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ARCS INTERSECTING AT MOST ONCE

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Abstract. We prove that on a punctured oriented surface with Euler characteristic $\chi < 0$, the maximal cardinality of a set of essential simple arcs that are pairwise non-homotopic and intersecting at most once is $2|\chi|(|\chi|+1)$. This gives a cubic estimate in $|\chi|$ for a set of curves pairwise intersecting at most once on a closed surface. We also give polynomial estimates in $|\chi|$ for sets of arcs and curves pairwise intersecting a uniformly bounded number of times. Finally, we prove that on a punctured sphere the maximal cardinality of a set of arcs starting and ending at specified punctures and pairwise intersecting at most once is $\frac{1}{2}|\chi|(|\chi|+1)$.

1 Introduction

Let S be a connected oriented punctured surface of finite type with Euler characteristic $\chi < 0$. We start with the following observation.

Remark 1.1. The maximal cardinality of a set of essential simple arcs on S that are pairwise non-homotopic and disjoint is $3|\chi|$. This follows from the fact that every such set can be extended to form an ideal triangulation of the surface. Fixing any hyperbolic metric, the surface S has area $2\pi|\chi|$, while an ideal triangle has area π. Thus there are $2|\chi|$ triangles, and these have $6|\chi|$ sides among which each arc appears twice.

The main result of the article is a similar formula for arcs pairwise intersecting at most once.

Theorem 1.2. *The maximal cardinality of a set $\mathcal{A}$ of essential simple arcs on S that are pairwise non-homotopic and intersecting at most once is*

$$f(|\chi|) = 2|\chi|(|\chi|+1).$$

Example 1.3. This bound is sharp. The surface S can be cut along disjoint arcs into an ideal polygon P. Since the Euler characteristic of P equals 1, the number of arcs we have cut along equals $1-\chi = |\chi|+1$, so that P is a $(2|\chi|+2)$–gon. We consider

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the union $\mathcal{A}$ of the set of all diagonals of P, which has cardinality $\frac{(2|\chi|+2)(2|\chi|-1)}{2}$, with the set of $|\chi|+1$ arcs we have cut along. In total,

$$|\mathcal{A}| = \frac{(2|\chi|+2)(2|\chi|-1)}{2} + (|\chi|+1) = (|\chi|+1)\big((2|\chi|-1)+1\big).$$

We deduce a (non-sharp) cubic bound on curves pairwise intersecting at most once. Let g be the genus of S. We now allow S to be closed. Let $f(\cdot)$ be the function from Theorem 1.2.

Theorem 1.4. *The cardinality of a set $\mathcal{C}$ of nonperipheral essential simple closed curves on S that are pairwise non-homotopic and intersecting at most once is at most*

$$g \cdot \big(2f(|\chi|)+1\big) + |\chi| - 1.$$

The question about the maximal cardinality of $\mathcal{C}$ was asked by Farb and Leininger. Apart from the intrinsic beauty of the problem, one motivation to compute this value is that it is one more than the dimension of the 1-*curve complex*, see [Suo07, Quest 1]. This complex is constructed by taking the flag span of the graph whose vertices correspond to curves on S and edges join the vertices corresponding to curves intersecting at most once, as opposed to disjoint curves for the usual curve complex. The 1-curve complex might have better properties than the usual one (like the Rips complex versus the Cayley complex of a hyperbolic group), in particular it might be contractible [Suo07, Quest 2] as opposed to the usual curve complex [Har86]. Another interpretation of the condition on $\mathcal{C}$, this time in the terms of the mapping class group of S, is that Dehn twists around the curves in $\mathcal{C}$ pairwise either commute or satisfy the braid relation.

For S a torus, $\mathcal{C}$ consists of at most 3 curves. For S a closed genus 2 surface, the maximal cardinality of $\mathcal{C}$ is 12, as proved by Malestein, Rivin, and Theran [MRT14]. They also gave examples of $\mathcal{C}$ with size quadratic in $|\chi|$ for arbitrary surfaces. Note that these sets are not the sets of systoles (shortest curves) for any hyperbolic metric as it was shown by Parlier that sizes of such sets grow subquadratically [Par13]. Until our work, the best upper bound for the cardinality of $\mathcal{C}$ was only exponential [MRT14]. Farb and Leininger obtained the lower quadratic and upper exponential bounds as well [Lei11].

The question of Farb and Leininger is a particular case of a problem studied by Juvan, Malnič and Mohar [JMM96], who allowed the curves to intersect at most a fixed number of times k. They proved that there exists an upper bound on the cardinality of $\mathcal{C}$ if the surface is fixed. We establish upper bounds polynomial in $|\chi|$ for the sets of either curves or arcs, where for the arcs we give also lower bounds of the same degree. We do not have, even for arcs and $k = 2$, a guess of an explicit formula.

Theorem 1.5. *The maximal cardinality of a set $\mathcal{A}$ of essential simple arcs on S that are pairwise non-homotopic and intersecting at most k times grows as a polynomial of degree $k+1$ in $|\chi|$.*

Corollary 1.6. *The maximal cardinality of a set $\mathcal{C}$ of nonperipheral essential simple closed curves on S that are pairwise non-homotopic and intersecting at most k times grows at most as a polynomial of degree k^2+k+1 in $|\chi|$.*

Note that in Corollary 1.6 the upper bound does not match Aougab's recent lower bound, which is polynomial of degree $\lfloor \frac{k+1}{2} \rfloor + 1$ [Aou14], improving the degree $\frac{k}{4}$ from [JMM96]. Thus in Corollary 1.6, similarly as in Theorem 1.4, there is still room for improvement. Our upper bound is also not efficient if we fix the surface and vary k, compared to [JMM96].

Finally, we prove the following variation of Theorem 1.2.

Theorem 1.7. *Let S be a punctured sphere, and let p, p' be its (not necessarily distinct) punctures. Then the maximal cardinality of a set $\mathcal{A}$ of essential simple arcs on S that are starting at p and ending at p', and pairwise intersecting at most once is*

$$\frac{1}{2}|\chi|(|\chi|+1).$$

Organisation. The outline of the proof of Theorem 1.2 and the deduction of Theorem 1.4 is given in Sect. 2. The two key lemmas are proved in Sect. 3. In Sect. 4 we prove Theorem 1.5 and Corollary 1.6. We prove Theorem 1.7 in Sect. 5.

2 Nibs

In this section, we give the proof of Theorem 1.2, up to two lemmas postponed to Sect. 3. Then we deduce Theorem 1.4. A rough idea is that we generalise the argument of Remark 1.1. We replace disjoint triangles of the triangulation with *nibs*, which have also area π but might (self-)intersect. However, we have a linear bound of the number of nibs at a point (Proposition 2.2).

As in the introduction, S is an oriented punctured surface with Euler characteristic $\chi < 0$. An *arc* is a map $(-\infty, \infty) \to S$ that converges to punctures at infinities. An arc is *simple* if it is embedded and *essential* if it is not homotopic into a puncture neighbourhood (called a *cusp*). Unless stated otherwise, all arcs are simple and essential and sets of arcs consist of arcs that are pairwise non-homotopic. Assume that we have a set $\mathcal{A}$ of arcs on S. We fix an arbitrary complete hyperbolic metric on S and assume that all arcs in $\mathcal{A}$ are realised as geodesics. In particular they are pairwise in *minimal position*, that is minimising the number of intersection points in their homotopy classes.

Definition 2.1. A *tip* τ of $\mathcal{A}$ is a pair (α, β) of oriented arcs in $\mathcal{A}$ starting at the same puncture and consecutive. That is to say that there is no other arc in $\mathcal{A}$ issuing from this puncture in the clockwise oriented cusp sector from α to β. We allow $\beta = \alpha$ or $\beta = \alpha^{-1}$.

Let $\tau = (\alpha, \beta)$ be a tip and let N_τ be an open abstract ideal hyperbolic triangle with vertices a, t, b. The tip τ determines a unique local isometry $\nu_\tau \colon N_\tau \to S$ sending

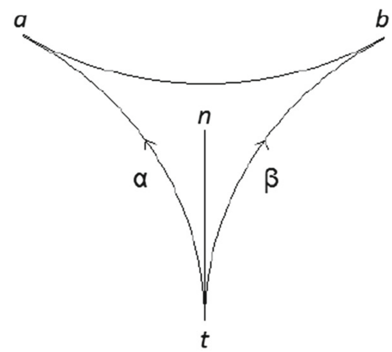

Figure 1: A tip, its nib and a slit

ta to α, tb to β and mapping a neighbourhood of t to the clockwise oriented cusp sector from α to β. We call ν_τ the *nib* of τ.

Proposition 2.2. *Suppose that the arcs in $\mathcal{A}$ pairwise intersect at most once. Let $\nu\colon N = \bigsqcup_\tau N_\tau \to S$ be the disjoint union of all the nibs ν_τ. Then for each $s \in S$ the preimage $\nu^{-1}(s)$ consists of at most $2(|\chi|+1)$ points.*

Here are the ingredients of the proof of Proposition 2.2.

Lemma 2.3. *Suppose that we have a partition of the set of punctures of S into $\mathcal{P}_1$ and $\mathcal{P}_2$. The maximal cardinality of a set of pairwise non-homotopic disjoint essential simple arcs that start in $\mathcal{P}_1$ and end in $\mathcal{P}_2$ is $2|\chi|$.*

Proof. Without loss of generality we can assume that the set of arcs is maximal. Then all the complementary components are ideal squares, so that each of them has area 2π. Since the area of S is $2\pi|\chi|$, there are $|\chi|$ such squares. Each square has 4 sides and each arc is a side of two squares. Thus there are $2|\chi|$ arcs. □

Definition 2.4. Let $n \in N_\tau$ be a point in the domain of a nib. The *slit* at n is the restriction of ν_τ to the geodesic ray in N_τ joining t with n. See Figure 1.

By a *geodesic ray* we will always mean a geodesic joining a point at infinity or a puncture with a point in the interior of $\mathbf{H}^2$ or S. We postpone the proofs of the following two lemmas to Sect. 3.

Lemma 2.5. *A slit is an embedding.*

Actually, it can be proved that if α and β are disjoint, then the entire nib embeds. Otherwise it is at worst 2 to 1. We emphasise that in Lemma 2.5 we do not assume that the arcs in $\mathcal{A}$ pairwise intersect at most once.

Lemma 2.6. *Suppose that the arcs in $\mathcal{A}$ pairwise intersect at most once. If for distinct $n, n' \in N$ we have $\nu(n) = \nu(n')$, then the images in S of the slits at n, n' are disjoint except at the endpoint.*

Proof of Proposition 2.2. Let S' be the surface obtained from S by introducing an additional puncture at s. Then the absolute value of the Euler characteristic of S' is $|\chi| + 1$. Consider the collection of all slits at preimages of s, and let $\mathcal{S}$ be the set of corresponding arcs on S'. The arcs in $\mathcal{S}$ are simple by Lemma 2.5 and pairwise disjoint by Lemma 2.6. The set of punctures of S' decomposes into $\mathcal{P}_1 = \{s\}$ and $\mathcal{P}_2$, which are the punctures inherited from the punctures of S. Each arc in $\mathcal{S}$ joins $\mathcal{P}_1$ with $\mathcal{P}_2$, hence by Lemma 2.3 we have $|\mathcal{S}| \leq 2(|\chi| + 1)$. □

Proof of Theorem 1.2. Each arc in $\mathcal{A}$ is the first arc of exactly two tips, depending on its orientation, so the area of N equals $2|\mathcal{A}|\pi$. The area of S equals $2\pi|\chi|$. By Proposition 2.2, the map $\nu\colon N \to S$ is at most $2(|\chi| + 1)$ to 1, so we have

$$2|\mathcal{A}|\pi \leq 2\pi|\chi| \cdot 2(|\chi| + 1).$$ □

Question 2.7. Example 1.3 does not exploit all configurations of $2|\chi|(|\chi| + 1)$ arcs pairwise intersecting at most once. An example of another configuration is obtained by viewing the four-punctured sphere S as the boundary of a tetrahedron T with vertices removed. Consider the following set of 12 arcs on S. The first six arcs are the six edges of T. Each of the other six arcs is obtained by taking the midpoint m of one of the edges and concatenating at m the medians of the two faces of T containing m.

What can one say about the space of all configurations of $2|\chi|(|\chi| + 1)$ arcs pairwise intersecting at most once?

We now deduce Theorem 1.4. A closed curve on S is *simple* if it is embedded, *essential* if it is homotopically nontrivial, and *nonperipheral* if it is not homotopic into a cusp. In what follows, unless stated otherwise, all curves are closed, simple, essential, nonperipheral and sets of curves consist of curves that are pairwise non-homotopic. We fix an arbitrary hyperbolic metric on S and realise all curves as geodesics, so that they are pairwise in minimal position.

Proof of Theorem 1.4. Note that if all the curves in $\mathcal{C}$ are separating, then they are disjoint. The theorem follows since a set of disjoint separating curves has cardinality bounded by $|\chi| - 1$. Henceforth we will assume that there is a non-separating curve in $\mathcal{C}$.

We prove the bound by induction on g assuming χ is fixed. If $g = 0$, then all curves are separating, a case which we have already discussed. If $g > 0$, choose a non-separating curve α in $\mathcal{C}$. Then $\mathcal{C} - \{\alpha\}$ partitions into the set $\mathcal{C}_{\mathrm{dis}}$ of curves disjoint from α and the set $\mathcal{C}_{\mathrm{int}}$ of curves intersecting α. Let S' be the punctured surface obtained from S by cutting along α. Note that the Euler characteristic of S' equals the Euler characteristic χ of S and the genus of S' is one less than that of S. By the inductive hypothesis we have $|\mathcal{C}_{\mathrm{dis}}| \leq (g-1)(2f(|\chi|) + 1) + |\chi| - 1$.

Each curve in $\mathcal{C}_{\mathrm{int}}$ is cut to an arc on S'. Two different curves cut to the same arc differ by a power of the Dehn twist D along α. Since curves differing by D^l intersect in l points, there might be at most 2 such curves in $\mathcal{C}_{\text{int}}$ for each arc on S'. Thus by Theorem 1.2 we have $|\mathcal{C}_{\text{int}}| \leq 2f(|\chi|)$. Hence

$$\begin{aligned}|\mathcal{C}| = 1 + |\mathcal{C}_{\text{dis}}| + |\mathcal{C}_{\text{int}}| &\leq 1 + (g-1)(2f(|\chi|)+1) + |\chi| - 1 + 2f(|\chi|) = \\ &= g \cdot (2f(|\chi|)+1) + |\chi| - 1. \qquad \square\end{aligned}$$

3 Slits

In this section we complete the proof of Theorem 1.2, by proving Lemmas 2.5 and 2.6 about slits.

Proof of Lemma 2.5. We identify the universal cover of S with the hyperbolic plane $\mathbf{H}^2$ and $\pi_1 S$ with the deck transformation group. Orient the circle at infinity. For points x, x' at infinity we will use the notation $[x, x']$ for the oriented sub-interval of the circle at infinity, and notation xx' for the geodesic arc inside $\mathbf{H}^2$. Choose a lift of ν_τ to $\mathbf{H}^2$, and identify its image with N_τ. Extend the slit to a ray γ joining t with the side ab of N_τ. We need to show that the interior of γ embeds in S.

Suppose that γ intersects its translate $g(\gamma)$ in a point O for some nontrivial $g \in \pi_1 S$. Without loss of generality assume that $g(O)$ is farther from $g(t)$ than O on $g(\gamma)$. Let q be the point at infinity that is the endpoint of the geodesic extension of $g(\gamma)$. See Figure 2. We analyse where are the fixed point(s) of g. Since O and $g(O)$ lie in that order on $g(t)q$, in the case where g is hyperbolic both fixed points lie on one side of $g(t)q$, and on that side the repelling point separates the attracting point from $g(t)$. Hence t also has to lie on that side, between the repelling point and $g(t)$. Without loss of generality assume that $t, g(t), q$ are cyclically ordered. We conclude that the fixed points lie in the interval $[q, t]$. Moreover, g restricts on the interval

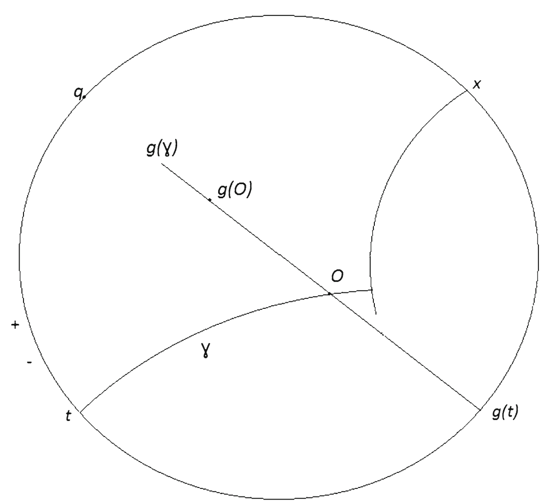

Figure 2: Here we must take $x = a$, since b lies on the wrong side of $g(t)q$

$[g(t), q]$ to an increasing function into $[g(t), t]$. The same holds in the case where g is parabolic.

There is $x \in \{a, b\}$ that lies in the interval $[g(t), q]$. Then the points $t, g(t), x, g(x)$ lie at the circle at infinity in that cyclic order. This means that the projection of the arc tx to S self-intersects, which is a contradiction with the fact that it belongs to $\mathcal{A}$. $\square$

In the proof of Lemma 2.6 we will need the following. We parametrise a ray $(-\infty, 0] \to S$ so that the limit at $-\infty$ is a puncture and the image of 0 is a point on S.

SUBLEMMA 3.1. *Let $H, H' \colon (-\infty, 0] \times [-1, 1]$ be homotopies of rays on S, such that*

- $H(0, \cdot) = H'(0, \cdot)$,
- *arcs $H(\cdot, y)$ and $H'(\cdot, y)$ are in minimal position on the surface $S - H(0, y)$ for $y = -1, 1$,*
- *rays $H(\cdot, y)$ and $H'(\cdot, y)$ are embedded for $y \in [-1, 1]$.*

Then the arcs $H(\cdot, -1)$ and $H'(\cdot, -1)$ on $S - H(0, -1)$ intersect the same number of times as the arcs $H(\cdot, 1)$ and $H'(\cdot, 1)$ on $S - H(0, 1)$.

Proof. We identify (up to homotopy) the surfaces $S - H(0, y)$ for $y \in [-1, 1]$ by pushing the puncture $H(0, y) = H'(0, y)$. The last condition, saying that all the rays are embedded, ensures that the arcs $H(\cdot, y)$ (respectively $H'(\cdot, y)$) on $S - H(0, y)$ can be identified. The assertion follows from the fact that the number of intersection points between arcs in minimal position for $y = -1, 1$ is a homotopy invariant. $\square$

Proof of Lemma 2.6. Let τ, τ' be tips with distinct $n \in N_\tau$, $n' \in N_{\tau'}$ and $\nu(n) = \nu(n')$. Note that by the requirement in Definition 2.1 of a tip that α and β are consecutive, the slits at n, n' are distinct rays on S. Choose lifts of $N_\tau, N_{\tau'}$ to $\mathbf{H}^2$ so that n is identified with n'. Extend the slits at n, n' to geodesic rays γ, γ' from t, t' to $ab, a'b'$. Without loss of generality suppose that the interval (t, t') at infinity contains the endpoints of the geodesic extensions of γ, γ'.

First consider the case where there are $x \in \{a, b\}, x' \in \{a', b'\}$ that lie in (t, t') so that $x' < x$, see the left side of Figure 3. Then the geodesic arcs $tx, t'x'$ intersect at a point m. Let $H(\cdot, \cdot)$ be the projection to S of the homotopy of geodesic rays joining tn to tm ending in the geodesic segment nm. By Lemma 2.5, these rays embed in S. Similarly let $H'(\cdot, \cdot)$ be the projection of the homotopy joining $t'n$ to $t'm$. Since the rays are geodesic, the arcs $H(\cdot, y), H'(\cdot, y)$ are in minimal position on $S - H(0, y)$. By Sublemma 3.1, if the slits at n, n' intersect outside the endpoint, then the projections to S of tx and $t'x'$ intersect at least once outside the projection of m, hence at least twice in total, which is a contradiction with the definition of $\mathcal{A}$.

The other case is that $a, b' \in [t', t]$ and $b \le a' \in (t, t')$, see the right side of Figure 3. If $a = t'$ or $b' = t$, then we have also the second equality, since α and β were required to be consecutive in Definition 2.1 of a tip. In this situation, let m be

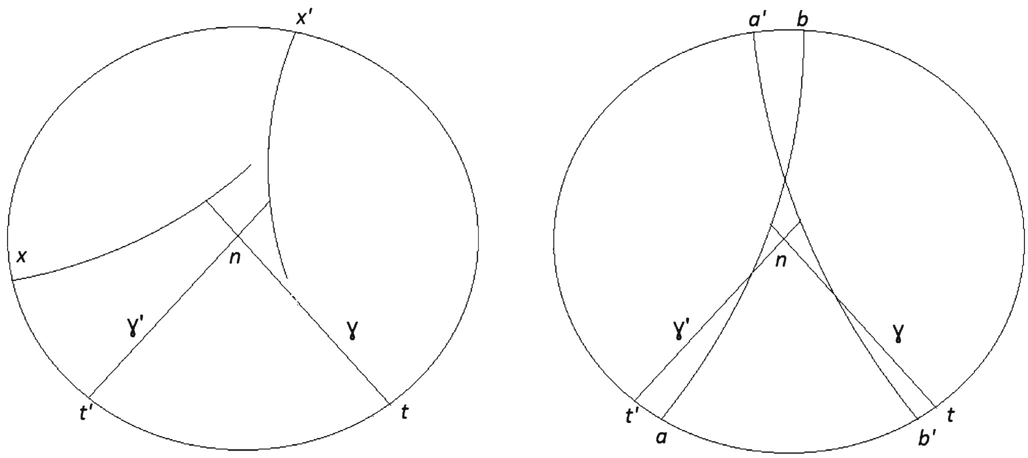

Figure 3: Instance of the first case with $x = a, x' = a'$ on the *left*, and the second case on the *right*

any point on the geodesic arc tt'. Construct H, H' as before. By Sublemma 3.1, if the slits at n, n' intersect outside the endpoint, then the projections to S of tm and $t'm$ intersect outside the projection of m. This contradicts the fact that the projection of $tt' = ta$ is in $\mathcal{A}$, hence simple.

Thus we can assume $t' < a < b' < t$. Then the geodesic arcs $ta, t'b'$ intersect. We denote their intersection point by m and apply Sublemma 3.1 as in the first case. □

The same proof gives also the following.

LEMMA 3.2. *Suppose that the arcs in $\mathcal{A}$ pairwise intersect at most $k \geq 1$ times. If for distinct $n, n' \in N$ we have $\nu(n) = \nu(n')$, then the images in S of slits at n, n' intersect at most $k - 1$ times outside the endpoint.*

QUESTION 3.3. A slit can be viewed as an interpolation between the arcs ta and tb of a nib. Is there an alternate proof of Lemmas 2.5 and 2.6 using convexity of the distance function?

4 Multiple Intersections

In this section we prove Theorem 1.5 and Corollary 1.6, starting with the lower bound in Theorem 1.5.

EXAMPLE 4.1. Consider a twice punctured sphere with punctures p, p' viewed as a cylinder. Let S be the surface obtained by putting on each of $k+1$ concentric circles separating p from p' a set of $\frac{|\chi|}{k+1}$ punctures. Let α be an arc joining p to p' crossing each of these concentric circles exactly once. Consider the set $\mathcal{A}$ of arcs joining p to p' crossing each of these concentric circles exactly once, and disjoint from α. Arcs in

$\mathcal{A}$ pairwise intersect at most k times, since there might be at most one intersection point in each sub-cylinder bounded by consecutive concentric circles. Each arc in $\mathcal{A}$ is determined by the locations of its intersection points with concentric circles, for each of which there are $\frac{|\chi|}{k+1}+1$ possibilities. Moreover, these locations determine the arc uniquely, except that an arc homotopic to α will be given two times, by the two extreme sets of locations. Thus

$$|\mathcal{A}| = \left(\frac{|\chi|}{k+1}+1\right)^{k+1} - 1.$$

Proof of Theorem 1.5. By Example 4.1 it remains to prove the upper bound. We prove it by induction on the number k of allowed intersections. Case $k=0$ is discussed in Remark 1.1. The inductive step from $k=0$ to $k=1$ has been performed in the proof of Theorem 1.2, and we now generalise it to any k. Suppose that we have proved that the maximal cardinality of a set of arcs on S pairwise intersecting at most $k \geq 0$ times is $\leq C(k)|\chi|^{k+1}$ for some constant $C(k)$ depending only on k. Let $\mathcal{A}$ be a set of arcs on S pairwise intersecting at most $k+1$ times. They give rise to $2|\mathcal{A}|$ tips. Let $s \in S$. By Lemmas 2.5 and 3.2 the slits at the preimage $\nu^{-1}(s)$ in the union of all the nibs form a set of simple arcs on $S-s$ pairwise intersecting at most k times. Thus by the inductive assumption the map ν is at most $C(k)(|\chi|+1)^{k+1}$ to 1. Hence the area $2|\mathcal{A}|\pi$ of the domain of ν is bounded by

$$2\pi|\chi| \cdot C(k)(|\chi|+1)^{k+1} \leq 2\pi C(k+1)|\chi|^{k+2},$$

for some $C(k+1)$, as desired. □

Proof of Corollary 1.6. We orient all the curves. We cut the surface along an arbitrary curve $\alpha \in \mathcal{C}$, possibly separating. The surface $S-\alpha$ is either connected of Euler characteristic χ or consists of two components with Euler characteristics χ_1, χ_2 satisfying $\chi_1+\chi_2=\chi$, hence $|\chi_1|^{k+1}+|\chi_2|^{k+1} < |\chi|^{k+1}$.

Let $\mathcal{C}_{\text{int}}$ denote the set of curves in $\mathcal{C}$ intersecting α. Each curve in $\mathcal{C}_{\text{int}}$ splits into $\leq k$ oriented arcs on $S-\alpha$. These arcs pairwise intersect at most k times, hence by Theorem 1.5 there is at most $C(k)|\chi|^{k+1}$ of them. Each curve in $\mathcal{C}_{\text{int}}$ can be reconstructed from an ordered k'-tuple of $k' \leq k$ of these arcs, so this gives $\leq k\big(C(k)|\chi|^{k+1}\big)^k$ possibilities, where the factor k comes from varying by a power D^l of the Dehn twist D around α with $l \leq k$. (Note that we do not need to consider partial twists since the k'-tuple of arcs is ordered.) Altogether this yields $|\mathcal{C}_{\text{int}}| \leq C'(k)|\chi|^{k(k+1)}$.

To bound $|\mathcal{C}-\mathcal{C}_{\text{int}}|$, we need to continue performing the cutting procedure. The number of cutting steps is bounded by the cardinality of the maximal set of disjoint curves on S, which equals $|\chi|-1+g < 2|\chi|$. Thus we obtain a polynomial bound of degree $1+k(k+1)$ in $|\chi|$ for $|\mathcal{C}|$. □

5 Punctured Spheres

In this section we prove Theorem 1.7. We start with giving the lower bounds. The construction will be different depending on whether the ending puncture p' is the same or distinct from the starting puncture p.

Example 5.1. Let S be a punctured sphere. We construct a set of $\frac{1}{2}|\chi|(|\chi|+1)$ arcs on S that are starting and ending at a specified puncture p, pairwise intersecting at most once.

View S as a punctured disc with the outer boundary corresponding to p and other $|\chi|+1$ punctures lying on a circle c parallel to the outer boundary, dividing c into $|\chi|+1$ segments. Up to homotopy, there are $|\chi|+1$ rays joining the centre of the disc with p and intersecting c only once, at $|\chi|+1$ possible segments of c. Consider $\binom{|\chi|+1}{2}$ arcs obtained by merging a pair of such rays. Up to homotopy they pairwise intersect at most once, at the centre of the disc.

Example 5.2. Let S be a punctured sphere. We construct a set $\mathcal{A}$ of $\frac{1}{2}|\chi|(|\chi|+1)$ arcs on S that are starting at a specified puncture p and ending at a distinct specified puncture p', pairwise intersecting at most once.

Consider a cylinder with boundary components corresponding to p, p', and place other $|\chi|$ punctures on an arc α joining p with p' dividing it into an ordered set of $|\chi|+1$ segments. There are $\binom{|\chi|+1}{2}$ choices of a pair $\alpha_1 < \alpha_2$ of such segments. For each such pair we construct the following arc in $\mathcal{A}$. We start at the starting point of α and follow it on its left avoiding the punctures until we find ourselves on the level of α_1. Then we cross α_1 and change to the right side of α. We stay on the right side until we reach α_2, where we switch again to the left side and continue till the ending point of α. It is easy to verify that arcs in $\mathcal{A}$ pairwise intersect at most once.

In the proof of the upper bound in Theorem 1.7 we need the following.

Lemma 5.3. *Suppose that we have l points on the unit circle in $\mathbf{R}^2$ and a set $\mathcal{I}$ of pairwise intersecting chords between them. We allow degenerate chords that are single points. Then $|\mathcal{I}| \leq l$.*

Up to considering separately the (easy) case where there is a degenerate chord, this lemma follows from the linear case of the thrackle conjecture, proved by Paul Erdös [Erd46]. This was pointed out to us by Sergey Norin. Here we give a proof found by Marcin Sabok.

Proof. We label the points cyclically by the set $\{0,\dots,l-1\} \subset \mathbf{N}$ and assign to each chord the interval of $\mathbf{R}$ with corresponding endpoints. We keep the notation $\mathcal{I}$ for this set of intervals. By Helly's Theorem, there is a point $i \in \{0,\dots,l-1\}$ belonging to all the intervals in $\mathcal{I}$. The centres of the intervals in $\mathcal{I}$ lie in the intersection of $\frac{1}{2}\mathbf{N}$ with $[\frac{i}{2}, i+\frac{l-1-i}{2}]$, which has cardinality l. Thus if $|\mathcal{I}| \geq l+1$, then by the pigeon principle two intervals in $\mathcal{I}$ have the same centres. But this means that their corresponding chords are disjoint. $\square$

Proof of Theorem 1.7. By Examples 5.1 and 5.2 it suffices to prove the upper bound, which we do by induction. We first treat the case where the arcs in $\mathcal{A}$ are required to start and end at the same puncture p.

For $|\chi| = 1$ the theorem is trivial as on the thrice-punctured sphere there is only one essential arc joining p with itself. Suppose that we have proved the theorem for smaller $|\chi|$, in particular for the surface $\bar{S}$ obtained from S by forgetting a puncture r distinct from p. For an arc $\alpha \in \mathcal{A}$ on S let $\bar{\alpha}$ be the corresponding possibly non-essential arc on $\bar{S}$. Let $\bar{\mathcal{A}}$ denote the union of all essential $\bar{\alpha}$. We will now analyse to what extent this map $\mathcal{A} \to \bar{\mathcal{A}}$ is well-defined and injective. To this end, we consider the following *exceptional* arcs in $\mathcal{A}$, and an associated collection $\mathcal{I}$ of rays on $\bar{S}$ from p to r and arcs from p to p passing through r.

The first instance of an *exceptional* arc α is when $\bar{\alpha}$ is non-essential. This happens if and only if α separates r from all other punctures. We include in $\mathcal{I}$ the unique ray on $\bar{S}$ joining p with r disjoint from α.

Secondly, observe that if $\bar{\alpha} = \bar{\alpha}'$ for $\alpha \neq \alpha'$, then since α and α' intersect at most once, they are in fact disjoint (up to homotopy) and bound a bigon with one puncture r. There is no third arc $\alpha'' \in \mathcal{A}$ with $\bar{\alpha}'' = \bar{\alpha}$, since at most one component of $S - \alpha \cup \alpha' \cup \alpha''$ contains r. We call such α and α' *exceptional* as well. We include in $\mathcal{I}$ the unique essential arc on $\bar{S}$ joining p with p and passing through r that is disjoint from both α and α'.

Summarising, the map $\mathcal{A} \to \bar{\mathcal{A}}$ is well-defined and injective outside the set of exceptional arcs of cardinality $|\mathcal{I}|$, where we avoid only one exceptional arc from each pair bounding a bigon with puncture r. By the inductive assumption we have $|\bar{\mathcal{A}}| \leq \frac{1}{2}(|\chi| - 1)|\chi|$, so it suffices to prove $|\mathcal{I}| \leq |\chi|$.

Let $\mathcal{H}$ be the set of arcs on S obtained from $\mathcal{I}$ by reintroducing the puncture r and thus splitting some arcs of $\mathcal{I}$ in half. Note that if some halves coincide, we keep only one copy in $\mathcal{H}$. Observe that the arcs in $\mathcal{H}$ are disjoint, since if we had two intersecting halves of two arcs in $\mathcal{I}$, then we could choose two corresponding exceptional arcs intersecting at least twice. See the left side of Figure 4. Arcs in $\mathcal{H}$ connect r with p, hence their complementary components are punctured bigons, thus of area $\geq 2\pi$. Hence there are at most $|\chi|$ such components and thus $|\mathcal{H}| \leq |\chi|$. We now intersect $\mathcal{H}$ with a small circle centred at r. Each arc in $\mathcal{I}$ is determined by a pair of points or a point of this intersection, and we connect them by a chord. These chords satisfy the hypothesis of Lemma 5.3, since otherwise we could choose two corresponding exceptional arcs intersecting at least twice. See the right side of Figure 4. By Lemma 5.3 we have $|\mathcal{I}| \leq |\chi|$, which finishes the proof in the case where $p' = p$.

We now consider the case where $p' \neq p$. The argument is the same, with the following modifications. The puncture r that we are forgetting is required to be distinct from both p and p'. All $\bar{\alpha}$ are essential, since they connect different punctures. However, it is no longer true that if $\bar{\alpha} = \bar{\alpha}'$ for $\alpha \neq \alpha'$, then α and α' are disjoint. Nevertheless, it is easy to see that they may intersect only in the configuration illustrated in Figure 5. In that case all arcs in $\mathcal{A}$ are disjoint from arc α'' from Figure 5,

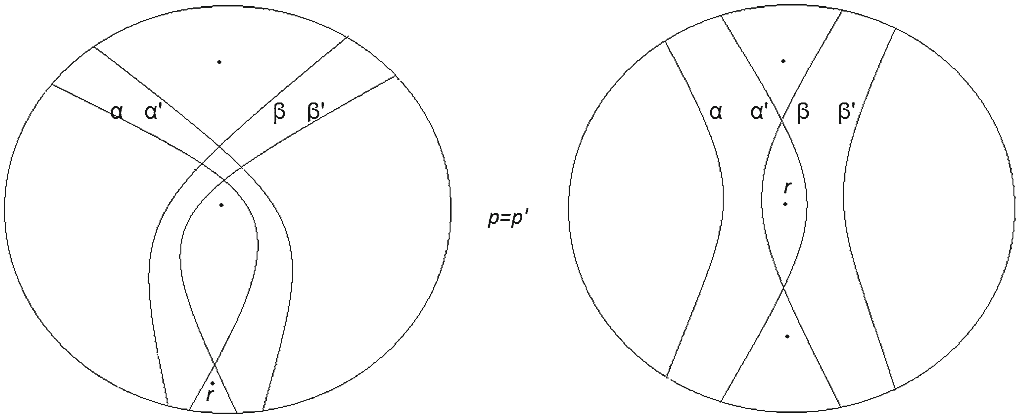

Figure 4: On the *left* why the halves of the arcs in $\mathcal{I}$ are disjoint, on the *right* why $\mathcal{I}$ satisfies the hypothesis of Lemma 5.3

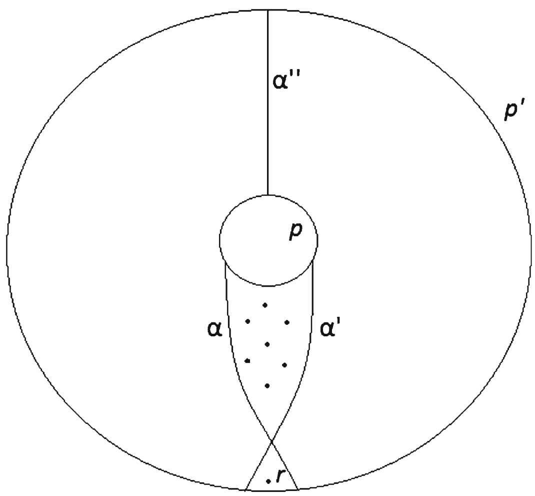

Figure 5: The unique configuration of three arcs with common $\bar{\alpha}$

thus without loss of generality we can assume $\alpha'' \in \mathcal{A}$. We then have $\bar{\alpha}'' = \bar{\alpha}$. Moreover, there are no other arcs in $\mathcal{A}$ with the same $\bar{\alpha}$. We call both α, α' *exceptional* and associate to them the bigons that they bound with α'' containing r.

We construct $\mathcal{I}$ and $\mathcal{H}$ as before. However, as pointed out to us by one of the referees, among the complementary components of $\mathcal{H}$, except for the punctured bigons, there is a square, possibly with no punctures, hence also only of area $\geq 2\pi$. Thus we can only conclude with $|\mathcal{H}| \leq |\chi| + 1$. We again intersect $\mathcal{H}$ with a small circle centred at r. We assign chords to arcs in $\mathcal{I}$ as before. Note that the points of the intersection of the circle with $\mathcal{H}$, depending on whether the arc in $\mathcal{H}$ joins r to p or p', are partitioned in two families $\mathcal{Q}, \mathcal{Q}'$ of consecutive points. We consider an additional chord joining the two outermost points of $\mathcal{Q}$. Since each of the other chords connects a point in $\mathcal{Q}$ to a point in $\mathcal{Q}'$, it intersects the additional chord. Hence the chords satisfy the hypothesis of Lemma 5.3. Thus $|\mathcal{I}|+1 \leq |\chi|+1$, as desired. □

QUESTION 5.4. Is there a way to make the proof of Theorem 1.2 work to prove Theorem 1.7 or vice-versa?

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