to it. In case of an error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of the form $10^n - 1$, where n is integer and positive,

   $\boxed{\texttt{YES}}$ (a) is divisible by 9;

   $\boxed{\texttt{NO}}$ (b) is prime;

   $\boxed{\texttt{YES}}$ (c) is odd.

*You can write only in the indicated places and only the words **YES** and **NO**. Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = \min(30, D + m/100),$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in the case when many candidates get the same number of "big" points.

Good luck!

1. A series of positive numbers $\sum_{n=1}^{\infty} a_n$ converges. It follows that

- [ ] (a) the limit $\lim_{n\to\infty} a_n$ exists;
- [ ] (b) the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is decreasing;
- [ ] (c) there exists a positive number n_0 such that $a_n < 1/n$ for all $n > n_0$.

2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is thrice differentiable and $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 1$. It follows that

- [ ] (a) f has a local extremum at point 0;
- [ ] (b) f is increasing in some neighborhood of 0;
- [ ] (c) $f(x) > 0$ for all $x \in (0, 10^{-7})$.

3. Let X be a linear space of dimension 10. There exist subspaces $U, V \subset X$ such that $\dim(U \cap V) = 1$ and

- [ ] (a) $X = U \oplus V$;
- [ ] (b) $\dim U > 6$ and $\dim V > 6$;
- [ ] (c) $\dim U = \dim V = 6$ and $X = U + V$.

4. Number 2 is an eigenvalue of a real square matrix A of order n. It follows that

- [ ] (a) number 2 is an eigenvalue of A^{T};
- [ ] (b) number 2 is an eigenvalue of A^2;
- [ ] (c) matrix $A - 2I_n$ is singular.

5. Some four rows of a matrix $A \in \mathbb{R}^{23,23}$ are linearly independent. It follows that

- [ ] (a) the dimension of the image of the linear map represented by A is less or equal to 20;
- [ ] (b) the dimension of the kernel of A is at least 20;
- [ ] (c) there exists a linearly independent subset of columns of A that has at least 19 elements.

6. Let X, Y be linear spaces and let $f\colon X \to Y$ and $g\colon Y \to X$ be linear maps satisfying $g \circ f = 1_X$, where $1_X(x) = x$ for every $x \in X$. It follows that

- [ ] (a) f is an isomorphism;
- [ ] (b) g is an isomorphism;
- [ ] (c) g is an epimorphism.

7. Let A, B be nonempty sets and let $f\colon A \to B$ be a function. It follows that

- [ ] (a) the inverse image of a nonempty set is a nonempty set, i.e. $Y \neq \emptyset$ implies that $f^{-1}(Y) \neq \emptyset$ for any $Y \subseteq B$;
- [ ] (b) the inverse images of disjoint sets are disjoint, i.e. $Y_1 \cap Y_2 = \emptyset$ implies that $f^{-1}(Y_1) \cap f^{-1}(Y_2) = \emptyset$ for any $Y_1, Y_2 \subseteq B$;
- [ ] (c) the inverse image of the codomain is the domain, i.e. $f^{-1}(B) = A$.

8. There exists an equivalence relation on $\{a, b\}^*$ with
   - [ ] (a) countably many finite equivalence classes;
   - [ ] (b) uncountably many finite equivalence classes;
   - [ ] (c) only infinite equivalence classes.

9. Three edges have been removed from the clique K_6. It follows that the remaining graph
   - [ ] (a) has no Euler cycle;
   - [ ] (b) has a Hamilton cycle;
   - [ ] (c) is nonplanar.

10. The number of permutations f of the set $\{1, 2, \ldots, 7\}$ such that
    - [ ] (a) f has exactly 4 fixed points, is $7 \cdot 6 \cdot 5$;
    - [ ] (b) $f(1) > \max(f(2), f(3))$, is $7!/3$;
    - [ ] (c) $\{1, f(1), f(f(1)), \ldots\} = \{1, 2, \ldots, 7\}$, is 720.

11. There exist positive integers r, n such that the number of solutions of the system of congruences $\{x \equiv 2 \pmod 3,\ x \equiv 3 \pmod 5,\ x \equiv r \pmod n\}$ in the set $\{0, 1, \ldots, 15n\}$ is
    - [ ] (a) 0, but the system has a solution in $\mathbb{N}$;
    - [ ] (b) 15;
    - [ ] (c) $3n$.

12. Let a_n be the number of nonnegative integer solutions x_1, x_2 of the equation $x_1 + 2x_2 = n$. It follows that the generating function of the sequence $\langle a_n \rangle_{n=0}^{\infty}$ is
    - [ ] (a) $\frac{1+x^2}{1-x}$;
    - [ ] (b) $\frac{1}{(1-x)^2(1+x)}$;
    - [ ] (c) $\frac{1}{(1-x)(1-2x)}$.

13. Two fair cubical dice are rolled, one red and one blue. Let A be the event that the red dice shows 6. Let B be the event that the blue dice shows 6. Let C be the event that the red and the blue dice show the same number. It follows that
    - [ ] (a) event C is more probable than event A;
    - [ ] (b) $P(C \mid A \cap B) = P(C)$;
    - [ ] (c) events A and C are independent.

14. Let X and Y be two independent random variables with expected value 10 and variance 1. It follows that
    - [ ] (a) $P(X \geq 20) \leq \frac{1}{200}$;
    - [ ] (b) $P(X \geq Y) \geq \frac{1}{2}$;
    - [ ] (c) $\mathrm{Var}(X - Y) = 2$.

15. Researcher performed a statistical test at the significance level $\alpha = 0.05$. The obtained value of the test statistic was 0.21. The critical region for the test was $(-\infty, -1.645)$. It follows that

- [ ] (a) the value of the test statistic did not belong to the critical region so the null hypothesis was supported;
- [ ] (b) assuming the null hypothesis was true, the researcher would support the null hypothesis with probability $1 - \alpha$;
- [ ] (c) without details about the null hypothesis and the alternative hypothesis we cannot tell if the null hypothesis should be rejected.

16. Assume that some cat catches X_i rats during i-th hour, where X_i is a random variable of Poisson distribution with the parameter $\lambda = 5$. For each pair of hours i, j the variables X_i, X_j are independent. It follows that

- [ ] (a) the distribution of the number of rats caught by the cat in the period of 6 hours is the Poisson distribution with the parameter $\lambda = 30$;
- [ ] (b) the probability that the cat does not catch any rat in the period of one hour is $e^{-2.5}$;
- [ ] (c) the variance of the number of rats that are caught in one hour by the cat is 5.

17. Data concerning orders in some restaurant was collected. Out of 1000 orders 500 contained apple pie and 300 contained tea. Estimated probability that apple pie and tea occurred together in the same order, resulting from this data, was 0.1. Assuming that the orders were independent it holds that

- [ ] (a) there were 100 orders in which apple pie and tea occurred together;
- [ ] (b) the probability of the event "tea is in the order" is 0.3;
- [ ] (c) the probability of the event "tea is in the order" given that "apple pie is in the order" is 0.5.

18. Consider the following optimization problem: given $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, find $s \in \mathbb{R}$ that minimizes $\|Ax - sx\|_2$. It follows that

- [ ] (a) if $x \neq 0$ then the problem has an unique solution;
- [ ] (b) the cost of finding the solution is $O(n)$;
- [ ] (c) if $x \neq 0$ then the normal equations for this problem are $x^{\mathrm{T}}sx = x^{\mathrm{T}}Ax$.

19. Let w^* be the best uniform polynomial approximation of degree at most 1 of the function $f(x) = \sin(x)$ on the interval $[0, \pi]$. It follows that

- [ ] (a) some alternating set for f and w^* contains 3 points from the interval $[0, \pi]$;
- [ ] (b) there exists a Lagrange interpolation polynomial of degree at most 1 with interpolation error equal to $\|f - w^*\|_\infty$;
- [ ] (c) the polynomial w^* has the form $w^*(x) = \pi/2$.

20. The input for all three algorithms mentioned below is an array of size n. In the random permutation model the expected time complexity of

- [ ] (a) Hoare selection algorithm is $\Theta(n \log \log n)$;
- [ ] (b) Quick Sort is $\Theta(n \log n)$;
- [ ] (c) Selection Sort is $\Theta(n\sqrt{n})$.

21. The *depth* of a node in a rooted tree is the number of *edges* on the path from the root to this node. The *height* of a rooted tree is the maximum of depths of its nodes. Consider the AVL tree T of height $d > 0$. It follows that

- [ ] (a) T has at least $2^d/1000$ nodes;
- [ ] (b) if we remove all the nodes of depth bigger than $d/2$, the remaining tree is a full binary tree;
- [ ] (c) there exists i, where $d/2 < i < d$, such that all nodes of depth i are non-leafs.

22. An example of a packet protocol is

- [ ] (a) IP;
- [ ] (b) UDP;
- [ ] (c) TCP.

23. Table `tab` was defined as

```
CREATE TABLE tab(a INT, b INT, c INT, PRIMARY KEY (a, b))
```

and its definition has not been modified. A valid content for `tab` consists of

- [ ] (a) rows `(1, 1, 1)` and `(1, 1, 2)`;
- [ ] (b) rows `(1, 1, 1)` and `(1, 2, 1)`;
- [ ] (c) rows `(1, 1, 1)` and `(2, 1, 1)`.

24. Let `R` be a table with two columns `X` and `Y`, containing n rows, where $n > 0$. Consider a SQL query

```
SELECT X, X FROM R
```

The result of this query

- [ ] (a) contains at least two rows;
- [ ] (b) is a table with exactly two columns;
- [ ] (c) contains at most n rows.

25. The logical block size in a file system is

- [ ] (a) the smallest file allocation unit;
- [ ] (b) determined dynamically depending on the file size;
- [ ] (c) a multiple of physical block size.

26. There are M > 1 writers and N > 1 readers running the following program:

```
semaphore S = N;

process writer() {
    while (true) {
        for (int i = 0; i < N; i++) P(S);
        // writing
        for (int i = 0; i < N; i++) V(S);
    }
}

process reader() {
    while (true) {
        P(S);
        // reading
        V(S);
    }
}
```

Let us assume that semaphore S is strongly fair. There is an execution that leads to

- [ ] (a) deadlock;
- [ ] (b) starvation of writers but not deadlock;
- [ ] (c) starvation of readers but not deadlock.

27. The product of two $n \times n$ matrices A and B is calculated in parallel by n^2 sequential processes $P(i, j)$ for $1 \leq i \leq n$, $1 \leq j \leq n$. Each process calculates the dot product of i-th row of A and j-th column of B in the linear cost.

- [ ] (a) The span of this computation is $\Theta(n)$.
- [ ] (b) The critical path length of this computation is $\Theta(n^2)$.
- [ ] (c) The work of this computation is $\Theta(n^3)$.

28. Assume that files a.py and b.py are both stored in the current folder and have the same contents, given below:

```
def sum(n):
    s, c = 0, 1
    while c <= n:
        s += c
        c += 1
    return s
```

After starting in the current folder the standard Python interpreter and entering first the command `import a, b` and then

- [ ] (a) the command `sum.a(sum.b(3))` the interpreter will print 21;
- [ ] (b) the command `a.sum(b.sum(3))` the interpreter will print 21;
- [ ] (c) the command `sum(sum(3))` the interpreter will print 21.

29. Here is a program in Python:

```
def f1(par):
    par = par + 1

def f2(par):
    par = par + "1"

def f3(par):
    par = par + [1]

i, s, lst = 0, "0", [0]
f1(i), f2(s), f3(lst)
print(f"{i=}, {s=}, {lst=}")
```

When executed the program prints the value of

- [ ] (a) the variable i equal to 1;
- [ ] (b) the variable s equal to '01';
- [ ] (c) the variable lst equal to [0, 1].

30. Here is a program in Python:

```
lst1, lst2, lst3, lst4 = [1, 2], [3, 4], [5, 6], [7, 8]

lst1 += lst4
lst2.append(lst4)
lst3 = [lst4]

print(f"{len(lst1)=}, {len(lst2)=}, {len(lst3)=}")
```

When executed the program prints the length of

- [ ] (a) the list lst1 equal to 4;
- [ ] (b) the list lst2 equal to 4;
- [ ] (c) the list lst3 equal to 4.