CHAPTER 27

ON A MATHEMATICAL MODEL FOR THICK SURFACES

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Motivated by previous work on elastic rods with self-contact, involving the concept of the global radius of curvature for curves (as defined by Gonzalez and Maddocks), we define the *global radius of curvature* $\triangle[X]$ for a wide class of parametric surfaces. It turns out that a positive lower bound $\triangle[X] \geq \theta > 0$ provides, naively speaking, the surface with a thickness of magnitude θ; it serves as an excluded volume constraint for X, prevents self-intersections, and implies that the image of X is an embedded C^1-manifold with a Lipschitz continuous normal. Taking into account possible applications to variational problems for embedded objects, we also obtain a convergence and a compactness result for such thick surfaces.

The main object of this note is to introduce and explain the crucial notions and results. Thus, we defer almost all proofs to our forthcoming paper.[22]

1. Introduction

Physical surfaces such as sheets of paper, thin elastic plates, pieces of cloth, or aluminium foil often undergo large deformations in space so that different parts of the same object touch each other. These self-contact phenomena can also be observed on various smaller length scales, especially in biological systems, e.g., pinched skin tissue, buckled membranes, or conformations of lipid vesicles under thermal influence. The underlying common feature of all these examples is that of a surface with a small but positive thickness reflecting the fact that interpenetration of matter is impossible. The mathematical modelling of the intuitively obvious mechanism of self-avoidance is a challenging task: one needs an analytically tractable notion of thickness for surfaces, which in particular should be accessible to variational methods in order to deal with energy minimization problems in the framework of nonlinear elasticity. Moreover, surfaces with positive thickness are embedded; hence a suitable notion of self-avoidance should also lead to a novel treatment of classical geometric boundary value problems such as the Plateau problem or free and semi-free problems in the class of *embeddings*. This would produce physically relevant solutions of fixed topological type without self-intersections — in contrast to the classical solutions, where one frequently encounters non-embedded solutions due to the geometry of the boundary configurations. (See the discussion on minimal surfaces in Ch. 4.10 of Dierkes et al.[4])

Our aim is to introduce a purely geometric notion of thickness for a large class of (nonsmooth) parametric surfaces suitable for the calculus of variations. Motivated by the second author's previous cooperations on elastic rods with self-contact[10,18,19,20], which involved the concept of the global radius of curvature for curves as suggested by Gonzalez and Maddocks,[9] we define the *global radius of curvature for surfaces.*

The idea can be sketched as follows. Take a continuous parametric surface $X : \mathbb{R}^2 \supset \mathbb{B}^2 \to \mathbb{R}^3$ of disk-type which possesses a tangent plane on a dense subset $G \subset \mathbb{B}^2$. (Note carefully that this is a very wide class of surfaces: the area of X can be infinite, and the "good" set G where the tangent plane exists might just be countable.) Consider the radii of all spheres touching the image $X(\overline{\mathbb{B}}^2)$ of X in one of the "good" points $X(w), w \in G$ and containing at least one other point of the surface. We define the infimum of these radii as the global radius of curvature $\triangle[X]$ of the surface X. It turns out that a positive lower bound on $\triangle[X]$ serves as an *excluded volume constraint* for the surface X.

In fact, one of our main results is that $\triangle[X] \geq \theta > 0$ implies that $X(\overline{\mathbb{B}}^2)$ is a $C^{1,1}$-manifold with boundary, where the domain size and the $C^{1,1}$-norms of the local graph representation of $X(\overline{\mathbb{B}}^2)$ are uniform and depend solely on the constant θ (Theorems 9 and 10). This result requires careful analysis of the normal in the interior and near the boundary, since the set $\mathbb{B}^2 \setminus G$ of bad points without a tangent plane is allowed to have full measure. In view of applications in the calculus of variations we prove that the excluded volume constraint in terms of the global radius of curvature is stable under pointwise convergence of parametrizations (see Theorem 13). Moreover, assuming a uniform upper bound on area and a uniform positive lower bound on the global radius of curvature for a family of surfaces we can prove the existence of a C^1-convergent subsequence to a limit manifold of class $C^{1,1}$ again with uniform control on the local graph representations (Theorem 14). This compactness result may be a crucial step towards the study of variational problems for embedded surfaces in geometry and non-linear elasticity. Let us also mention that our results carry over to arbitrary co-dimension and are not restricted to disk-type surfaces.

An alternative approach to prevent a surface from self-intersecting is to introduce explicit repulsive forces between pairs of points on the surface. Based on this idea Kusner and Sullivan[12] suggested a Möbius invariant *knot energy* for k-dimensional submanifolds in $\mathbb{R}^d$ without boundary. These highly singular potential energies, however, require some regularization to account for adjacent points on the surface and, apart from the one-dimensional case of knot energies for curves[17,7,11], there are no analytical results regarding existence of minimizers or their regularity. Banavar, Gonzalez, Maddocks and Maritan[1] proposed so-called *many-body potentials*, replacing the Euclidean distance between two points by geometric multipoint functions on curves, or *tangent-point distances* for surfaces as Lagrangians for multiple integrals, in order to avoid the technical difficulties arising from the singularities in the potential, and to introduce an intrinsic length-scale for thickness. Although not stated explicitly in their paper[1] Banavar et. al clearly had the concept of global curvature for smooth surfaces based on tangent-point distances in mind from which their many-body potentials arise. Apart from numerical investigations for tube-like chains in the protein science[2] based on this idea, however, there presently are, to the best of our knowledge, no analytical results in this direction, with one exception: For a particular example of a three-body potential, the so-called *total Menger curvature* on one-dimensional sets, there is a remarkable regularity result of Léger[13] motivated by removability problems for bounded analytical functions in the complex plane (i.e., Vitushkin's conjecture and its solution by David). Léger proved that a Borel set E with bounded total Menger curvature and with positive and finite one-dimensional Hausdorff measure is actually the union of Lipschitz graphs apart from a measure zero set. He also claimed an analogous result for higher dimensional objects without giving the proof. Another contribution to thickness of surfaces in terms of the classical injectivity radius and the geometric focal distance for $C^{1,1}$-smooth submanifolds without boundary is given by the work of Durumeric[5]. He proves among other things a compactness result based on Gromov's compactness theorem, and he provides upper bounds on the diffeomorphism and isotopy types for $C^{1,1}$-submanifolds with a uniform lower bound on the injectivity radius.

There are other papers that investigate the structure of surfaces under various weak assumptions imposed on geometric quantities. Semmes[21] considered hypersurfaces M^d in $\mathbb{R}^{d+1}$ whose normals have small norm in the space BMO of functions of bounded mean oscillation (such surfaces can twist and spiral, and be far from being graphs). He proved that each such M is a chord-arc surface with small constant, i.e. for each $x \in M$ and each $R > 0$, the intersection of the ball $B_R(x) := \{y \in \mathbb{R}^{d+1} : |y - x| < R\}$ with M stays close to the hyperplane that passes through x and is perpendicular to the mean value of the normal, $n_{x,r} = ⨍_{B_R(x)} n(y)dy$, taken w.r.t. the surface measure on M. Toro[23] proved that surfaces with generalized fundamental form in L^2 are Lipschitz manifolds (as a consequence, the graph of every function $u \in W^{2,2}(\Omega)$, where $\Omega \subset \mathbb{R}^2$, can be parametrized by a bi-Lipschitz map). Her work was later generalized by Müller and Šverák[16] who gave a sharp condition on the L^2-norm of the second fundamental form, guaranteeing that a complete, connected, noncompact surface immersed in $\mathbb{R}^d$ is embedded.

For surfaces S homeomorphic to $\mathbb{R}^2$ these results where sharpened by Bonk and Lang[3], who, to answer a conjecture of Fu[8], considered a very rich class of Alexandrov surfaces, with a notion of integral curvature defined as a signed measure μ on S (if S is smooth, then for each $A \subset S$ the value $\mu(A)$ is equal to the integral of Gaussian curvature over A w.r.t. the surface measure). They proved that if $\mu^+(S) < 2\pi$ and $\mu^-(S)$ is finite, then S is bi-Lipschitz equivalent to the plane. The bound 2π is sharp.

Our work is also related to Federer's notion of *sets of positive reach* introduced in his seminal paper[6] on curvature measures. In fact, Section 4 of that paper provides valuable tools for the proofs of our convergence and compactness results, see Theorems 13 and 14.

The presentation is structured as follows: In Section 2 we give the precise definitions of the class of admissible surfaces, of the global radius of curvature for surfaces and provide simple analytical and topological consequences. In Section 3 we first discuss a priori estimates for the normal line depending only on a positive lower bound for the global radius of curvature, and next use these estimates to describe the structure of the image of a thick surface X; this image turns out to be a $C^{1,1}$-manifold. Finally, Section 4 contains the convergence and the compactness results.

2. Basic definitions

2.1. *Admissible mappings*

We work with parametric surfaces $X\colon \overline{\mathbb{B}}^2 \to \mathbb{R}^3$, continuous up to the boundary of the unit disk. We also require X to be differentiable in the classical sense at all points $w \in G$, where G is a dense subset of $\mathbb{B}^2$ (which of course may depend on X), and we impose the condition

$$\operatorname{rank} DX(w) = 2 \qquad \text{for all } w \in G, \tag{2.1}$$

so that the tangent plane $T_wX\colon = X(w)+DX(w)(\mathbb{R}^2)$ is a well defined two-dimensional affine plane. Each such mapping X will be called *admissible*. The class of all admissible mappings is denoted by $\mathcal{A}(\mathbb{B}^2,\mathbb{R}^3)$. It is clear that $X \in \mathcal{A}(\mathbb{B}^2,\mathbb{R}^3)$ could *a priori* have infinite area. On the other hand, if $X \in C^1(\mathbb{B}^2,\mathbb{R}^3) \cap C^0(\overline{\mathbb{B}}^2,\mathbb{R}^3)$ is an immersion, then X is admissible.

If $X \in \mathcal{A}(\mathbb{B}^2,\mathbb{R}^3)$ is differentiable in the classical sense at w and $\operatorname{rank} DX(w) = 2$, then w is called a *good parameter*; hence $G \subset \mathbb{B}^2$ is the set of good parameters. Note that if Σ is an arbitrary two-dimensional Riemannian manifold with or without boundary then the class $\mathcal{A}(\Sigma,\mathbb{R}^3)$, and in fact also $\mathcal{A}(\Sigma,\mathbb{R}^d)$, where $d \geq 3$, can be defined in a similar way.

Remark. To give an example of a well investigated class of (nonsmooth) mappings where most of the above assumptions are automatically satisfied, we recall here the definition and a handful of properties of n-absolutely continuous functions (the class of these functions is denoted by AC^n). In our setting, $n = 2$. Let $\Omega \subset \mathbb{R}^n$. One says (cf. Malý[14]) that $f \in AC^n(\Omega,\mathbb{R}^d)$ whenever for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $k \in \mathbb{N}$ and every finite family of pairwise disjoint balls $B_1,\dots,B_k$ in Ω the following is satisfied:

$$\sum_{i=1}^{k} \mathcal{L}^n(B_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{k} \Big(\operatorname*{osc}_{B_i} f\Big)^n < \varepsilon,$$

where osc f stands for the sum of oscillations of all coordinates of f, and $\mathcal{L}^n$ denotes the Lebesgue measure.

Obviously, such mappings are continuous. Malý proves that such f's are also almost everywhere differentiable in the classical sense and have weak gradients in L^n (so that for $n = 2$ the area of f is finite!). Moreover, for $d \geq n$ the Lusin condition is satisfied, i.e. $\mathcal{H}^n(f(E)) = 0$ whenever $\mathcal{L}^n(E) = 0$, and the area formula holds.

Thus, for $\Omega \subset \mathbb{R}^2$, $AC^2(\Omega, \mathbb{R}^3)$ is a proper subspace of $W^{1,2}(\Omega, \mathbb{R}^3)$. The latter space contains discontinuous mappings; it also contains mappings which are continuous but nowhere differentiable in the classical sense. On the other hand, for bounded Ω we have

$$\bigcup_{p>2} W^{1,p}(\Omega, \mathbb{R}^3) \subset AC^2(\Omega, \mathbb{R}^3), \tag{2.2}$$

so that the class AC^2 is indeed larger than any of the Sobolev spaces $W^{1,p}$, $p > 2$. In fact, this inclusion is proper: consider the map $f\colon B_1(0) \to \mathbb{R}^2 \subset \mathbb{R}^3$ defined by $w \mapsto f(w) = \big(|w| \log(1 + |w|^{-1})\big)^{-1} w$. One can check that $f \in AC^2 \setminus W^{1,p}$ for every $p > 2$.

2.2. *Global radius of curvature for surfaces*

Let X be an admissible mapping, i.e., $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$. For all *good* $w' \in \mathbb{B}^2$ we set

$$n(w') := \frac{X_u(w') \wedge X_v(w')}{|X_u(w') \wedge X_v(w')|}.$$

Then for $w \in \mathbb{B}^2$ and $w' \in G \subset \mathbb{B}^2$ we define

$$\begin{aligned} &r(X(w); X(w'), DX(w')) \\ &: = \begin{cases} 0 \quad \text{if } X(w) = X(w'), \\ \infty \quad \text{if } X(w) \in T_{w'}X \text{ and } X(w) \neq X(w'), \\ \dfrac{|X(w) - X(w')|}{2\left|n(w') \cdot \frac{X(w)-X(w')}{|X(w)-X(w')|}\right|} \quad \text{in the remaining cases.} \end{cases} \end{aligned} \tag{2.3}$$

In plain words, $r(x, y, p)$ is the radius of the unique sphere through the points $x, y \in \mathbb{R}^3$ tangent to the affine plane $y + p(\mathbb{R}^2)$, where p is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ with rank $p = 2$. This radius becomes infinite when the vector $x - y \neq 0$ lies in the plane $p(\mathbb{R}^2)$, and is set to be zero if $x = y$.

Definition 1: For arbitrary $w \in \mathbb{B}^2$ we call

$$\rho[X](w)\colon = \inf_{\text{good } w' \in \mathbb{B}^2} r(X(w); X(w'), DX(w'))$$

the *global radius of curvature of X at w*, and

$$\triangle[X]\colon = \inf_{w\in\mathbb{B}^2} \rho[X](w)$$

the *global radius of curvature of X.*

The intuitive idea behind this concept is that a positive lower bound $\theta > 0$ on $\triangle[X]$ will allow us to place a pair of open balls of radius at least θ at each point of the surface "touching" the surface from both sides without intersecting it. From this we can infer that any surface X with $\triangle[X] \geq \theta$ satisfies the *excluded volume constraint* as described in the introduction. In particular, $X(\mathbb{B}^2)$ is an embedded surface in $\mathbb{R}^3$. Of course, all this requires proof, especially since *a priori* only good points $X(w)$, $w \in G$, — hence a possibly only countable set of surface points! — can be used in this construction. In the next section we describe just the backbone of the reasoning and refer to our paper[22] for more details.

As a first consequence of Definition 1 let us note that for any $w \in \mathbb{B}^2$ with $\rho[X](w) > 0$ one has $X(w) \neq X(w')$ for all $w' \in G$. Consequently, if $\triangle[X] > 0$, then $X(w) = X(\tilde{w})$ implies that either $w = \tilde{w}$ or that both w and $\tilde{w}$ are "bad" parameters (i.e., $w, \tilde{w} \in \mathbb{B}^2 \setminus G$).

Moreover, if $w' \in G$ and $\triangle[X] \geq \theta > 0$ then the two open balls B_1, B_2 of radius θ centered at $X(w') \pm \theta n(w')$ do not intersect the surface $X(\mathbb{B}^2)$, since otherwise we could find a point $X(w)$ such that $r(X(w); X(w'), DX(w')) < \theta$ contradicting our assumption on $\triangle[X]$. We shall sometimes call B_1, B_2 *excluded* or *forbidden* balls.

Remark. Let us note that for admissible mappings $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^d)$, $d > 3$, the global radius of curvature $\triangle[X]$ can be defined analogously. The definition of $r(x, y, p)$ remains unchanged. There is, however, one notable difference. For every good parameter w' in the domain we have now — instead of two excluded touching balls centered at $X(w') \pm \theta n(w')$ — a forbidden region

$$U_{w'} = \bigcup_{q\in S_{\theta,w'}} B_\theta(q),$$

where the set of centers

$$S_{\theta,w'} := \mathbb{S}_\theta^{d-1}(X(w')) \cap N_{w'}X$$

is given by the intersection of the round $(d-1)$-dimensional sphere $\mathbb{S}_\theta^{d-1}(X(w')) = \{s \in \mathbb{R}^d : |s - X(w')| = \theta\}$ with the affine normal space

$N_{w'}X = X(w') + \big(DX(w')(\mathbb{R}^2)\big)^{\perp}$. Thus, $U_{w'}$ looks, roughly speaking, like a thick degenerate doughnut. We have $\dim N_{w'}X = d-2$ for good w', therefore $S_{\theta,w'}$ is in fact a $(d-3)$-dimensional sphere in $N_{w'}X$. (Note that for $d=3$ the centers of B_1, B_2 do form a zero dimensional sphere contained in the normal line.) Analogously to the codimension 1 case, $X(\mathbb{B}^2)\cap U_{w'}$ is empty.

3. Structure of thick surfaces

3.1. *Interior continuity of the normal*

Let $X\colon \mathbb{B}^2 \to \mathbb{R}^3$ be an admissible surface with $\triangle[X] \geq \theta > 0$, and let $\varrho \in (0,\theta)$. Assume that $w \in \mathbb{B}^2$ is a good parameter, i.e., $w \in G$. Let

$$\ell(w) :\, = \big\{X(w) + tn(w) \mid t \in \mathbb{R}\big\}$$

be the (affine) normal line to X at w. We set

$$\begin{aligned} d(w) &:= \operatorname{dist}\big(X(w), X(\partial\mathbb{B}^2)\big),\\ C_\varrho(w) &:= \{p \in \mathbb{R}^3 \,|\, \operatorname{dist}(p, \ell(w)) = \varrho\}, \end{aligned}$$

and write π_w to denote the orthogonal projection onto the affine tangent plane T_wX.

To prove that the normal direction to X is uniformly continuous on compact subsets of the disk, we show first that at every good point w the image of X stretches away from $X(w)$ in all directions parallel to T_wX, as long as the distance from $X(w)$ is comparable to θ. Intuitively, the surface cannot fold abruptly at length scales much smaller than θ: close to *every* straight line through $X(w)$ in T_wX intersected with $B_r(X(w))$ we see points of the surface, as long as $r < \theta \leq \triangle[X]$ and the boundary $X(\partial\mathbb{B}^2)$ is far away.

Definition 2: We say that X has the *ϱ-stretching property* at $w \in G \subset \mathbb{B}^2$ iff

$$\pi_w\big(C_\varrho(w) \cap X(\mathbb{B}^2) \cap B_{2\varrho}(X(w))\big)$$

is a circle of radius ϱ in the tangent plane T_wX.

Lemma 3: *Assume that $\triangle[X] \geq \theta > 0$. If w is a good parameter and $\varrho \in (0, \varrho_0]$, where $\varrho_0 :\, = \min\big(\theta, d(w)/2\big)$, then X has the ϱ-stretching property at w.*

Since the proof of this lemma is short and contains a topological argument which (a) shows how the definition of good parameters is used and (b) is applied later, in a modified version, to control the behavior of the normal at the boundary, we give the details.

Proof. Fix $w \in G$ and $\varrho \in (0, \varrho_0]$. Without loss of generality we assume that $X(w) = O = (0,0,0) \in \mathbb{R}^3$ and the normal $n(w) = (0,0,1)$. Let $B_1 = B_\theta(0,0,\theta)$ and $B_2 = B_\theta(0,0,-\theta)$; then $X(\mathbb{B}^2) \cap (B_1 \cup B_2) = \emptyset$. Since $\operatorname{rank} DX(w) = 2$, the curve $X(\partial B_\delta(w))$ is, for some sufficiently small $\delta \in (0,\varrho)$, linked with the normal line $\ell(w)$. Now, let $I \subset C_\varrho(w) \backslash (B_1 \cup B_2)$ be a fixed (but otherwise arbitrary) vertical line segment contained in $B_{2\varrho}(O)$ and having its endpoints on ∂B_1 and ∂B_2.

To show that $X(\mathbb{B}^2) \cap I$ is nonempty, consider a homotopy $(\gamma_s)_{s\in[0,1]}$ from $\gamma_0 = X(\partial B_\delta(w))$ to $\gamma_1 = X(\partial \mathbb{B}^2)$, defined as a composition of X with a homotopy from $\partial B_\delta(w)$ to $\partial \mathbb{B}^2$ in $\overline{\mathbb{B}}^2 \setminus B_\delta(w)$. Let σ be the closed curve consisting of I and two straight segments that join the endpoints of I to $O = X(w)$. The curves γ_0 and σ are linked, whereas γ_1 and σ are not linked, for otherwise we would have $\operatorname{dist}(X(w), \gamma_1) < \varrho$, contrary to the definition of ϱ_0. (See the figure below.)

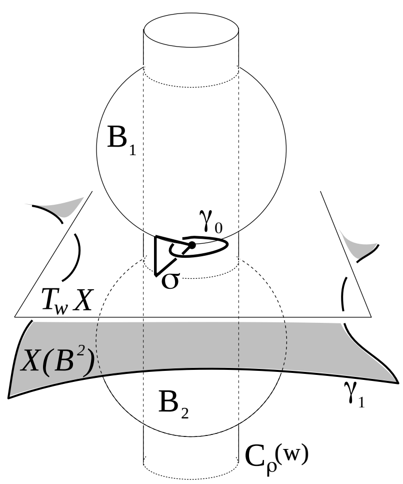

Figure 1. Touching tangent balls B_1 and B_2, and the curves $\sigma, \gamma_0, \gamma_1$.

It follows that γ_s must, for some $s \in (0,1)$, contain some point $p \in \sigma$. Certainly $p \notin B_1 \cup B_2$. Moreover, $p \neq 0 = X(w)$ since w is a good parameter. Thus, $p \in I$. This completes the proof of the lemma. □

Next, we obtain the following.

Lemma 4: *Let $\triangle[X] \geq \theta > 0$. If $w, w' \in \mathbb{B}^2$ are good parameters such that*

$$|X(w) - X(w')| < \min\bigl(\theta, d(w)/2\bigr)$$

and $\alpha(w, w') \in [0, \frac{\pi}{2}]$ is the angle between the normal directions at w and w', then

$$\alpha(w, w') \leq \frac{5\pi}{\theta}|X(w) - X(w')|. \tag{3.1}$$

The proof is lengthy but elementary. It follows from the stretching property that $\alpha(w, w')$ cannot be too large, for otherwise the forbidden balls associated to w' would contain some point in $X(\mathbb{B}^2)$ close to $T_w X$.

Since the estimate of Lemma 4 is uniform, and the set of good parameters is dense, we immediately obtain the following.

Corollary 5: *The normal direction has a continuous extension to all $w \in \mathbb{B}^2$ and the estimate* (3.1) *holds for all w, w' such that $|X(w) - X(w')| \leq \min\bigl(\theta, d(w)/2\bigr)$.*

Since now we can speak of an affine normal line $\ell(w)$ at every point $X(w)$, $w \in \mathbb{B}^2$, we can associate to *each* point on the surface a pair of open balls of radius θ touching the surface without intersecting it:

Corollary 6: *Let $w \in \mathbb{B}^2$. Then*

$$X(\mathbb{B}^2) \cap B_1 = X(\mathbb{B}^2) \cap B_2 = \emptyset$$

for the two open balls B_1, B_2 centered on the normal line $\ell(w)$ with radius θ, and touching each other at $X(w)$.

The proof is a simple *reductio ad absurdum* based on Hausdorff convergence of excluded balls: if there were some points of the surface in B_1 or B_2, then one of the excluded balls B_1^j, B_2^j associated to $w_j \in G$, where $w_j \to w$ as $j \to \infty$, would contain these points for j sufficiently large, a contradiction. As in Corollary 5 we need here that the set G of good parameters be dense in $\mathbb{B}^2$.

3.2. *Continuity of the normal at the boundary*

From now on we assume that X is an admissible mapping with a rectifiable boundary contour $X(\partial\mathbb{B}^2)$ and with $\triangle[X] \geq \theta > 0$. Moreover we suppose that the global radius of curvature $\triangle[X\big|_{\partial\mathbb{B}^2}]$ of the curve $X(\partial\mathbb{B}^2)$ (as defined in Gonzalez et al.[10], p. 35) is bounded below by θ. Note carefully that from now on $\triangle[\cdot]$ is used to denote two closely related but formally different notions. We always distinguish the argument in brackets, to avoid misunderstanding.

Theorem 7: Let $w \in \partial\mathbb{B}^2$. If $(w_j)_{j=1,2,\ldots} \subset G \subset \mathbb{B}^2$ is a sequence of good parameters such that $w_j \to w$ as $j \to \infty$ and the normal vectors

$$n(w_j) \colon = \frac{X_u \wedge X_v(w_j)}{|X_u \wedge X_v(w_j)|} \stackrel{j\to\infty}{\longrightarrow} \nu \in S^2,$$

then for every good parameter $w' \in \mathbb{B}^2$ such that $|X(w') - X(w)| \leq \theta/100$ we have

$$\alpha(w, w') \leq \frac{100}{\theta}|X(w) - X(w')|, \tag{3.1}$$

where $\alpha(w, w') \in [0, \frac{\pi}{2}]$ is the angle between the affine normal line $\ell(w')$ and the line $\ell(w) = \{X(w) + t\nu \mid t \in \mathbb{R}\}$. In particular, $\ell(w)$ does not depend on the choice of the sequence $(w_j) \subset G$.

As before, by density, this theorem implies the following.

Corollary 8: *The normal direction (and therefore T_wX) has a continuous extension to all $w \in \overline{\mathbb{B}}^2$ and the estimate*

$$\alpha(w, w') \leq \frac{500}{\theta}|X(w) - X(w')| \tag{3.2}$$

holds for all $w, w' \in \overline{\mathbb{B}}^2$ such that $|X(w) - X(w')| \leq \theta/400$.

Moreover, for all $w \in \overline{\mathbb{B}}^2$ we have

$$X(\mathbb{B}^2) \cap B_1 = X(\mathbb{B}^2) \cap B_2 = \emptyset$$

for the two open balls B_1, B_2 centered on the normal line $\ell(w)$ with radius θ, and touching each other at $X(w)$.

We omit the proofs. The main idea behind the proof of Theorem 7 is that if the surface contains a point $X(w')$, $w' \in \mathbb{B}^2$, such that $\delta := |X(w) - X(w')|$ does not exceed, say, $\theta/100$, then it contains also lots of other points $X(w'')$ lying very close to some half-circle of radius δ, centered at $X(w)$ and perpendicular to ν. This is the boundary counterpart of the stretching property from the previous section. Next, one shows that the normal direction at $X(w')$ must be close to ν, for otherwise the excluded balls associated to w' would contain one of the points $X(w'')$ constructed in the first step of the proof. This part of the argument is tedious but reduces to elementary geometry.

3.3. *Structure of the image*

For $p = X(w) \in X(\overline{\mathbb{B}}^2)$, $\pi_w : \mathbb{R}^3 \to T_w X$ the orthogonal projection, and $\ell(w)$ the affine normal line passing through p let

$$V_\varrho(p) : = \{q \in \mathbb{R}^3 \mid \operatorname{dist}(q, \ell(w)) < \varrho,\ |q - \pi_w(q)| < \varrho\}$$

denote a solid open cylinder with axis parallel to $\ell(w)$, centered at $p = X(w)$, with radius $\varrho > 0$ and height 2ϱ.

Theorem 9: Let $\triangle[X] \geq \theta > 0$ and let $w \in G \subset \mathbb{B}^2$ be such that

$$\operatorname{dist}(X(w), X(\partial\mathbb{B}^2)) > 2\sigma\theta\,,$$

where $\sigma \in (0, 1/100]$ can be chosen at will. Then $X(\mathbb{B}^2) \cap V_{\sigma\theta}(X(w))$ is a graph of a function $g \in C^{1,1}(B^2_{\sigma\theta}(0), \mathbb{R})$ with $\|g\|_{C^{1,1}} \leq C/\theta$, where C is some absolute constant.

Thus, loosely speaking, a portion of the surface contained in a cylinder of size comparable to θ is a graph of a $C^{1,1}$ function. The norm of this function is estimated inversely proportional to θ.

Remark. In fact, the assumption $\triangle[X] \geq \theta$ is not applied directly in the proof. What matters is the existence of excluded touching balls for every point in $X(\mathbb{B}^2)$, as given in Corollary 6, and Lipschitz continuity (w.r.t. to distances measured in the image) of the normal direction $\ell(w)$. These two facts imply that the intersection of $X(\mathbb{B}^2)$ with a neighbourhood of $X(w)$ is a graph of some function g, and that g is everywhere differentiable and Lipschitz. Lipschitz continuity of Dg follows then from a simple trick, using again Lemma 4. Thus, Theorem 9 applies to any continuous surface for which the excluded balls exist at every point in the image such that the line joining their centers varies in a Lipschitz continuous way. The original parametrization is not really important here.

Theorem 10: Let $p \in X(\partial\mathbb{B}^2)$, where $X : \overline{\mathbb{B}}^2 \to \mathbb{R}^3$ is an admissible surface with $\triangle[X] \geq \theta$ such that $\triangle[X\big|_{\partial\mathbb{B}^2}] \geq \theta$ for some $\theta > 0$. Assume that $X\big|_{\partial\mathbb{B}^2}$ is weakly monotone. Then, there exists a function $g \in C^{1,1}(B^2_{\theta/300}(0))$ such that $\|g\|_{C^{1,1}} \leq C/\theta$,

$$X(\overline{\mathbb{B}}^2) \cap V_{\theta/300}(p) = \mathrm{Graph}\,(g\big|_{\Omega^+})\,,$$

where $B^2_{\theta/300}(0)$ is a disk in T_wX, and $\Omega^+ = \{(x, y) \in B^2_{\theta/300}(0) \mid y > \psi(x)\}$ for some function ψ of class $C^{1,1}(\mathbb{R})$ with $\psi(0) = \psi'(0) = 0$ and $\|\psi\|_{C^{1,1}} \leq C/\theta$.

(A familiar) example. An open rotational cylinder with two hemispheres of the same radius attached at both ends shows that $C^{1,1}$ regularity is optimal for thick surfaces (with pairs of excluded touching balls existing at every point in the image). This particular surface fails to be C^2 at all points where the hemispheres meet the cylinder.

The theorems of this section show how strong in fact the assumption $\triangle[X] \geq \theta > 0$ is. Even if the global curvature radius of the boundary curve $\gamma := X(\partial\mathbb{B}^2)$ is positive, γ can be badly knotted. However, if we know *in addition* that γ bounds a surface $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ with $\triangle[X] \geq \theta > 0$, then it follows from Theorems 9 and 10 that γ cannot be knotted! This is vaguely reminiscent of the famous Fáry–Milnor theorem[15] relating curvature to topology.

4. Convergence and compactness

Following Federer[6], we define the *reach* of a set $A \subset \mathbb{R}^d$ as the supremum of those $r \in \mathbb{R}$ for which every $x \in B_r(A)$ has a unique next point $a \equiv \Pi_A(x) \in A$, such that $\mathrm{dist}\,(x, A) = |x - \Pi_A(x)|$. If the set $A \subset \mathbb{R}^d$ is closed, then the map $\Pi_A(.) : B_{\mathrm{reach}\,(A)}(A) \to A$ is continuous (cf. Thm. 4.8(4) in Federer[6]). For $A \subset \mathbb{R}^d$ and $a \in A$ one defines the *tangent cone* T_aA as

$$\begin{aligned} T_aA := \{v \in \mathbb{R}^d \,|\, v = 0 \\ &\text{or } \forall \epsilon > 0\ \exists b \in A \cap B_\epsilon(a) \setminus \{a\} \text{ such that } \left|\tfrac{b-a}{|b-a|} - \tfrac{v}{|v|}\right| < \epsilon\}, \end{aligned}$$

which reduces to the classical (linear, not affine!) tangent plane $T_a\Sigma$, if $A = \Sigma \subset \mathbb{R}^d$ is a C^1-submanifold in $\mathbb{R}^d$. Thm. 4.18 in Federer[6] characterizes closed sets of positive reach by an inequality reflecting a uniform second order contact between the set and its tangent cone:

Theorem 11: For a closed set $A \subset \mathbb{R}^d$ and $t > 0$ one has $\mathrm{reach}(A) \geq t$ if and only if $2\,\mathrm{dist}\,(b-a, T_aA) \leq |b-a|^2/t$ for all $a, b \in A$.

Returning to our setting of admissible surfaces $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$, and invoking Lemma 3 (iii) of Gonzalez et al.[10], we easily prove the following characterization.

Lemma 12: *Let $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ with $\triangle[X] > 0$ and $\triangle[X\big|_{\partial\mathbb{B}^2}] > 0$ and $\theta > 0$. Then the following two statements are equivalent:*

(i) $\triangle[X] \geq \theta$ *and* $\triangle[X\big|_{\partial\mathbb{B}^2}] \geq \theta$,
(ii) $\mathrm{reach}(X(\overline{\mathbb{B}}^2)) \geq \theta$ *and* $\mathrm{reach}(X(\partial\mathbb{B}^2)) \geq \theta$.

Federer proves the following: *if $K \subset \mathbb{R}^d$ is compact, then for each $t > 0$ the set*

$$\{A \subset K, A \neq \emptyset, \mathrm{reach}(A) \geq t\}$$

is compact with respect to the Hausdorff distance. (See Remark 4.14 in his paper[6].) Applying this compactness theorem, we prove the following two results.

Theorem 13: Let a sequence $(X_j)_j \subset \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ satisfy $\triangle[X_j] \geq \theta > 0$ and $\triangle[X_j\big|_{\partial\mathbb{B}^2}] \geq \theta$, and let $X_j\big|_{\partial\mathbb{B}^2}$ be weakly monotone parametrizations of the boundary curves $X_j(\partial\mathbb{B}^2)$. Assume also that $X \in C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$ and that $X_j(\partial\mathbb{B}^2)$ converges to $X(\partial\mathbb{B}^2)$ in Hausdorff distance.

If $X_j(w) \to X(w)$ as $j \to \infty$ for all w belonging to some dense subset of $\mathbb{B}^2$, then

(1) the X_j are uniformly bounded;
(2) the sets $A_j :\ = X_j(\overline{\mathbb{B}}^2)$ converge in Hausdorff distance to a $C^{1,1}$-manifold A with $\mathrm{reach} A \geq \theta$, and we have $A = X(\overline{\mathbb{B}}^2)$.

Theorem 14: Let $X_j : \mathbb{B}^2 \to \mathbb{R}^3$ be a sequence of admissible surfaces with $\triangle[X_j] \geq \theta > 0$. Assume moreover that:

(1) $\sup_j \mathcal{H}^2(X_j(\overline{\mathbb{B}}^2)) \leq M < +\infty$;

(2) $X_j\big|_{\partial\mathbb{B}^2}$ are weakly monotone parametrizations of rectifiable Jordan curves with global radius of curvature uniformly bounded below by θ, and there exists some $R > 0$ such that each of the curves $X_j(\partial\mathbb{B}^2)$ contains a point $p_j \in B_R(0)$.

Then one can select a subsequence j' such that $A_{j'} \colon = X_{j'}(\overline{\mathbb{B}}^2)$ converge in Hausdorff distance to A, A is a $C^{1,1}$-manifold with boundary, and the nearest point projection $\pi_A \colon B_\theta(A) \to A$ is well defined.

Remarks. 1. It follows from our proofs that in both theorems above the limit manifold A is also equipped with local graph representations whose norms and sizes are uniformly controlled by θ, as described in Section 3.3. Moreover, Corollary 6 holds for A, i.e., we have two touching balls B_1 and B_2 at every point of A, and $A \subset \mathbb{R}^3 \setminus (B_1 \cup B_2)$.
2. The uniform area bound in the assumptions of Theorem 14 is satisfied when $\sup_j \|X_j\|_{W^{1,2}} \le M < +\infty$. The second part of assumption (2), i.e. the existence of a point $p_j \in X_j(\partial\mathbb{B}^2) \cap B_R(0)$, is obviously satisfied when the boundary contours converge to a fixed curve or are themselves fixed (as often encountered in the calculus of variations).
3. Using variants of Theorem 14 for closed surfaces of arbitrary fixed genus, one can prove that in the class of all $C^{1,1}$-surfaces Σ_g that have thickness (i.e., reach in the sense of Federer) greater than or equal to $\theta > 0$ there exists an *ideal* representative, i.e. a surface with minimal area.

Sketch of proof of Theorem 14.
Step 1. All $A_j = X_j(\overline{\mathbb{B}}^2)$ are contained in a fixed ball $B_R(0) \subset \mathbb{R}^3$. To see this, fix j and consider the covering $\{B_{\sigma_0\theta}(p) \mid p \in A_j\}$ of A_j. (Since each A_j is a $C^{1,1}$-manifold with uniform control on the local graph representation, cf. Theorems 9 and 10, we can find two absolute constants K and σ_0 such that, for each $j = 1, 2, \ldots$,

$$K^{-1}\sigma^2\theta^2 \le \mathcal{H}^2\big(B_{\sigma\theta}(p) \cap A_j\big) \le K\sigma^2\theta^2 \qquad (4.1)$$

whenever $\sigma \le \sigma_0$. In plain words, pieces of the surface in a ball of radius $\delta \lesssim \theta$ have their area comparable to the area of a flat disk of radius δ.) Apply Vitali's lemma to this covering, to obtain a (possibly finite) sequence of pairwise disjoint balls $B_{\sigma_0\theta}(p_k)$, where $p_k \in A_j$, such that $\{B_{5\sigma_0\theta}(p_k) \mid k = 1, 2, \ldots\}$ is a covering of A_j. Take N of these balls. Invoking (4.1), we obtain

$$NK^{-1}\sigma_0^2\theta^2 \le \sum_{k=1}^{N} \mathcal{H}^2(B_{\sigma_0\theta}(p_k) \cap A_j) \le \mathcal{H}^2(X_j(\overline{\mathbb{B}}^2)) \le M.$$

This yields $N \le KM\sigma_0^{-2}\theta^{-2}$, and a uniform bound for diam $(X_j(\overline{\mathbb{B}}^2))$ follows. Together with assumption (2) this implies that there is an $\tilde{R} > 0$ such that $\bigcup_j X_j(\overline{\mathbb{B}}^2) \subset B_{\tilde{R}}(0)$.

Step 2. Applying Federer's compactness theorem for A_j with reach$(A_j) \ge \theta$ by Lemma 12, we select a subsequence such that $A_j \to A$ in Hausdorff distance and the reach of A is greater or equal to θ.

Step 3. Take a small number $\varepsilon > 0$ and consider the covering of $B_{\varepsilon\theta}(A)$ given by $\{B_{\varepsilon\theta}(p) \mid p \in A\}$. Apply Vitali's lemma and select a (possibly finite) sequence $B_{\varepsilon\theta}(p_k)$, $k \ge 1$, such that

$$A \subset \bigcup_{k\ge 1} B_{5\varepsilon\theta}(p_k)\,.$$

We can assume that dist $(A_j, A) \le \varepsilon\theta/2$ for all j. Moreover, we can find points $p_{jk} \in A_j$, $j = 1, 2, \ldots$, such that $p_{jk} \to p_k$ as $j \to \infty$. For each j and each k, we have $B_{\varepsilon\theta/2}(p_{jk}) \subset B_{\varepsilon\theta}(p_k)$; thus, for each fixed j the balls $B_{\varepsilon\theta/2}(p_{jk})$, where $k \ge 1$, are pairwise disjoint. An argument analogous to the one carried out in Step 1 above shows that for a fixed small ε the index k can take only finitely many values, say $k = 1, \ldots, N$ with some $N = N(\varepsilon, M, \theta)$.

For each fixed j the balls $B_{6\varepsilon\theta}(p_{jk})$, $k = 1, \ldots, N$, form a covering of $B_{\varepsilon\theta}(A)$ and of $B_{\varepsilon\theta/2}(A_j)$. Let $n_{jk} \in \mathbb{S}^2$ be normal to $A_j = X_j(\mathbb{B}^2)$ at $p_{jk} =: X_j(w_{jk})$. Selecting finitely many subsequences, we may assume that

$$n_{jk} \to \nu_k \in \mathbb{S}^2 \qquad \text{as } j \to \infty,\ k = 1, \ldots, N,$$

and moreover that

$$|n_{jk} - \nu_k| < \delta \qquad \text{for all } j = 1, 2 \ldots \text{ and } k = 1, \ldots, N.$$

Now, fixing δ and ε sufficiently small, we can apply the results of the previous section to conclude that $A_j \cap B_{10\varepsilon\theta}(p_k)$ is contained in a graph of a function $g_{jk}\colon B^2_{20\varepsilon\theta} \to \mathbb{R}$ such that $g_{jk} \in C^{1,1}$, $\|g_{jk}\|_{C^{1,1}} \le \text{const}/\theta$, where $B^2_{20\varepsilon\theta}$ is a fixed disk in the plane passing through p_k and perpendicular to ν_k (i.e. in the plane which is the limit of $T_{w_{jk}}X_j$ as $j \to \infty$), $k = 1, \ldots, N$.

Since the Lipschitz norms of all Dg_{jk} are uniformly bounded, by the Arzela–Ascoli theorem we may assume that $g_{jk} \to g_k$ as $j \to \infty$ in the

C^1-topology, for each fixed $k = 1, \dots, N$. Moreover, each g_k is of class $C^{1,1}$ since the Lipschitz condition for Dg_{jk} is preserved in the limit $j \to \infty$.

Now, an arbitrary point $p \in A$ belongs to $B_{5\varepsilon\theta}(p_k)$ for some k. It follows from previous considerations that $A \cap B_{5\varepsilon\theta}(p)$ is a $C^{1,1}$-graph. □

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