TWIST-RIGID COXETER GROUPS

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Abstract. We prove that two angle-compatible Coxeter generating sets of a given finitely generated Coxeter group are conjugate provided one of them does not admit any elementary twist. This confirms a basic case of a general conjecture which describes a potential solution to the isomorphism problem for Coxeter groups.

MSC: 20F10, 20F55

Keywords: Coxeter group, isomorphism problem, twist, rigidity, hierarchy

1. Introduction

A subset S of a group W is called a **Coxeter generating set** if there is a Coxeter matrix $(m_{s,s'})_{s,s'\in S}$ such that the relations $(ss')^{m_{s,s'}} = 1$ provide a presentation of W. A group admitting a Coxeter generating set is called a **Coxeter group**. In this article we consider only finitely generated Coxeter groups; this hypothesis will not be repeated anymore. Any Coxeter generating set of such a group is automatically finite.

It is a basic and natural problem to determine all possible Coxeter generating sets for a given Coxeter group W. Finding an algorithmic way to describe these would actually solve the isomorphism problem for Coxeter groups, which as of today remains open. Substantial progress in this direction has been accomplished in recent years (see [Müh06] for a 2006 survey), providing in particular some important reduction steps which we shall now briefly outline.

Given a fixed Coxeter generating set S for W, an **S-reflection** (or a **reflection**, if the dependence on the generating set S does not need to be emphasised) is an element conjugate to some element of

*Supported by the Belgian Fund for Scientific Research (FNRS).

†Partially supported by MNiSW grant N201 012 32/0718 and the Foundation for Polish Science.

S. Two Coxeter generating sets S and R for W are called **reflection-compatible** if the set of S-reflections is contained in the set of R-reflections. Then the sets of S- and R-reflections coincide (see Corollary A.2 in Appendix A). We shall further say that S and R are **angle-compatible** if every **spherical** pair $\{s, s'\} \subset S$ (*i.e.* every pair generating a finite subgroup) is conjugate to some pair $\{r, r'\} \subset R$ (in particular if we put $s = s'$ this implies that S and R are reflection-compatible). Saying that S and R are angle-compatible means exactly that S is *sharp-angled* with respect to R, in the language of [Müh06] and [MM08]. However, since this implies that conversely every spherical pair $\{r, r'\} \subset R$ is conjugate to some pair $\{s, s'\} \subset S$ (see Corollary A.4), we prefer a symmetric way of phrasing that.

Reflection- and angle-compatibility are well illustrated by the simplest class of Coxeter groups, namely finite dihedral groups. Indeed, the dihedral group of order 20 is isomorphic to the direct product of $\mathbf{Z}/2$ with the dihedral group of order 10; the corresponding Coxeter generating sets are not reflection-compatible. Moreover, identifying this group with the automorphism group of a regular decagon, every generating pair of reflections is a Coxeter generating set, and any two such pairs are reflection-compatible. However, they are angle-compatible if and only if the associated pairs of axes intersect at the same angle.

A first motivation to consider the notion of angle-compatibility comes from the following basic observation.

Remark. The group $\mathrm{Aut}(W)$ contains a finite index subgroup all of whose elements map every Coxeter generating set to an angle-compatible one.

Indeed, the Coxeter group W has finitely many conjugacy classes of finite subgroups; in particular, there are finitely many conjugacy classes of pairs of elements generating a finite subgroup. The finite index normal subgroup of $\mathrm{Aut}(W)$ preserving the conjugacy class of each of these pairs satisfies the desired property.

A deeper reason to consider angle-compatible Coxeter generating sets comes from the fact that, by the main results of [HM04] and [MM08] there are certain explicit operations which transform any two Coxeter generating sets into angle-compatible ones (see Appendix B). It is further conjectured in [Müh06] that, up to conjugation, any two angle-compatible Coxeter generating sets may be obtained from one another by a finite sequence of *elementary twists*, a notion which was introduced in [BMMN02]. If this conjecture is confirmed, it implies in particular that the isomorphism problem is decidable in the class of Coxeter groups (see [MM08, Corollary 1.1]). The main goal of this paper is to prove a basic case of this conjecture. In order to provide a precise statement, we first recall the definition of elementary twists in detail.

Let $S \subset W$ be a Coxeter generating set. Given a subset $J \subset S$, we denote by W_J the subgroup of W generated by J. We call J **spherical** if W_J is finite. If J is spherical, let w_J denote the longest element of W_J. We say that two elements $s, s' \in S$ are **adjacent** if $\{s, s'\}$ is spherical. Given a subset $J \subset S$, we denote by $J^\perp$ the set of those elements of $S \setminus J$ which commute with J. A subset $J \subset S$ is **irreducible** if it is not of the form $K \cup K^\perp$ for some non-empty proper subset $K \subset J$.

Let $J \subset S$ be an irreducible spherical subset and assume that $S \setminus (J \cup J^\perp)$ is a union of two subsets A and B such that a and b are not adjacent for all $a \in A$ and $b \in B$. This simply means that W splits as an amalgamated product over $W_{J \cup J^\perp}$. Note that A and B are in general not uniquely determined by J.

We then consider the map $\tau \colon S \to W$ defined by

$$\tau(s) = \begin{cases} s & \text{if } s \in A, \\ w_J s w_J & \text{if } s \in S \setminus A, \end{cases}$$

which is called an **elementary twist**. The relevance of this notion was first highlighted in [BMMN02, Theorem 4.5], where it is shown that any elementary twist τ transforms S into another Coxeter generating set for W. Notice that τ might not extend to an automorphism of W; however it does extend provided w_J lies in the centre of W_J (for more details see [BMMN02, Section 4]).

A Coxeter generating set S is called **twist-rigid** if it does not admit any elementary twist. The purpose of this paper is to study Coxeter groups admitting some twist-rigid Coxeter generating set; these are called **twist-rigid** Coxeter groups. Observe that if S is a Coxeter generating set for W which admits an elementary twist τ, then S and $\tau(S)$ are two non-conjugate angle-compatible Coxeter generating sets. Our main result is the following converse.

Theorem 1.1. *Let S and R be angle-compatible Coxeter generating sets for a group W. If S is twist-rigid, then S and R are conjugate.*

By the Remark above, Theorem 1.1 has the following immediate consequence.

Corollary 1.2. *If a Coxeter group W is twist-rigid, then* $\mathrm{Out}(W)$ *is finite.*

Using the main results of [HM04] and [MM08] we also obtain the following, which we prove in Appendix B.

Corollary 1.3. *Let W be a twist-rigid Coxeter group.*

(i) All Coxeter generating sets for W are twist-rigid.

(ii) There is an algorithm which, given a Coxeter matrix (associated with a Coxeter generating set) for W, produces as an output representatives of all conjugacy classes of Coxeter generating sets for W, in terms of words in the original generators.

(iii) In particular, this algorithm produces as an output all possible Coxeter matrices for W. Hence the isomorphism problem is decidable in the class of twist-rigid Coxeter groups.

Note that a twist-rigid Coxeter group may admit more than one conjugacy class of Coxeter generating sets (and even two Coxeter generating sets with different Coxeter matrices). In fact, the combination of Theorem 1.1 with the main results of [HM04] and [MM08] leads to a precise description of those Coxeter groups which admit a unique conjugacy class of Coxeter generating sets. However, we do not formulate this condition explicitly, since it is technical and not particularly illuminating. Similarly, we could extract from the same combination of results a precise description of those twist-rigid Coxeter groups whose Coxeter generating sets admit only one Coxeter matrix.

Theorem 1.1 has been previously proved under various special assumptions, see [Müh06] for a 2006 state of art (but note that the announcement of [Müh06, Theorem 3.7] was too optimistic at the time). Since then, the only contributions known to us are [Car06] and [RT08].

Outline of the proof strategy. We now sketch the overall strategy governing the proof of Theorem 1.1. Our approach is inspired by [MW02]. Let S and R be two angle-compatible Coxeter generating sets for W. We call the Davis complex associated with S the **reference** Davis complex, and we denote it by $\mathbb{A}_{\mathrm{ref}}$. The Davis complex associated with R is called the **ambient** Davis complex, and is denoted by $\mathbb{A}_{\mathrm{amb}}$.

Since S and R are reflection-compatible, the set of S-reflections coincides with the set of R-reflections and its elements are called simply reflections. We denote by $\mathcal{Y}_r$ the wall in $\mathbb{A}_{\mathrm{ref}}$ fixed by a reflection r, and by $\mathcal{W}_r$ the wall in $\mathbb{A}_{\mathrm{amb}}$ fixed by r. Since two walls intersect non-trivially if and only if the associated reflections generate a finite group, it follows that the assignment $\mathcal{Y}_r \mapsto \mathcal{W}_r$ preserves the parallelism relation.

The Coxeter generating sets S and R are conjugate if and only if the set $\{\mathcal{W}_s\}_{s\in S}$ is **geometric** in the sense that it consists of walls containing the panels of some given chamber of $\mathbb{A}_{\mathrm{amb}}$. In order to show that $\{\mathcal{W}_s\}_{s\in S}$ is geometric, it is enough to construct a set $\{\Phi_s\}_{s\in S}$ of half-spaces in $\mathbb{A}_{\mathrm{amb}}$ satisfying the following. First we require that the boundary wall of each Φ_s equals $\mathcal{W}_s$ (shortly, Φ_s is a half-space **for** s). Second, we require that for all $s, s' \in S$ the pair $\{\Phi_s, \Phi_{s'}\}$ is **geometric**, *i.e.* the set $\Phi_s \cap \Phi_{s'}$ is a fundamental domain for the $\langle s, s'\rangle$-action on $\mathbb{A}_{\mathrm{amb}}$. The fact that these two conditions imply that $\{\mathcal{W}_s\}_{s\in S}$ is geometric is originally due to J.-Y. Hée [Hée90] and was established independently by Howlett–Rowley–Taylor [HRT97, Theorem 1.2] (see also [CM07, Fact (1.6)] for yet another proof as well as some additional references).

In view of the above discussion, proving Theorem 1.1 boils down to establishing the following.

Theorem 1.4. *Let S and R be angle-compatible Coxeter generating sets for a group W. Assume that S is twist-rigid. Then there exists a set of half-spaces $\{\Phi_s\}_{s\in S}$ in $\mathbb{A}_{\mathrm{amb}}$ such that Φ_s is a half-space for s and $\{\Phi_s, \Phi_{s'}\}$ is geometric for all $s, s' \in S$*

For example, if s and t are two elements of S which generate an infinite dihedral group, we need to define Φ_s as the unique half-space for s containing the wall $\mathcal{W}_t$, which we denote by $\Phi(\mathcal{W}_s, \mathcal{W}_t)$. In particular, if $t' \in S$ is another element with the same property, we need to verify that $\mathcal{W}_t$ and $\mathcal{W}_{t'}$ lie on the same side of $\mathcal{W}_s$. In order to address this compatibility issue, we shall view the walls $\mathcal{W}_t$ and $\mathcal{W}_{t'}$ as part of a larger family of walls parallel to $\mathcal{W}_s$. This family is parametrised by a certain set of data which we call *markings*.

More precisely, a marking μ with *core* $s \in S$ is a pair $\mu = ((s, w), m)$, where $w \in W$, $m \in S$, which satisfies a number of conditions depending on the combinatorics and the geometry of $\mathbb{A}_{\mathrm{ref}}$.

We will consider two particular types of markings. One type will be *complete markings*, which will give rise to walls parallel to $\mathcal{W}_s$, like in the example above. Namely, we require (in particular) that the wall $\mathcal{Y}^\mu = w\mathcal{Y}_m$ is parallel to $\mathcal{Y}_s$, so that $\mathcal{W}^\mu = w\mathcal{W}_m$ is parallel to $\mathcal{W}_s$. Hence every complete marking $\mu = ((s, w), m)$ with core $s \in S$ induces a choice of half-space $\Phi^\mu_s = \Phi(\mathcal{W}_s, \mathcal{W}^\mu)$ in $\mathbb{A}_{\mathrm{amb}}$.

We will also take advantage of the fact that Theorem 1.4 has been already proved for 2-spherical Coxeter groups [CM07]. (Recall that $J \subset S$ is called **2-spherical** if all of its two element subsets are spherical and a Coxeter group is **2-spherical** if it admits a Coxeter generating set S which is 2-spherical.) In particular, each 2-spherical but non-spherical subset $J \subset S$ containing s gives rise to a natural choice of a half-space for s. Consider a marking $\mu = ((s, w), m)$ and let $j_1 \dots j_n$ be a word of minimal length representing w. Even if $\mathcal{W}^\mu = j_1 \dots j_n \mathcal{W}_m$ intersects $\mathcal{W}_s$, once the *support* $J = \{s, j_1, \dots, j_n\} \subset S$ is irreducible 2-spherical but non-spherical, there is a natural choice of a half-space Φ^μ_s for s (see Corollary 2.6). A marking μ of this type is called *semi-complete*.

We next introduce a relation on the set of all complete and semi-complete markings with common core, which we call a *move*. We show that two markings μ, μ' related by a move induce the same choice of a half-space, namely we show $\Phi^\mu_s = \Phi^{\mu'}_s$. In the special case of complete markings μ, μ' the reason for this is that $\mathcal{W}^\mu$ and $\mathcal{W}^{\mu'}$ intersect.

The markings discussed at the beginning, where $\mathcal{W}^\mu = \mathcal{W}_t$, for some $t \in S$ (*i.e.* markings for which $w = 1$) are part of a class of particularly well behaved *good markings*. They have, in particular, the property that w is uniquely determined by the support J.

The major part of the proof is then to show that any two good markings with common core are related to one another by a sequence of moves. The moves can be essentially read from the Coxeter diagram of (W, S), and this allows us to appeal to graph-theoretic arguments. Since S is assumed to be twist-rigid, there is 'enough space' to move around the diagram of (W, S). We use the formalism of Masur–Minsky hierarchies to assemble all the data.

In the present context, a *hierarchy* is a system of geodesics in the Coxeter diagram of (W, S) between a pair of good markings. It is constructed in a seemingly arbitrary way, but later reveals highly organised structure. It admits a resolution into a sequence of *slices* which give rise to good markings related by moves.

However, a new type of moves pops up out of this procedure; to handle it we need to leave the setting of good markings and consider complete markings in full generality.

We stress that the assumption that S is twist-rigid is effectively used only in two places in the proof. We use it to prove the existence of a hierarchy with a given main geodesic (Lemma 6.5) and in a similar situation in Section 8 (Lemma 8.4).

Furthermore, the hypothesis that S and R are angle-compatible (and not just reflection-compatible) is used only in the proof of Proposition 5.2 and in the case where $S = S' \cup S'^{\perp}$ for some spherical S'.

Organisation of the article. In Section 2 we recall some basic facts on Coxeter groups and Davis complexes. In Section 3 we define *complete, semi-complete* and *good markings* and describe how they determine choices of half-spaces. In Theorem 3.15 we claim that this choice does not depend on the marking μ, provided that μ is good. We next prove Theorem 3.15 in Section 4, by means of *moves*. However, we leave two crucial graph-theoretic results, namely Theorem 4.5 and Proposition 4.6, for Sections 6–8. We prove Theorem 1.4 (the main result) in Section 5.

In Section 6 we describe how to connect a pair of good markings by a *hierarchy*. Then, in Section 7, we show how to resolve a hierarchy into a sequence of *slices*, which gives rise to a sequence of good markings related by moves. In Section 8 we finally deal with the last type of moves.

We include Appendix A, where we explain why the relations of reflection- and angle-compatibility are symmetric. In Appendix B we provide a proof of Corollary 1.3.

Acknowledgements. The writing of this paper was initiated during the stay of both authors at the Institut des Hautes Études Scientifiques, and continued at the Hausdorff Research Institute for Mathematics. We are grateful for the hospitality of both institutes.

2. Preliminaries on Coxeter groups

In this section we collect some basic facts on Coxeter groups. Let W be a group with a Coxeter generating set S and let $\mathbb{A}$ be the associated Davis complex.

The gallery distance between chambers c, c' of the Davis complex $\mathbb{A}$ is denoted by $d(c, c')$. By the **distance** of a chamber c to a wall $\mathcal{Y}$, we mean the minimal gallery distance from c to a chamber containing a panel contained in $\mathcal{Y}$; it is denoted by $d(c, \mathcal{Y})$.

If $w \in W$ and c is chamber of $\mathbb{A}$, we denote by $w.c$ the image of c under the action of w. For $w \in W$ let $\ell(w)$ denote the word-length of w; in other words, $\ell(w) = d(w.c_0, c_0)$, where c_0 is the identity chamber of $\mathbb{A}$.

Lemma 2.1 ([Bou68, Chapitre IV, Exercice 22])**.** *Let $w \in W$. The set of $s \in S$ satisfying $\ell(ws) < \ell(w)$ is spherical.*

For $s \in S$, we denote by α_s the positive root in the Tits representation corresponding to s (see *e.g.* [BH93, Section 1]).

Lemma 2.2 ([BH93, Lemma 1.7])**.** *Let $\mathcal{Y}$ be a wall with associated positive root α and let $s \in S$. Then we have*

$$d(s.c_0, \mathcal{Y}) = \begin{cases} d(c_0, \mathcal{Y}) + 1 & \text{if } \langle \alpha, \alpha_s \rangle > 0, \\ d(c_0, \mathcal{Y}) & \text{if } \langle \alpha, \alpha_s \rangle = 0, \\ d(c_0, \mathcal{Y}) - 1 & \text{if } \langle \alpha, \alpha_s \rangle < 0. \end{cases}$$

The following is a consequence of [Deo82, Proposition 5.5] (see also [Kra09, Proposition 3.1.9]).

Proposition 2.3. *If $I \subset S$ is irreducible and non-spherical, then its centraliser in W coincides with $W_{I^\perp}$.*

We also need the following, known as the Parallel Wall Theorem.

Theorem 2.4 ([BH93, Theorem 2.8])**.** *For any (W, S) there exists a constant n such that the following holds. For any wall $\mathcal{Y}$ and a chamber c at gallery distance at least n from $\mathcal{Y}$, there is another wall separating c from $\mathcal{Y}$.*

We finish this section with the following known fact; as explained in the introduction, it is a basic ingredient of the proof of Theorem 1.1.

Theorem 2.5 ([CM07, Main Result (1.1)])**.** *Let S and R be reflection-compatible Coxeter generating sets for a group W. If S is irreducible 2-spherical and non-spherical, then S and R are conjugate.*

By Theorem A.1 this yields the following.

Corollary 2.6. *Let R be a Coxeter generating set for W. Let $S \subset W$ be a Coxeter generating set for a subgroup $W_S \subset W$, consisting of R-reflections. If S is irreducible 2-spherical and non-spherical, then*

there is a unique fundamental domain for the W_S-action on the Davis complex of W associated with R, which is adjacent to all s-invariant walls, over $s \in S$. In particular, if Φ_s, for $s \in S$, denotes the half-space for s containing this fundamental domain, then for every $\{s, s'\} \subset S$ the pair $\{\Phi_s, \Phi_{s'}\}$ is geometric.

3. Choices of half-spaces

Recall that we simultaneously work in the reference Davis complex $\mathbb{A}_{\text{ref}}$ associated with the generating set S, where the wall fixed by a reflection $r \in W$ is denoted by $\mathcal{Y}_r$, and in the ambient Davis complex $\mathbb{A}_{\text{amb}}$, where the wall fixed by r is denoted by $\mathcal{W}_r$. The word-length $\ell(\cdot)$ will be always measured with respect to the set S.

The aim of this section is to introduce the notions of *complete*, *semi-complete* and *good markings*. Each such marking μ with *core* $s \in S$ will determine a half-space Φ_s^μ for s in the ambient Davis complex $\mathbb{A}_{\text{amb}}$.

First, we give a rough definition of a *complete marking*. We start at c_0, the identity chamber of $\mathbb{A}_{\text{ref}}$. We consider a gallery issuing from c_0 which moves away from $\mathcal{Y}_s$. We stop at the first wall we cross which does not intersect $\mathcal{Y}_s$, and we denote it by $\mathcal{Y}^\mu$. If the type of the last panel is $m \in S$, then $\mathcal{Y}^\mu = w\mathcal{Y}_m$, where $w.c_0$ is the last chamber of our gallery. We call m the *marker* and (s, w) the *base*.

We now give the precise definitions.

Definition 3.1. A **base** is a pair (s, w) with $s \in S$ and $w \in W$ satisfying

(i) $d(w.c_0, \mathcal{Y}_s) = \ell(w)$, and
(ii) every wall which separates $w.c_0$ from c_0 intersects $\mathcal{Y}_s$.

Note that condition (i) is equivalent to $d(w.c_0, sw.c_0) = 2\ell(w) + 1$. We call s the **core** of the base. The **support** of the base is the smallest subset $J \subset S$ such that W_J contains both s and w. A base is **irreducible, spherical,** 2**-spherical**, *etc*, if its support is irreducible, spherical, 2-spherical, *etc*.

Remark 3.2. Let (s, w) be a base and let $j_1 \dots j_n$ be a word (over S) of minimal length representing w. Then the support J of (s, w) equals $\{s, j_1, \dots, j_n\}$.

(i) By Lemma 2.2, condition (i) in Definition 3.1 is equivalent to $\langle j_{i-1} \dots j_1 \alpha_s, \alpha_{j_i} \rangle < 0$, for all $1 \leq i \leq n$. (Condition (ii) is equivalent to this expression being greater than -1.)
(ii) In particular, if we write $\alpha^i = j_i \dots j_1 \alpha_s = \sum_{j \in J} \lambda_j^i \alpha_j$, then, since $\alpha^i = \alpha^{i-1} - 2\langle \alpha^{i-1}, \alpha_{j_i} \rangle \alpha_{j_i}$, the coefficients λ_j^i are nondecreasing in i.
(iii) By (i), if $j \in S \setminus (J \cup J^\perp)$, then (s, wj) satisfies condition (i) in Definition 3.1.

Remark 3.3. By Theorem 2.4, for fixed (W, S) the value of $\ell(w)$ in Definition 3.1 is bounded.

Below we show that the support of a base must be 2-spherical of a very specific type.

Definition 3.4. A 2-spherical subset $J \subset S$ is **tree-2-spherical** if its Dynkin diagram is a union of trees.

Lemma 3.5. *Any base (s, w) is irreducible tree-2-spherical (*i.e. *its support J is irreducible tree-2-spherical).*

Proof. Let $j_1 \dots j_n$ be a word of minimal length representing w. If $J = \{s, j_1, \dots, j_n\}$ is reducible, and i is the least index for which $\{s, \dots j_i\}$ is reducible, then the distances from $j_1 \dots j_{i-1}.c_0$ and $j_1 \dots j_i.c_0$ to $\mathcal{Y}_s$ are equal, violating condition (i) in Definition 3.1.

If J is not tree-2-spherical, then let i be the least index such that $J' = \{s, j_1, \dots j_i\}$ is not tree-2-spherical. We claim that $j_1 \dots j_{i-1}\mathcal{Y}_{j_i}$ does not intersect $\mathcal{Y}_s$, contradicting condition (ii) in Definition 3.1.

Indeed, by Remark 3.2(ii) one can show inductively that all λ_j^{i-1}, for $j \in J' \setminus \{j_i\}$, equal at least 1. Since J' is not tree-2-spherical, there are at least two elements $j \in J' \setminus \{j_i\}$ such that $\langle \lambda_j^{i-1}\alpha_j, \alpha_{j_i}\rangle \leq -\frac{1}{2}$, or at least one with $\langle \lambda_j^{i-1}\alpha_j, \alpha_{j_i}\rangle \leq -1$. Hence $\langle \alpha^{i-1}, \alpha_{j_i}\rangle \leq -1$ and consequently $\mathcal{Y}_{j_i}$ does not intersect $j_{i-1} \dots j_1\mathcal{Y}_s$, as required. □

Throughout most of the article we will be only discussing the following special kind of a base.

Definition 3.6. Assume that (s, w) is a base satisfying $w = j_1 \dots j_n$ where all j_i are pairwise different and different from s. We call such a base **simple**.

If $J \subset S$ is irreducible spherical and $s \in J$, then there exists a simple base with support J and core s. Namely, it suffices to order the elements of $J \setminus \{s\}$ into a sequence (j_i) so that for every $1 \leq i \leq n$ the set $\{s, j_1, \dots, j_i\}$ is irreducible. Every $(s, j_1 \dots j_i)$ is a base by inductive application of Remark 3.2(iii).

Lemma 3.7. *Two simple bases with common core and common support are equal.*

Proof. Let (s, w) and (s, w') be two simple bases with common core and support. Let $w = j_1 \dots j_n$ and $w' = j_{\pi(1)} \dots j_{\pi(n)}$, where π is a permutation of the set $\{1, \dots, n\}$. To re-order the j_i's and prove $w' = w$ it suffices to show that if for some $1 < i \leq n$ the element $j_{\pi(i-1)}$ does not commute with $j_{\pi(i)}$, then we have $\pi(i-1) < \pi(i)$.

First observe that, by condition (i) in Definition 3.1, the sets $T_i = \{s, j_1, \dots, j_i\}$ and $T_i' = \{s, j_{\pi(1)}, \dots, j_{\pi(i)}\}$ are irreducible for every $1 \leq i \leq n$ (as in the proof of Lemma 3.5). By Lemma 3.5 the Dynkin diagram of J is a tree, hence the Dynkin diagrams of all the T_i and T_i' are subtrees.

If $j_{\pi(i-1)}$ does not commute with $j_{\pi(i)}$, then $j_{\pi(i-1)}$ separates $j_{\pi(i)}$ from s in the tree corresponding to T_i'. In particular $j_{\pi(i-1)}$ separates $j_{\pi(i)}$ from s in the tree corresponding to the entire J. Hence $j_{\pi(i-1)}$ belongs to $T_{\pi(i)}$ and consequently we have $\pi(i-1) < \pi(i)$, as desired. □

Finally, we define a *complete marking*.

Definition 3.8. A **marking** is a pair $((s,w),m)$, where (s,w) is a base (the **base** of the marking) and $m \in S \setminus J^\perp$ (m is called the **marker**), where J is the support of (s,w).

We say that the marking is **complete**, if $w\mathcal{Y}_m$ does not intersect $\mathcal{Y}_s$, *i.e.* (s,wm) does not satisfy condition (ii) in Definition 3.1.

The **core** and the **support** of the marking μ are the core and the support of its base. The marking μ is **simple** if its base is simple and $m \notin J$.

Remark 3.9.

(i) By Remark 3.2(ii), if $((s,w),m)$ is a complete marking and $j \in S$ is such that $\ell(wj) > \ell(w)$ and (s,wj) is a base, then $((s,wj),m)$ is a complete marking.

(ii) In particular, in view of Remark 3.2(iii), we have the following. If $((s,w),m)$ is a complete marking with support J, and $j \in S \setminus (J \cup J^\perp)$ is such that $((s,w),j)$ is not a complete marking, then $((s,wj),m)$ is a complete marking.

We describe how complete markings with core s determine half-spaces for s in the ambient Davis complex $\mathbb{A}_{\mathrm{amb}}$.

Definition 3.10. Let $\mu = ((s,w),m)$ be a complete marking with core s. Denote $\mathcal{W}^\mu = w\mathcal{W}_m$. We define $\Phi^\mu_s = \Phi(\mathcal{W}_s, \mathcal{W}^\mu)$ (which, as in the introduction, denotes the half-space for s containing $\mathcal{W}^\mu$ in $\mathbb{A}_{\mathrm{amb}}$).

There is another way to determine half-spaces for s.

Definition 3.11. A marking $((s,w),m)$ with support J is **semi-complete** if $J \cup \{m\}$ is irreducible 2-spherical but non-spherical. We then define Φ^μ_s to be the half-space for s in $\mathbb{A}_{\mathrm{amb}}$ given by Corollary 2.6. If a marking is at the same time complete and semi-complete, then by Corollary 2.6 this coincides with Definition 3.10.

Note that a complete marking might not be semi-complete; we have decided to use this term to underline the fact that we are treating complete and semi-complete markings similarly.

Remark 3.12. If (s,w) is a base with support J and $m \in S \setminus J^\perp$ is such that $J \cup \{m\}$ is non-spherical, then $((s,w),m)$ is a semi-complete or complete marking. This follows from the fact that if $J \cup \{m\}$ is not 2-spherical, then by Remark 3.2(ii) the wall $\mathcal{Y}_m$ does not intersect $w^{-1}\mathcal{Y}_s$.

Definition 3.13. Finally, a **good** marking is a complete or semi-complete simple marking with spherical base.

We point out that, under mild hypothesis, good markings exist.

Lemma 3.14. *Let S be an irreducible non-spherical Coxeter generating set for W. Then for every $s \in S$ there exists a good marking with core s.*

Proof. Let $J \subset S$ be a maximal irreducible spherical subset containing s. Any simple marking with support J and with marker in $S \setminus (J \cup J^{\perp})$ is good, by Remark 3.12. □

The main element of the proof of Theorem 1.4 is the following.

Theorem 3.15. *Let S and R be reflection-compatible Coxeter generating sets for W. Assume that S is twist-rigid. Let $s \in S$. We consider all (if there are any) good markings μ with core s. Then the half-space Φ_s^μ (in $\mathbb{A}_{\mathrm{amb}}$) does not depend on μ.*

We explain the proof and the consequences of Theorem 3.15 in the next sections.

Observe that in Theorem 3.15 we only assume that S and R are reflection-compatible and we do not require them to be angle-compatible.

Although in the statement of Theorem 3.15 we consider only a restricted family of markings, namely the good markings, the other more general markings will come up in the proof.

4. Moves

In this Section we describe the ingredients of the proof of Theorem 3.15.

Definition 4.1. Let $((s,w),m),((s,w'),m')$ be complete or semi-complete markings with common core and supports J, J'. We say that they are related by **move**

- **M1** if $w = w'$, the markers m and m' are adjacent, and both markings are complete,
- **M2** if there is $j \in S$ such that $w = w'j$ and moreover m equals m' and is adjacent to j.
- **M3** if $J \cup \{m\} \cup J' \cup \{m'\}$ is 2-spherical,
- **M4** if $((s,w),m)$ is complete, for some maximal spherical subset K of J we have $K \subset m^{\perp}$, and $J = J' \cup \{m'\}$.

The half-spaces Φ_s^μ below are chosen in $\mathbb{A}_{\mathrm{amb}}$, as in Definitions 3.10 and 3.11.

Lemma 4.2. *If markings μ and μ' are related by one of moves M1–M4, then $\Phi_s^\mu = \Phi_s^{\mu'}$, where s is the common core of μ and μ'.*

Proof. The argument depends on the type of the move.

M1 Since $\mathcal{W}_m$ intersects $\mathcal{W}_{m'}$, it follows that $\mathcal{W}^\mu = w\mathcal{W}_m$ intersects $\mathcal{W}^{\mu'} = w\mathcal{W}_{m'}$. They are both disjoint from $\mathcal{W}_s$, hence they lie in the same half-space for s.

M2 If any of the markings is not complete, then they are also related by move M3, see below.

Assume now that both markings are complete. Since $\mathcal{W}_j$ intersects $\mathcal{W}_m$, it follows that $j\mathcal{W}_m$ intersects $\mathcal{W}_m$. Hence $\mathcal{W}^{\mu'} = w'\mathcal{W}_m = wj\mathcal{W}_m$ intersects $\mathcal{W}^\mu = w\mathcal{W}_m$ and they lie in the same half-space for s.

M3 This case follows immediately from Corollary 2.6.

M4 Since J is non-spherical and $K \subset J$ is maximal spherical, the fixed point set of wW_Kw^{-1} in $\mathbb{A}_{\text{amb}}$ is disjoint from $\mathcal{W}_s$ and is contained in the half-space $\Phi_s^{\mu'}$ (Corollary 2.6). Moreover, $\mathcal{W}^\mu = w\mathcal{W}_m$ intersects this fixed point set, hence we have $\Phi_s^\mu = \Phi_s^{\mu'}$.

□

In view of Lemma 4.2, in order to prove Theorem 3.15, it is enough to prove the following.

Theorem 4.3. *Let S be a twist-rigid Coxeter generating set for W. Let μ and μ' be good markings with common core $s \in S$. Then there is a sequence, from μ to μ', of complete or semi-complete markings such that each two consecutive ones are related by one of moves M1–M4.*

Observe that this is a statement concerning only the reference Davis complex $\mathbb{A}_{\text{ref}}$. The proof of Theorem 4.3 consists of two pieces. The first one is the following, which we prove in Sections 6–7.

Definition 4.4. Let $((s, w), m)$ and $((s, w'), m')$ be good markings with common core. We say that they are related by **move N1** if $w = w'$ and m and m' are adjacent.

Theorem 4.5. *Let $(W, S), \mu, \mu'$ be as in Theorem 4.3. Then there is a sequence, from μ to μ', of good markings such that each two consecutive ones are related by move N1, M2 or M3.*

The second ingredient is the following, which helps us to resolve move N1. We prove it in Section 8.

Proposition 4.6. *Let S be a twist-rigid Coxeter generating set for W. Let μ be a complete marking with non-spherical base. Then there is a sequence of complete markings from μ to a semi-complete (possibly not complete) marking μ' with support J' containing J satisfying the following. Each two consecutive markings in the sequence are related by move M1, M2 or M4, where move M4 may appear only as the last one.*

Notice that since μ and μ' are related by moves, they have in particular the same core.

We demonstrate how those two ingredients fit together to form the following.

Proof of Theorem 4.3. By Theorem 4.5, it suffices to prove Theorem 4.3 under the assumption that good markings μ and μ' are related by move N1. If both μ, μ' are semi-complete, then they are related by move M3. On the other hand, if they are both complete, then they are related by move M1. Hence without loss of generality we can restrict to the case where $\mu = ((s, w), m)$ is complete and $\mu' = ((s, w), m')$ is not complete.

By Remark 3.9(ii), the pair $\nu = ((s, wm'), m)$ is another complete marking and it is related to μ by move M2. The base of ν is non-spherical.

Now we apply Proposition 4.6 to ν. We obtain a semi-complete marking ν', related to ν by a sequence of moves M1, M2 and M4, with support containing $J \cup \{m'\}$, where J is the support of μ. Hence ν' is related to μ' by move M3. □

Remark 4.7. The hypothesis that S is twist-rigid in both Theorem 4.5 and Proposition 4.6 can be weakened, see Remarks 7.7 and 8.6.

5. Proof of the main theorem

In this section we deduce Theorem 1.4 from Theorem 3.15. Before we do that we point out the following, which does not require a proof (see Figure 1).

Lemma 5.1. *Let $\{s, s'\} \subset W$ be conjugate to some spherical non-commuting pair $\{r, r'\} \subset R$. Suppose a wall $\mathcal{W}$ (in $\mathbb{A}_{\mathrm{amb}}$) intersects at least one of $\mathcal{W}_s, \mathcal{W}_{s'}$ and none of $s\mathcal{W}_{s'}, s'\mathcal{W}_s$. Then the pair*

$$\{s\Phi(s\mathcal{W}_{s'}, \mathcal{W}), s'\Phi(s'\mathcal{W}_s, \mathcal{W})\}$$

is geometric. (If $m_{s,s'} = 3$, then we do not need the hypothesis that $\mathcal{W}$ intersects at least one of $\mathcal{W}_s, \mathcal{W}_{s'}$.)

Proof of Theorem 1.4. Without loss of generality we may assume that S is irreducible. If S is spherical, then the theorem follows from [CM07, Proposition 11.7].

Otherwise, since S is non-spherical, by Lemma 3.14 each $s \in S$ is a core of a good marking μ. Hence we can put $\Phi_s = \Phi_s^\mu$ and by Proposition 3.15 this does not depend on the choice of μ. It remains to prove the following. We stress that we do not need to assume anymore that S is twist-rigid.

Proposition 5.2. *Let S and R be angle-compatible Coxeter generating sets for W. Assume that S is irreducible and non-spherical, and let $s, s' \in S$. Suppose that there are half-spaces $\Phi_s, \Phi_{s'}$ for s, s' in $\mathbb{A}_{\mathrm{amb}}$,*

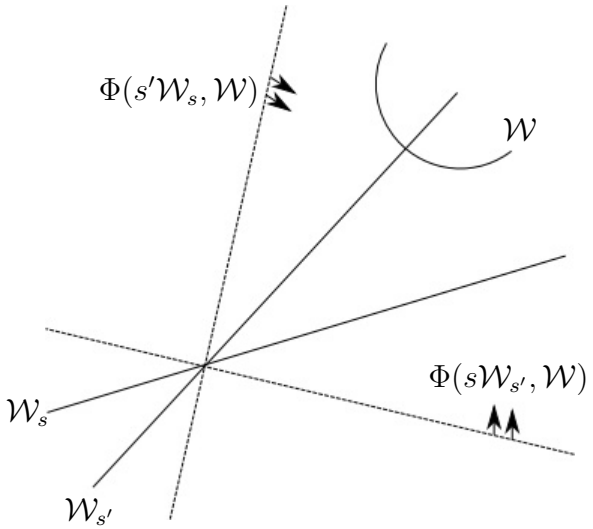

FIGURE 1. Configuration of walls from Lemma 5.1

such that for all good markings μ, μ' with respective cores s, s' we have $\Phi_s = \Phi_s^\mu$ and $\Phi_{s'} = \Phi_{s'}^{\mu'}$. Then the pair $\{\Phi_s, \Phi_{s'}\}$ is geometric.

Proof. If s and s' are not adjacent, then we can consider complete markings $\mu = ((s,1),s')$, $\mu' = ((s',1),s)$ and we obtain $\Phi_s^\mu = \Phi(\mathcal{W}_s, \mathcal{W}_{s'})$, $\Phi_{s'}^{\mu'} = \Phi(\mathcal{W}_{s'}, \mathcal{W}_s)$, as desired. Hence we may assume that s, s' are adjacent. We may also assume that they do not commute.

Denote the union of $\{s\}$ with the set of all elements from S adjacent to s by $B(s)$. If there is $t \in S$ outside $B(s) \cup B(s')$, then we proceed as follows. Let Σ be the union of the two acute-angled sectors between $\mathcal{W}_s$ and $\mathcal{W}_{s'}$. Since the choices of half-spaces coming from the markings $((s,s'),t)$ and $((s,1),t)$ coincide, it follows that $\mathcal{W}_t$ is contained in $\Sigma \cup s'\Sigma$. Analogously, since the choices of half-spaces coming from the markings $((s',s),t)$ and $((s',1),t)$ coincide, $\mathcal{W}_t$ is contained in $\Sigma \cup s\Sigma$. Since we have $(\Sigma \cup s'\Sigma) \cap (\Sigma \cup s\Sigma) = \Sigma$, we obtain $\mathcal{W}_t \subset \Sigma$, and consequently $\{\Phi(\mathcal{W}_s, \mathcal{W}_t), \Phi(\mathcal{W}_{s'}, \mathcal{W}_t)\}$ is geometric, as desired. We assume henceforth $S = B(s) \cup B(s')$.

Moreover, if there is an irreducible 2-spherical but non-spherical subset J of S containing s and s', then we take some maximal irreducible spherical $K \subset J$ containing s, s' and any $m \in J \setminus (K \cup K^\perp)$. We consider semi-complete simple markings μ, μ' with support K, marker m, and cores s, s'. The pair of half-spaces $\{\Phi_s^\mu, \Phi_{s'}^{\mu'}\}$ is geometric by Corollary 2.6. Hence we can assume from now on that any irreducible 2-spherical subset J of S containing s and s' is spherical.

Claim. There exist complete simple markings $\mu = ((s, s'j_2 \dots j_n), m)$, $\mu' = ((s', sj_2 \dots j_n), m)$ with common spherical support $J = \{s, s', j_2, \dots, j_n\}$ such that the common marker m satisfies the following.

At least one of s, s' commutes with $\{j_2, \dots, j_n\}$ and its invariant wall (*i.e.* $\mathcal{W}_s$ or $\mathcal{W}_{s'}$) intersects $\mathcal{W}_m$.

Before we justify the claim, let us show how it implies the proposition. We verify the hypothesis of Lemma 5.1 for $\mathcal{W} = j_2 \dots j_n \mathcal{W}_m$. If, say, s commutes with $\{j_2, \dots, j_n\}$ and $\mathcal{W}_s$ intersects $\mathcal{W}_m$, then $\mathcal{W}_s$ also intersects $\mathcal{W}$.

On the other hand, since μ and μ' are complete, $\mathcal{W}$ does not intersect $s\mathcal{W}_{s'}$ and $s'\mathcal{W}_s$. Hence, by Lemma 5.1, the pair formed by $\Phi^{\mu}_{s} = \Phi(\mathcal{W}_s, s'\mathcal{W}) = s'\Phi(s'\mathcal{W}_s, \mathcal{W})$ and $\Phi^{\mu'}_{s'} = \Phi(\mathcal{W}_{s'}, s\mathcal{W}) = s\Phi(s\mathcal{W}_{s'}, \mathcal{W})$ is geometric.

We now justify the claim. If $B(s) \neq B(s')$, then this is obvious, we take $J = \{s, s'\}$ and m outside $B(s) \cap B(s')$. Otherwise, we pick a maximal irreducible spherical subset $K \subset S$ containing s, s' and an element $m \in S \setminus (K \cup K^{\perp})$. By our discussion m is adjacent to both s and s', but not adjacent to some $t \in K$. Let $J \subset K$ be the union of $\{s, s'\}$ with the set of all vertices in the t component of the Dynkin diagram of $K \setminus \{s, s'\}$. Either s or s' is a leaf in the Dynkin diagram of J. This implies the claim. □

This ends the proof of Theorem 1.4. However, we still need to prove Theorem 4.5 and Proposition 4.6, which we do in the remaining sections. □

6. Hierarchies

Our goal for this and the next section is to prove Theorem 4.5. First we need to assemble the connectivity data of the Coxeter diagram of (W, S), and we do it via the *hierarchy* formalism. This formalism was invented in a different context by Masur–Minsky [MM00, Section 4]. Where convenient, we preserve the original names, notation and structure of the exposition.

The core of all our markings, throughout this and the next section is a fixed $s \in S$, and all markings are simple with spherical bases. Hence any marking is uniquely determined by its irreducible spherical support $J \ni s$ and its marker $m \in S \setminus (J \cup J^{\perp})$ (see Definition 3.6 and Lemma 3.7). Hence we may allow ourselves to write (J, m) instead of $((s, w), m)$. In this and the next section, the only place where we will use the hypothesis that S is twist-rigid, will be in the proof of Lemma 6.5.

Below, a **path** in $T \subset S$ is a sequence of elements from T such that each two consecutive ones are adjacent. A path is **geodesic** in T if its length is minimal among paths in T with the same endpoints.

Definition 6.1 (compare [MM00, Definition 4.2])**.** Let $J \subset S$ be irreducible spherical. A **geodesic** k with **domain** J is a triple

$$k = ((k_0, \dots k_n), I_k, T_k),$$

where $(k_0, \dots, k_n)$ is a geodesic path in $S \setminus (J \cup J^\perp)$, with $n \geq 0$, and $I_k = (J_{I_k}, m_{I_k}), T_k = (J_{T_k}, m_{T_k})$ are markings satisfying the following.

We require that either $I_k = (J, k_0)$ or I_k is good and $J \cup \{k_0\} \subset J_{I_k}$. Similarly, we require that either $T_k = (J, k_n)$ or T_k is good and $J \cup \{k_n\} \subset J_{T_k}$.

We allow the domain J to be the empty set but we then require $n = 0$ and we put $J^\perp = \emptyset$.

We denote the domain J by $D(k)$. We call k_i the **vertices** (lying) on k, where k_0 is the **first** vertex, k_n is the **last** vertex, and k_i, k_{i+1} are **consecutive**. The **length** of k equals n. We call I_k (resp. T_k) the **initial** (resp. the **terminal**) marking of k. If $I_k = (J, k_0)$ (resp. $T_k = (J, k_n)$) we call it **trivial**.

Remark 6.2 (compare [MM00, Lemma 4.10])**.** Let k be a geodesic. Then for every spherical subset $L \subset S$ there are at most two vertices from L (lying) on k, and if there are exactly two, then they are consecutive.

Definition 6.3 (compare [MM00, Definition 4.3])**.** Let $J \subset S$ be irreducible spherical. The set J is a **component domain** of a geodesic b if for some i we have $D(b) \cup \{b_i\} = J$.

A component domain J of a geodesic b is **directly subordinate backward** to b (we denote this by $b \swarrow J$) if $i > 0$ or I_b is not trivial.

A geodesic k is **directly subordinate backward** to a geodesic b (we denote this by $b \swarrow k$) if

- $b \swarrow D(k)$, and
- $I_k = \begin{cases} (D(k), b_{i-1}) & \text{if } i > 0, \\ I_b & \text{if } i = 0. \end{cases}$

Analogously, a component domain $J = D(f) \cup \{f_i\}$ of a geodesic f of length n is **directly subordinate forward** to f (we denote this by $J \searrow f$) if $i < n$ or T_f is not trivial. A geodesic k is **directly subordinate forward** to a geodesic f (we denote this by $k \searrow f$) if $D(k) \searrow f$, and moreover $T_k = (D(k), f_{i+1})$ if $i < n$ or $T_k = T_f$ if $i = n$.

Definition 6.4 (compare [MM00, Definition 4.4])**.** A **hierarchy** is a set H of geodesics satisfying the following properties.

(i) There is a distinguished **main** geodesic $g \in H$ with empty domain (with a single vertex s on g) and good initial and terminal markings.

(ii) For any irreducible spherical subset J of S with $b \swarrow J \searrow f$, where $b, f \in H$, there is a unique geodesic $k \in H$ satisfying $D(k) = J$ and $b \swarrow k \searrow f$.

(iii) For any geodesic $k \in H \setminus \{g\}$, there are geodesics $b, f \in H$ satisfying $b \swarrow k \searrow f$.

Lemma 6.5 (compare [MM00, Theorem 4.6])**.** *Assume that S is twist-rigid. Then for any geodesic g as in Definition 6.4(i), there is a hierarchy such that g is its main geodesic.*

Proof. We follow the proof in [MM00]. We call a set H of geodesics satisfying properties (i), (iii), and the uniqueness part of property (ii) a **partial hierarchy**. The set of partial hierarchies in which g is the main geodesic is non-empty, since $\{g\}$ is a partial hierarchy. We claim that there exists a maximal partial hierarchy in which g is the main geodesic.

To justify the claim, it is enough to bound uniformly (above) the number of geodesics in any such partial hierarchy H. We bound by induction on i the number of geodesics with domain of cardinality i. For $i = 0$ there is only one such geodesic, since for any such geodesic $k \in H$ we have by property (iii) a sequence $g = b^n \swarrow \ldots \swarrow b^1 \swarrow b^0 = k$ of geodesics with increasing domains, which implies $n = 0$ and $k = g$.

The number of geodesics with domain of cardinality $i+1$ is bounded by the square of the number of geodesics with domain of cardinality i times the number of irreducible spherical subsets of S of cardinality $i + 1$: indeed, by property (iii) for each geodesic $k \in H$ there are $b, f \in H$ satisfying $b \swarrow k \searrow f$ and by the uniqueness part of property (ii) b, f and $D(k)$ determine k uniquely.

This proves the claim that there exists a maximal partial hierarchy in which g is the main geodesic.

Now we prove that a maximal partial hierarchy H is already a hierarchy. Otherwise we would have some irreducible spherical subset $J \subset S$ and geodesics $b, f \in H$ satisfying $b \swarrow J \searrow f$, but no geodesic $k \in H$ with $D(k) = J$ and $b \swarrow k \searrow f$.

Suppose $J = D(b) \cup b_i$. If $i > 0$, then we denote $K_I = \{b_{i-1}\}$. Otherwise, we put $K_I = J_{I_b} \setminus (J \cup J^\perp)$ if it is non-empty and $K_I = \{m_{I_b}\}$ otherwise. Similarly, suppose $J = D(f) \cup f_{i'}$, where the length of f equals n'. If $i' < n'$, then we denote $K_T = \{f_{i'+1}\}$. Otherwise, we put $K_T = J_{T_f} \setminus (J \cup J^\perp)$ if it is non-empty and $K_T = \{m_{T_f}\}$ otherwise.

Since S is twist-rigid, there is a geodesic path (k_j) from some element of K_I to some element of K_T in $S \setminus (J \cup J^\perp)$ (possibly of length 0). We define $I_k = (J, b_{i-1})$ in case $i > 0$ and put $I_k = I_b$ otherwise. Similarly, we define $T_k = (J, f_{i'+1})$ in case $i' < n'$ and put $T_k = T_f$ otherwise.

Hence we have constructed a geodesic $k = ((k_j), I_k, T_k)$ with domain J satisfying $b \swarrow k \searrow f$. Thus $H \cup \{k\}$ is a partial hierarchy, which contradicts maximality of H. This proves that a maximal partial hierarchy is a hierarchy and ends the proof of the lemma. □

Note that although we have assumed that S does not admit any elementary twist, we have only used the fact that S does not admit an elementary twist with J containing the fixed element s of S.

From now on, throughout this and the next section, we assume that we are given a hierarchy H with main geodesic g as in the assertion of Lemma 6.5.

Definition 6.6 (compare [MM00, Section 4.3])**.** Let J be a component domain. Then its **backward sequence** is

$$\Sigma^-(J) = \{k \in H \colon D(k) \subset J \text{ and we have that } I_k \text{ is good or } m_{I_k} \notin J\}.$$

Its **forward sequence** is

$$\Sigma^+(J) = \{k \in H \colon D(k) \subset J \text{ and we have that } T_k \text{ is good or } m_{T_k} \notin J\}.$$

Lemma 6.7 (compare [MM00, Lemma 4.12])**.** *We have $\Sigma^-(J) = \{b^i\}_{i=0}^n$, where the b^i form a sequence $g = b^n \swarrow \ldots \swarrow b^0$. If for some $b \in \Sigma^-(J)$ all the vertices on b are outside J, then $b = b^0$. An analogous statement holds for $\Sigma^+(J)$.*

Proof. We follow again [MM00]. For the first assertion, since for every geodesic $k \in H$ we have a sequence $g = b^n \swarrow \ldots \swarrow b^0 = k$, it is enough to prove the following.

(i) If $k \in \Sigma^-(J)$ and $b \swarrow k$, then $b \in \Sigma^-(J)$.
(ii) If $k, k' \in \Sigma^-(J)$ and $b \swarrow k,\ b \swarrow k'$, then $k = k'$.

(i) Let $k \in \Sigma^-(J)$ and $b \swarrow k$. Then $D(b) \subset D(k) \subset J$. If I_b is trivial, then so is I_k and $m_{I_k} \notin J$ is a vertex on b which precedes the unique element of $D(k) \setminus D(b) \subset J$. By Remark 6.2 we have $m_{I_b} \notin J$. Hence $b \in \Sigma^-(J)$.

(ii) We prove this assertion together with the analogous one for $\Sigma^+(J)$ by induction on $|J|$. Suppose $k, k' \in \Sigma^-(J)$ and $b \swarrow k,\ b \swarrow k'$. If $|J| = 1$, *i.e.* if $J = \{s\}$, then $b = g$ and $k \searrow b,\ k' \searrow b$, hence by the uniqueness part of property (ii) of a hierarchy we have $k = k'$. Otherwise, let j be the vertex from J on b nearest to b_0. Since $k, k' \in \Sigma^-(J)$, we have $D(k) = D(b) \cup \{j\} = D(k')$. If $|D(k)| < |J|$, then k, k', b belong to $\Sigma^-(D(k))$ and we have $k = k'$ by the induction hypothesis. Otherwise we have $J = D(k)$, hence $k, k' \in \Sigma^+(J)$. By the induction hypothesis, by the analogous statement for $\Sigma^+(J)$, we have $f \in H$ with $k \searrow f, k' \searrow f$. By the uniqueness part of property (ii) of a hierarchy we obtain $k = k'$.

For the second assertion note that if we have $b \swarrow k$ and $k \in \Sigma^-(J)$, then the unique element of $D(k) \setminus D(b)$ which is a vertex on b belongs to J. □

Corollary 6.8 (compare [MM00, Lemma 4.15])**.** *Let J be a component domain with $H \ni b \swarrow J$. Then b is uniquely determined by J. Analogously, if $J \searrow f \in H$, then f is uniquely determined by J. In particular, if $k \in H$, then $D(k)$ uniquely determines k.*

Proof. If $b \not\perp J$, then $b \in \Sigma^-(J)$ and by Lemma 6.7 the cardinality of $D(b)$ determines b uniquely. The last statement follows from the uniqueness part of property (ii) of a hierarchy. □

Proposition 6.9 (compare [MM00, Lemma 4.21 and Theorem 4.7(3)])**.** *Let J be a component domain. If $J \not\pitchfork f \in H$, then there is $b \in H$ with $b \not\perp J$ (and* vice versa*). In particular, there is $k \in H$ with $D(k) = J$.*

Proof. The second assertion follows from the existence part of property (ii) of a hierarchy.

We prove Proposition 6.9 by induction on $|J|$. The case where $|J| = 1$ is immediate. Otherwise, let k be the element of $\Sigma^-(J)$ with the largest domain. First assume that there is a vertex from J on k and let $k_i \in J$ be such a vertex with the least index i. We have $k \not\perp D(k) \cup \{k_i\}$, hence we are done if $D(k) \cup \{k_i\} = J$ (actually, this cannot happen, because then the geodesic with support J is also in $\Sigma^-(J)$). Otherwise, we apply the inductive hypothesis to $D(k) \cup \{k_i\} \subsetneq J$. We obtain a geodesic $h \in H$ with domain $D(k) \cup \{k_i\}$ satisfying $k \not\perp h$. Then we have $h \in \Sigma^-(J)$, which contradicts the choice of k.

Now we consider the case where all the vertices on k are outside J. Then we have $k \in \Sigma^+(J)$ and, by Lemma 6.7 applied to $\Sigma^+(J)$, the geodesic k has the largest domain among the geodesics in $\Sigma^+(J)$. Since we have $f \in \Sigma^+(J)$, $|D(f)| = |J| - 1$, and $k \neq f$, it follows that $D(k)$ equals J. By property (iii) of a hierarchy there is $b \in H$ with $b \not\perp D(k)$. □

7. Slices

In this section we show how to resolve a hierarchy into a sequence of *slices*, compare [MM00, Section 5]. The slices give rise to good markings related by moves N1, M2 and M3 and we conclude with the proof of Theorem 4.5.

We assume we are given a fixed hierarchy H with main geodesic g with a single vertex s. All markings and notation are as in the previous section.

Definition 7.1. A **slice** is a pair (k, m), where $k \in H$ is a geodesic and m is a vertex on k such that $D(k) \cup \{m\}$ is non-spherical.

The marking **associated** to the slice (k, m) is the pair $(D(k), m)$. By Remark 3.12 this marking is good.

We define the **initial slice** in the following way. Let $k^0 = g$. For $i \geq 0$, while the first vertex k_0^i on the geodesic k^i does not equal $m_{I_{k^i}} = m_{I_g}$, we define k^{i+1} to be the geodesic in H whose domain is $D(k^i) \cup \{k_0^i\}$ — its existence is guaranteed by Theorem 6.9 and its uniqueness by Corollary 6.8. The initial slice is the last geodesic k of this sequence together with its first vertex $m = m_{I_g}$. Analogously we define the **terminal slice**.

Remark 7.2. The good marking associated to the initial slice equals I_g (the initial marking of the main geodesic). The marking associated to the terminal slice equals T_g.

Definition 7.3. We say that the slice (k', m') is a **successor** of the slice (k, m) if we have one of the following configurations:

(i) $k = k'$ and m, m' are consecutive vertices on k, or
(ii) $k \swarrow k'$ and $m' = m$ is the first vertex on k', or
(iii) $k \searrow k'$ and $m = m'$ is the last vertex on k, or
(iv) there is $h \in H$ satifying $k \searrow h \swarrow k'$, m is the last vertex on k, m' is the first vertex on k', and m', m are consecutive vertices on h.

Remark 7.4. The terminal slice has no successor. The initial slice is not a successor of any slice.

Theorem 7.5. *For each slice which is not the terminal slice there exists a unique successor. Each slice which is not the initial slice is a successor of a unique other slice.*

Proof. Let (k, m) be a slice. We prove that (k, m) has a successor or is the terminal slice.

We first assume that m is not the last vertex on k. Let m' be the vertex on k following the vertex m. If $D(k) \cup \{m'\}$ is non-spherical, then (k, m') is a slice. Slices (k, m) and (k, m') are in configuration (i) of Definition 7.3, in particular (k, m') is a successor of (k, m). If $D(k) \cup \{m'\}$ is spherical, then we have $k \swarrow D(k) \cup \{m'\}$ and by Theorem 6.9 there is a geodesic $k' \in H$ with $D(k') = D(k) \cup \{m'\}$ and $k \swarrow k'$. Then the first vertex on k' is m and the slice (k', m) is a successor of (k, m) in configuration (ii).

We now assume that m is the last vertex on k. Since we have $k \neq g$, there is $k' \in H$ satisfying $k \searrow k'$. We consider first the case where m is a vertex on k'. Let m' be the unique element of $D(k) \setminus D(k')$ which is the vertex on k' preceding m. If $D(k') \cup \{m\}$ is non-spherical, then (k', m) is a slice which is a successor of (k, m) in configuration (iii). Otherwise we have $k' \swarrow D(k') \cup \{m\}$ and by Proposition 6.9 there is a geodesic k'' with $D(k'') = D(k') \cup \{m\}$ and $k' \swarrow k''$. The first vertex on k'' equals m'. The pair (k'', m') is a slice, since $D(k'') \cup \{m'\} = D(k) \cup \{m\}$ is non-spherical. Then (k'', m') is a successor of (k, m) in configuration (iv).

It remains to consider the case where m is not on k'. Consider the geodesics satisfying $k = k^n \searrow k' = k^{n-1} \searrow \ldots \searrow k^0 = g$. Then for every $n \geq i > 0$ the unique element of $D(k^i) \setminus D(k^{i-1})$ is the last vertex on k^{i-1} and $T_{k^{i-1}}$ is not trivial. By Corollary 6.8, the slice (k, m) is the terminal slice. This completes the proof that every slice has a successor or is the terminal slice.

Following the same scheme and using Corollary 6.8 instead of Proposition 6.9 we obtain that a successor is unique. Analogously, every slice which is not the initial slice is a successor of a unique other slice. □

In view of Remark 7.4, Theorem 7.5 has the following immediate consequence.

Corollary 7.6 (compare [MM00, Proposition 5.4])**.** *There is a (unique) sequence of slices from the initial slice to the terminal slice, such that for each pair of consecutive elements, the second slice is a successor of the first slice.*

We are now prepared for the following.

Proof of Theorem 4.5. Let μ, μ' be two different good markings with core $s \in S$. Since S is twist-rigid, by Lemma 6.5 there is a hierarchy H with main geodesic g with a single vertex s and $I_g = \mu, T_g = \mu'$. By Remark 7.2 and Corollary 7.6 it is now enough to justify that if a slice (k', m') is a successor of a slice (k, m), then their associated good markings $\nu = (D(k), m),\ \nu' = (D(k'), m')$ are related by move N1, M2 or M3.

If $(k, m), (k', m')$ are in configuration (i) of Definition 7.3, then ν, ν' are related by move N1. If $(k, m), (k', m')$ are in configuration (ii) or (iii), then ν, ν' are related by move M2. Finally, if $(k, m), (k', m')$ are in configuration (iv), then ν, ν' are related by move M3. □

Remark 7.7. Observe that in the above proof we have only once used the hypothesis that S is twist-rigid, to guarantee the existence of the appropriate hierarchy (Lemma 6.5). However, the proof of Lemma 6.5 just requires that S does not admit an elementary twist with J containing s. Hence in the statement of Theorem 4.5 we could replace the hypothesis that S is twist-rigid with the above weaker hypothesis. However, we do not need this stronger result.

8. The last move

In this section we complete the proof of the main theorem by proving Proposition 4.6. We consider bases and markings in full generality, as defined in Section 3.

Definition 8.1. Let (s, w) be a base with support J. The **shadow** of the base (s, w) is the set of those elements $j \in J$ which satisfy $d(wj.c_0, \mathcal{Y}_s) \leq d(w.c_0, \mathcal{Y}_s)$. We denote the shadow by $\widetilde{J}$.

Lemma 8.2. *The shadow $\widetilde{J}$ is spherical (possibly reducible).*

Proof. Let $I \subset \widetilde{J}$ be the set of $j \in \widetilde{J}$ satisfying $d(wj.c_0, \mathcal{Y}_s) = d(w.c_0, \mathcal{Y}_s)$. By Lemma 2.2, the set I commutes with the reflection $r = wsw^{-1}$. By condition (i) in Definition 3.1 we have $\ell(r) = 2\ell(w) + 1$. Hence r does not lie in any W_K for a proper subset $K \subset J$. Then Proposition 2.3

guarantees that I is spherical. Denote by w_I the longest element of W_I. Elements $j \in \widetilde{J}$ with $d(wj.c_0, \mathcal{Y}_s) < d(w.c_0, \mathcal{Y}_s)$ satisfy $\ell(rj) < \ell(r)$. Since $w_I r = r w_I$, the shadow $\widetilde{J}$ is contained in (and in fact equals) the set of $j \in J$ satisfying $\ell(w_I rj) < \ell(w_I r)$. Hence, by Lemma 2.1, $\widetilde{J}$ is spherical. □

We have the following generalisation of Remark 3.9(ii), which follows from Remark 3.9(i).

Remark 8.3. If $((s,w),m)$ is a complete marking, j is an element of $S \setminus (\widetilde{J} \cup J^\perp)$ and $((s,w),j)$ is not a complete marking, then $((s,wj),m)$ is a complete marking.

Below we use the following terminology. Let T be a subset of S. A **component** of T is maximal subset $T' \subset T$ such that each two elements of T' are connected by a path in T. A subset $J \subset T$ **separates** T if $T \setminus J$ has at least two non-empty components. A subset $J \subset T$ **weakly separates** T if $J \cup J^\perp$ separates T. According to this terminology, the set S is twist-rigid if there is no irreducible spherical subset $J \subset S$ which weakly separates S.

Lemma 8.4. *Assume that S is twist-rigid. Let $J \subset S$ be irreducible 2-spherical and non-spherical. Let $K \subset J$ be spherical (possibly reducible). Then for every $m \in S \setminus (J \cup J^\perp)$ we have the following:*

(i) m is in the same component of $S \setminus (K \cup J^\perp)$ as $J \setminus K$, or
(ii) m is not adjacent to any element of $J \setminus K$, m belongs to $K^\perp$, and $J \cup \{m\}$ is twist-rigid.

Proof. We show that if any of the three elements of assertion (ii) does not hold, then we have assertion (i). First, obviously if m is adjacent to some element of $J \setminus K$, then we have assertion (i).

Second, if $m \notin K^\perp$, then we have $m \notin K'^\perp$ for some irreducible $K' \subset K$ satisfying $K \subset K' \cup K'^\perp$. Since J is irreducible, $J \setminus (K' \cup K'^\perp)$ is non-empty. Since K' does not weakly separate S, there is a path from m to an element of $J \setminus (K' \cup K'^\perp)$ outside $K' \cup K'^\perp \supset K \cup J^\perp$, and we have assertion (i).

Otherwise, if $m \in K^\perp$ but $J \cup \{m\}$ is not twist-rigid, then there exists some irreducible spherical subset $L \subset J \cup \{m\}$ which weakly separates $J \cup \{m\}$. We must have $m \notin L \cup L^\perp$ and $K \subset L \cup L^\perp$, because J is 2-spherical. Since L does not weakly separate S, there is a path from m to some element of the non-empty set $J \setminus (L \cup L^\perp) \subset J \setminus K$ outside $L \cup L^\perp \supset K \cup J^\perp$. This again yields assertion (i). □

Lemma 8.5. *In the case of assertion (ii) in Lemma 8.4, the set K is a maximal spherical subset of J.*

Proof. If there is a spherical subset $L \subset S$ with $K \subsetneq L \subset J$, then we have $L \neq J$, since J is non-spherical. Let $L' \subset L$ be irreducible

satisfying $L \subset L' \cup L'^{\perp}$ and containing an element outside K. Then $L' \cup L'^{\perp}$ does not contain m, contains $L \supset K$, but does not contain some other vertex in J, by irreducibility of J. Hence L weakly separates $J \cup \{m\}$. Contradiction. □

We are now ready for the following.

Proof of Proposition 4.6. Let $\mu = ((s, w), m)$ be the complete marking with non-spherical support J which we want to relate by moves to some semi-complete marking μ' with support J' containing J. We prove Proposition 4.6 by (backward) induction on $\ell(w)$. By Remark 3.3, for $\ell(w)$ large enough the content of Proposition 4.6 is empty. Suppose we have verified Proposition 4.6 for $\ell(w) = k + 1$. Assume now $\ell(w) = k$. By Lemma 8.2, the shadow $\widetilde{J} \subset J$ of (s, w) is spherical.

Since S is twist-rigid, we are in position to apply Lemma 8.4, with $K = \widetilde{J}$. First assume that we are in the case of assertion (ii) of Lemma 8.4 and thus we also have the conclusion of Lemma 8.5. Let $\mu' = ((s, w'), m')$ be any marking with support J' satisfying $J' \cup \{m'\} = J$. The marking μ' is semi-complete since J is irreducible non-spherical. Then μ and μ' are related by move M4 and we are done.

Now assume that we are in the case of assertion (i) of Lemma 8.4. Then there is a path $(h_0 = m, h_1, \ldots, h_l)$ in $S \backslash (\widetilde{J} \cup J^{\perp})$, where $h_l \in J \backslash \widetilde{J}$. Let i be the least index such that $J \cup \{h_i\}$ is 2-spherical (possibly $i = l$). Then for $1 \leq i' < i$ the complete markings $((s, w), h_{i'-1}), ((s, w), h_{i'})$ are related by move M1.

If $((s, w), h_i)$ is complete, then also $((s, w), h_{i-1}), ((s, w), h_i)$ are related by move M1. Then we can put $\mu' = ((s, w), h_i)$. If $((s, w), h_i)$ is not complete, then $((s, w), h_{i-1})$ is related by move M2 to $\nu = ((s, wh_i), h_{i-1})$, which is a complete marking by Remark 8.3.

The word-length of wh_i equals $k+1$ and we can apply Proposition 4.6 with $\ell(w) = k + 1$. We obtain that ν is related by a sequence of moves M1, M2 and M4 (only allowed as the last move) to a marking $\mu' = ((s, w'), m')$ with support J' such that $J' \cup \{m'\}$ is 2-spherical and satisfies $J' \supset J \cup \{h_i\} \supset J$. This finishes the proof of Proposition 4.6 for $n = k$. □

Remark 8.6. In the above argument, we have only once used the hypothesis that S is twist-rigid, in the proof of Lemma 8.4. However, we could just require that there is no irreducible spherical subset $K \subset S$ which weakly separates S and together with s is contained in some irreducible 2-spherical non-spherical subset $J \subset S$. This allows to weaken the hypothesis of Proposition 4.6. Again, we do not need this stronger result.

Appendix A. Reflection- and angle-compatibility

In this appendix we prove that the relations of reflection- and angle-compatibility are symmetric. Let R be a Coxeter generating set for a group W and let $\mathbb{A}$ be the associated Davis complex.

Translating the language of the root systems into the language of the Davis complex, the main result of [Deo89] may be phrased as follows.

Theorem A.1. *Let $S \subset W$ be some set of R-reflections and let $W_S \subset W$ be the subgroup generated by S. Define $\bar{S}$ as the set of all conjugates under W_S of elements of S. Let $C \subset \mathbb{A}$ denote a connected component of the space obtained by removing from $\mathbb{A}$ every wall associated to an element of $\bar{S}$. Let S_C be the subset of $\bar{S}$ of R-reflections in walls adjacent to some chamber in C.*

Then S_C is a Coxeter generating set for W_S and (the closure of) C is a fundamental domain for the W_S-action on $\mathbb{A}$. In particular W_S is a Coxeter group.

Corollary A.2. *Let $S \subset W$ be a Coxeter generating set such that every element of S is conjugate to some element of R. Then every element of R is conjugate to some element of S.*

Proof. Indeed, if $W_S = W$, then the set C consists of exactly one chamber. □

In order to obtain a similar statement concerning angle-compatibility, we record the following well-known fact.

Lemma A.3. *Given a pair of R-reflections $\{s, s'\} \subset W$ generating a finite subgroup, there is a spherical pair $\{r, r'\} \subset R$ such that $W_{\{s,s'\}}$ is conjugate to a subgroup of $W_{\{r,r'\}}$.*

In other words, every finite reflection subgroup of rank 2 is contained in a finite parabolic subgroup of rank 2.

Proof. Since $V = W_{\{s,s'\}}$ is finite, it is contained in some finite parabolic subgroup of W. We may thus assume without loss of generality that W is finite. Let $\mathbb{S}$ denote the underlying sphere of the corresponding Coxeter complex. The fixed point set $\mathbb{S}^V$ of V in $\mathbb{S}$ has codimension at most 2, since it contains the intersection of two equators. Therefore, the parabolic subgroup generated by all the reflections fixing $\mathbb{S}^V$ pointwise contains V and has rank at most 2. □

Corollary A.4. *Let $S \subset W$ be a Coxeter generating set such that every spherical pair of elements of S is conjugate to some pair of elements of R. Then every spherical pair of elements of R is conjugate to some pair of elements of S.*

Proof. By Corollary A.2, every element of R is an S-reflection. Therefore, given a pair $\{r, r'\} \subset R$, Lemma A.3 (with the roles of S and R

interchanged) yields a pair $\{s, s'\} \subset S$ such that $W_{\{r,r'\}}$ is conjugate to a subgroup of $W_{\{s,s'\}}$. By hypothesis the pair $\{s, s'\}$ is conjugate to some pair $\{t, t'\} \subset R$. In particular $W_{\{r,r'\}} \subset wW_{\{t,t'\}}w^{-1}$ for some $w \in W$. Since $W_{\{r,r'\}}$ and $W_{\{t,t'\}}$ are parabolic subgroups of the same rank, we deduce successively that we have $W_{\{r,r'\}} = wW_{\{t,t'\}}w^{-1}$ and then $\{r, r'\} = u\{t, t'\}u^{-1}$ for some $u \in W$. The result follows since the pairs $\{s, s'\}$ and $\{t, t'\}$ are conjugate. □

Appendix B. Reflection- and angle-deformations of twist-rigid Coxeter generating sets

The goal of this appendix is to prove Corollary 1.3. We present the basic facts from [HM04] and [MM08] needed for that purpose.

Let S be a Coxeter generating set for W. Following [HM04, Definition 5], we say that an element $\tau \in S$ is a **pseudo-transposition** if there is some $J \subset S$ such that the following conditions hold.

- **PT1:** The set J contains τ and for every $s \in S \setminus J$ either s and τ are not adjacent or s belongs to $J^\perp$.
- **PT2:** There is an odd number k such that J is of type C_k or $I_2(2k)$, and in the first case τ is the unique element of J commuting with all other elements of J except for one with which τ generates the dihedral group of order 8.

Suppose that τ is a pseudo-transposition. We then define ρ to be the longest word w_J of W_J, which is an involution and is central in W_J. Let a be the unique element of J different from τ and not commuting with τ. We set $\tau' = \tau a \tau$. Finally, we define $S' = S \cup \{\tau', \rho\} \setminus \{\tau\}$. It is shown in [HM04, Lemma 6] that S' is also a Coxeter generating set. We say that S' is an **elementary reduction** of S.

Lemma B.1. *Let S' be an elementary reduction of S. Then S is twist-rigid if and only if S' is twist-rigid.*

Proof. For $L \subset S \cup S'$, we denote, exceptionally, by $L^\perp$ the set of all elements of $S \cup S' \setminus L$ commuting with L. Note that if $L \subset S$ (resp. $L \subset S'$), then the set $S \setminus (L \cup L^\perp)$ (resp. $S' \setminus (L \cup L^\perp)$) is independent of whether we consider the usual or the exceptional definition of $L^\perp$.

Assume first that S is twist-rigid and let $L \subset S'$ be an irreducible spherical subset. We have to show that $S' \setminus (L \cup L^\perp)$ is **connected**, *i.e.* that it has only one connected component (for this and the other definitions see Section 8).

We denote $J' = J \cup \{\tau'\} \setminus \{\tau\}$, which is an irreducible spherical subset of S'. The set $K = J' \setminus \{\tau'\} = J \setminus \{\tau\}$ is also irreducible. Since S is twist-rigid, K does not weakly separate S and hence τ belongs to the unique connected component of $S \setminus (K \cup K^\perp)$. Since all elements of S adjacent to τ lie in $K \cup K^\perp$, we have $S \setminus (K \cup K^\perp) = \{\tau\}$ and consequently $S' \setminus (K \cup K^\perp) = \{\tau'\}$.

Thus every element of K is adjacent to every other element of S'. Hence there is no loss of generality in assuming $K \subset L \cup L^\perp$. Since K is irreducible, there are two cases to consider: either we have $K \subset L$ or $K \subset L^\perp$.

If $K \subset L$, then either we have $\tau \in L$ which implies $L = J'$, or else, in view of $S' = J' \cup K^\perp$, we have $L = K$. In the latter case $S' \setminus (L \cup L^\perp)$ is a singleton. In the former case we have $S' \setminus (L \cup L^\perp) = S \setminus (J \cup J^\perp)$ and this set is connected since S is twist-rigid. Thus, if $K \subset L$, then $S' \setminus (L \cup L^\perp)$ is connected, as desired.

If $K \subset L^\perp$, we first assume $\rho \in L$. Then $L = \{\rho\}$ and we are done because $S' \setminus (L \cup L^\perp) = S' \setminus (J' \cup J'^\perp) = S \setminus (J \cup J^\perp)$, which is connected since S is twist-rigid. We now assume $\rho \in L^\perp$. Then we also have $\tau' \in L^\perp$ which implies $L \subset S$. Moreover, then the set $S' \setminus (L \cup L^\perp)$ coincides with $S \setminus (L \cup L^\perp)$, which is connected since S is twist-rigid, and we are done.

Finally, it remains to consider the case where $K \subset L^\perp$ and ρ belongs to $S' \setminus (L \cup L^\perp)$. Then we also have $\tau' \in S' \setminus (L \cup L^\perp)$, hence L is contained S. It suffices to show that $S' \setminus (L \cup L^\perp \cup \{\rho\})$ is connected. The bijection from $S' \setminus (L \cup L^\perp \cup \{\rho\})$ onto $S \setminus (L \cup L^\perp)$, which maps τ' to τ and restricts to the identity outside $\{\tau'\}$, preserves the adjacency relation. Hence the connectedness of $S' \setminus (L \cup L^\perp \cup \{\rho\})$ follows from the connectedness of $S \setminus (L \cup L^\perp)$.

Similar arguments show that, conversely, if S' is twist-rigid and L is an irreducible spherical subset of S, then $S \setminus (L \cup L^\perp)$ is connected. □

A Coxeter generating set is called **reduced** if it does not contain any pseudo-transposition. For any Coxeter generating set S there is a sequence $S = S_1, \dots, S_n$ of Coxeter generating sets, where $n \leq |S|$, such that every S_{i+1} is an elementary reduction of S_i, and S_n is reduced ([HM04, Proposition 7]).

Theorem B.2 ([HM04, Theorem 1])**.** *Let R be a reduced Coxeter generating set for W. There is an explicit finite subgroup $\Sigma \subset \mathrm{Aut}(W)$ such that for each reduced Coxeter generating set S for W, there is some $\alpha \in \Sigma$ such that $\alpha(S)$ and R are reflection-compatible.*

This result is supplemented by the following. We use freely the terminology of [MM08].

Theorem B.3 ([MM08, Theorem 2])**.** *Let S and R be reflection-compatible Coxeter generating sets for W. Then there is a sequence $S = S_1, \dots, S_n$ of Coxeter generating sets, where $n \leq |S|$, such that every S_{i+1} is a J_i-deformation of S_i, for some $J_i \subset S_i$, and S_n and R are angle-compatible.*

Although in general J_i-deformations do not have to extend to automorphisms of W, this is in fact the case if S is twist-rigid.

Lemma B.4. *Let S be a twist-rigid Coxeter generating set for W. Then any J-deformation of S extends to an automorphism of W.*

Proof. The only J-deformation which might *a priori* not extend to an automorphism of W is described in [MM08, Section 7.6]. By [MM08, Definition 7.6], the sets $\{s\}$ and $J^\infty \setminus \{r\}^\perp$ fall into two distinct connected components of $S \setminus (\{r\} \cup \{r\}^\perp)$. Thus if S is twist-rigid, then $J^\infty \subset \{r\}^\perp$.

Moreover, we have $S = K \cup J^\perp \cup J^\infty$ by [MM08, Lemma 7.7] and every vertex of $J^\perp$ adjacent to some vertex of J^∞ actually belongs to $\{t\}^\perp$ by Condition (TWt) from [MM08, Definition 7.6]. It follows that the sets $\{s\}$ and $J^\infty \setminus \{t\}^\perp$ fall into two distinct connected components of $S \setminus (\{t\} \cup \{t\}^\perp)$. Thus if S is twist-rigid, then $J^\infty \subset \{t\}^\perp$.

We infer that if S is twist-rigid, then J^∞ must be contained in $\{r,t\}^\perp$. In view of [MM08, Lemma 7.14] the Coxeter generating set S and its J-deformation $\delta(S)$ have the same Coxeter matrix. □

We are now ready for the following.

Proof of Corollary 1.3. Let R be a twist-rigid Coxeter generating set for W. A sequence of elementary reductions transforms R into a reduced Coxeter generating set R'. By Lemma B.1 the set R' is twist-rigid.

Let now S be any other Coxeter generating set for W. Let S' be a reduced Coxeter generating set obtained from S by a sequence of elementary reductions. By Theorem B.2 there is an automorphism $\alpha \in \Sigma \subset \mathrm{Aut}(W)$ such that $\alpha(S')$ and R' are reflection-compatible. By Theorem B.3 and Lemma B.4 there is a sequence of J-deformations which extend to automorphisms of W transforming $\alpha(S')$ into a Coxeter generating set S'' such that S'' and R' are angle-compatible. By Theorem 1.1, the set R' is conjugate to S'' and consequently also to $\alpha(S')$ and to S'. Then by Lemma B.1 the set S is twist-rigid. This proves assertion (i).

A conjugate of the set S can be obtained from R' by composing with an element of Σ and an *a priori* bounded number of J-deformations and operations inverse to elementary reductions. This yields assertion (ii) and in particular assertion (iii). □

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