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GAGLIARDO–NIRENBERG INEQUALITIES WITH A BMO TERM

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Abstract

We give a simple direct proof of the interpolation inequality $\|\nabla f\|^2_{L^{2p}} \leqslant C\|f\|_{\mathrm{BMO}}\|f\|_{W^{2,p}}$, where $1 < p < \infty$. For $p = 2$ this inequality was obtained by Meyer and Rivière via a different method, and it was applied to prove a regularity theorem for a class of Yang–Mills fields. We also extend the result to higher derivatives, sharpening all those cases of classical Gagliardo–Nirenberg inequalities where the norm of the function is taken in L^∞ and other norms are in L^q for appropriate $q > 1$.

1. *Introduction*

In various branches of analysis, in particular in the theory of partial differential equations, it often happens that one can estimate a function and its derivatives of a given (high) order. Such estimates might result, for example, from a PDE that satisfies a maximum principle or is supplemented with pointwise constraints for solutions. It is then important to be able to derive good estimates for the intermediate derivatives. To this end, one usually applies various interpolation inequalities, in particular the Gagliardo–Nirenberg inequalities

$$\|\nabla^j f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|^{1-\theta}_{L^r(\mathbb{R}^n)}\|\nabla^\ell f\|^{\theta}_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

where $\theta = j/\ell \in (0,1)$ and $q^{-1} = \theta p^{-1} + (1-\theta)r^{-1}$, $1 \leqslant p, r \leqslant \infty$. In this paper we show that for $r = \infty$ and $p > 1$, (1.1) can be improved. In other words, the condition that f be bounded can be relaxed to $f \in \mathrm{BMO}$, where BMO denotes the space of functions of *bounded mean oscillation*. (All the necessary function spaces are defined at the end of the introduction.) Now, BMO also contains unbounded functions; moreover, in any dimension n there exists a sequence of smooth compactly supported functions ψ_j such that $\|\psi_j\|_{\mathrm{BMO}} \to 0$ as $j \to \infty$, whereas $\max|\psi_j| = 1$ for all j. These new inequalities are thus applicable to a wider class of functions, and even for the old class of functions they yield sharper conclusions.

Our work is inspired by Meyer and Rivière's paper [**11**]. These authors (see [**11**, Theorem 1.4]) prove the inequality

$$\|\nabla f\|^2_{L^4(\Omega)} \leqslant C(\Omega)\|f\|_{\mathrm{BMO}(\Omega)}\|f\|_{W^{2,2}(\Omega)}, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and they apply (1.2) to obtain a regularity theorem for a class of Yang–Mills connections. The advantages of (1.2) are clear: in numerous problems in nonlinear PDE, including for example harmonic maps and H-systems for $n = 2$, and other conformally invariant problems, one can obtain imbeddings into BMO without being able to use imbeddings into L^∞.

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An extension of (1.2) to other exponents $p > 1$ (see inequality (1.4) below) has recently been applied by Pumberger [**12**] to topics in partial regularity for stationary harmonic and J-holomorphic maps.

Kałamajska and Milani [**8**, p. 240] give a similar inequality in one dimension, namely,

$$\|u^{(j)}\|_{L^q(\mathbb{R})} \leqslant C\|u\|_{\mathrm{BMO}(\mathbb{R})}^{1-j/s}\|u\|_{H^s(\mathbb{R})}^{j/s} \tag{1.3}$$

for $0 \leqslant j \leqslant [s]$ and $q = 2s/j$.

The proof of (1.2) given in [**11**] is rather advanced: the authors use Besov spaces and their relation to BMO, Littlewood–Paley decomposition, and Whitney–like smoothings of Sobolev functions, combining all these tools in a tricky way. The proof of (1.3) in [**8**], suggested by H. Triebel, is similar in spirit: one has to invoke a number of results characterizing Triebel–Lizorkin spaces $F^s_{p,q}$ and their duals (see, for example, Triebel [**15**] or Adams and Hedberg [**1**] for definitions of these spaces; we shall not pursue that matter further). For related interpolation inequalities, see for instance [**3**], [**4**], [**9**], and [**10**].

Our main goal is to present a different, straightforward proof of (1.2). It relies on the duality of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and BMO. (This is not so surprising; if some quantity should be estimated by a product of two norms, one of them in BMO, then it is natural to look for a duality argument, and the Hardy space appears immediately.) The rest is reduced to simple applications of the Sobolev inequality, the Hardy–Littlewood maximal theorem and standard approximation arguments.

We also combine this reasoning with the classical Gagliardo–Nirenberg inequalities (1.1) to establish a full family of multiplicative inequalities that are similar to (1.2) and correspond to all those cases of (1.1) where $r = \infty$ and $p > 1$.

Here is the precise statement of these results. For the sake of simplicity we restrict our attention to functions defined on the whole space $\mathbb{R}^n$. It is clear that one may use standard extension methods to obtain local variants of both theorems.

THEOREM 1.1. *If $f \in W^{2,p}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, $p > 1$, then $\nabla f \in L^{2p}(\mathbb{R}^n)$ and*

$$\|\nabla f\|_{L^{2p}}^2 \leqslant C\|f\|_{\mathrm{BMO}}\|\nabla^2 f\|_{L^p} \tag{1.4}$$

for some constant $C = C(n,p)$.

THEOREM 1.2. *Assume that $f \in W^{k,p}(\mathbb{R}^n)$ for some $p > 1$ and $1 \leqslant m < k$, $m,k \in \mathbb{N}$. If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then $\nabla^m f \in L^q(\mathbb{R}^n)$ for $q := (k/m)p$ and*

$$\|\nabla^m f\|_{L^q} \leqslant C\|f\|_{\mathrm{BMO}}^{1-\theta}\|\nabla^k f\|_{L^p}^{\theta}, \qquad \text{where } \theta = m/k, \tag{1.5}$$

for some constant $C = C(k,m,p)$.

The method that we use to obtain Theorem 1.1 has a notable advantage when compared to the application of Littlewood–Paley decomposition, namely that one can modify it to obtain other sharp nonlinear variants of Gagliardo–Nirenberg inequalities. For example, for each $p \geqslant 2$ there exists a constant $C = C(n,p)$ such that

$$\int_{\mathbb{R}^n} |\nabla f|^{p+2}\,dx \leqslant C(n,p)\|f\|_{\mathrm{BMO}}^2 \int_{\mathbb{R}^n} |\nabla f|^{p-2}|\nabla^2 f|^2\,dx$$

for all smooth compactly supported f. Note that $|\nabla f|^{p-2}$ plays the role of a weight. This inequality can be applied to obtain quantitative gradient bounds for solutions of a wide class of nonlinear elliptic systems of the general form

$$-\operatorname{div}(|\nabla f|^{p-2}\nabla f) = G(x, f, \nabla f)$$

where the right-hand side G has critical growth; that is, $|G(x, f, \nabla f)| \leqslant \text{const}\,|\nabla f|^p$ (see the paper by T. Rivière and the author [**13**]).

NOTATION. Barred integrals always denote averages; that is, $⨍_A f\,dx = |A|^{-1}\int_A f\,dx$. Sometimes we also write $f_A = ⨍_A f\,dx$.

For a domain $\Omega \subset \mathbb{R}^n$, $W^{k,p}(\Omega)$ denotes the Sobolev space of all those functions in $L^p(\Omega)$ all of whose distributional partial derivatives up to order k also belong to $L^p(\Omega)$. For $p = 2$ the space $W^{k,2}(\mathbb{R}^n)$ is often denoted by $H^k(\mathbb{R}^n)$, and can be equivalently defined using the Fourier transform. In other words,

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in L^2 \text{ and } \|u\|^2_{H^k} : = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{k/2} |\hat{u}(\xi)|^2\, d\xi < \infty.$$

This definition makes sense for all $k > 0$. In what follows, we employ only standard, well-known properties of Sobolev spaces that can be found in [**2**] or [**16**]. In particular, we use the Sobolev inequality

$$\left(⨍_{B_r} |u - u_{B_r}|^p\, dx\right)^{1/p} \leqslant Cr\left(⨍_{B_r} |\nabla u|^{p_*}\, dx\right)^{1/p_*}, \tag{1.6}$$

where B_r is an arbitrary ball of radius r in $\mathbb{R}^n$, and $p_* := np/(n+p)$ for $p \geqslant n/(n-1)$.

BMO($\mathbb{R}^n$) stands for the space of functions of bounded mean oscillation, see for example [**14**, Chapter 4], with the seminorm

$$\|f\|_{\mathrm{BMO}} := \sup_Q \left(⨍_Q |f(y) - f_Q|\, dy\right), \tag{1.7}$$

the supremum being taken over all cubes in $\mathbb{R}^n$. (One can replace the average f_Q by any other constant c_Q; this does not affect the definition.) It is well known that BMO contains unbounded functions but their singularities are, roughly speaking, of logarithmic type. In particular,

$$|f_{Q_1} - f_{Q_2}| \leqslant C(n)\|f\|_{\mathrm{BMO}} \left(1 + \log \frac{|Q_2|}{|Q_1|}\right) \tag{1.8}$$

whenever $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $Q_1 \subset Q_2$.

The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ consists of all those $g \in L^1(\mathbb{R}^n)$ for which

$$g_* := \sup_{\varepsilon > 0} |\varphi_\varepsilon * g| \in L^1(\mathbb{R}^m).$$

Here and below, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for a fixed $\varphi \in C_0^\infty(B(0,1))$ with $\varphi \geqslant 0$ and $\int \varphi(y)\, dy = 1$. The definition does not depend on the choice of φ (see [**7**]). $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space with the norm $\|g\|_{\mathcal{H}^1} = \|g\|_{L^1} + \|g_*\|_{L^1}$. Finally, $(\mathcal{H}^1(\mathbb{R}^n))^* = \mathrm{BMO}(\mathbb{R}^n)$; see [**6**], [**7**], or [**14**, Chapter 4]. In particular,

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\, dx\right| \leqslant C(n)\|f\|_{\mathrm{BMO}}\|g\|_{\mathcal{H}^1} \tag{1.9}$$

whenever f is bounded and $g \in \mathcal{H}^1$; see [**14**, pp. 142–143].

Primes are used to denote Hölder conjugate exponents; that is, $p' = p/(p-1)$ for $p \geqslant 1$, and so on. Finally, the letter C stands for a general constant that may change its value even in a single string of estimates.

2. *The proofs*

The crucial difficulty is to prove Theorem 1.1 for smooth, compactly supported functions. Here we apply a trick that is very similar to that used, for example, by Coifman *et al.* [**5**] to prove that the Jacobian $\det Df$ of an arbitrary mapping $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ belongs to the Hardy space. That is,

$$\int |\nabla f|^{2p} = \left| \int f \, \mathrm{div}(|\nabla f|^{2p-2} \nabla f) \right| \qquad \text{for all } f \in C_0^\infty.$$

The latter integral can be bounded by an application of (1.9), *provided that* one can obtain an estimate of $g := \mathrm{div}(|\nabla f|^{2p-2}\nabla f)$ in the Hardy space $\mathcal{H}^1$. To prove such an estimate, we employ the definition of $\mathcal{H}^1$ given above, and we bound all the convolutions $g * \varphi_\varepsilon$ applying the Sobolev inequality (1.6) and the Hardy–Littlewood maximal theorem. The details of this reasoning are given below.

Theorem 1.2 follows from Theorem 1.1 by an inductive argument that uses classical Gagliardo–Nirenberg inequalities.

Proof of Theorem 1.1.

Step 1. Let $f \in C_0^\infty(\mathbb{R}^n)$. Integrating by parts, we write

$$\int_{\mathbb{R}^n} |\nabla f|^{2p} \, dx = - \int_{\mathbb{R}^n} f \, \mathrm{div}\big(|\nabla f|^{2p-2} \nabla f\big) \, dx. \tag{2.1}$$

We claim that $g := \mathrm{div}(|\nabla f|^{2p-2}\nabla f) \in \mathcal{H}^1(\mathbb{R}^n)$ and, moreover, that

$$\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C(n,p) \|\nabla f\|_{L^{2p}}^{2p-2} \|\nabla^2 f\|_{L^p}. \tag{2.2}$$

Once this estimate has been established, (2.1) combined with (1.9) yields

$$\begin{aligned} \|\nabla f\|_{L^{2p}}^{2p} &\leqslant C(n) \|g\|_{\mathcal{H}^1(\mathbb{R}^n)} \|f\|_{\mathrm{BMO}} \\ &\leqslant C(n,p) \|\nabla f\|_{L^{2p}}^{2p-2} \|\nabla^2 f\|_{L^p} \|f\|_{\mathrm{BMO}}, \end{aligned}$$

and we obtain (1.4) for smooth f. Thus, it is enough to prove (2.2). Recall the elementary inequality

$$\big| |X|^{t-2} X - |Y|^{t-2} Y \big| \leqslant C(t) |X - Y| (|X|^{t-2} + |Y|^{t-2}), \tag{2.3}$$

which is valid for all $t \geqslant 2$, $X, Y \in \mathbb{R}^n$. Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We set $m = \fint_{B(x,\varepsilon)} \nabla f \, dy$ and we estimate $g * \varphi_\varepsilon(x)$, using (2.3), and then Hölder's and Sobolev's inequalities as follows:

$$\begin{aligned} |g * \varphi_\varepsilon(x)| &= \left| \int_{B(x,\varepsilon)} \varphi_\varepsilon(x-y) \, \mathrm{div}\big(|\nabla f|^{2p-2} \nabla f - |m|^{2p-2} m\big)(y) \, dy \right| \\ &\leqslant \frac{C}{\varepsilon} \fint_{B(x,\varepsilon)} \big| |\nabla f|^{2p-2} \nabla f - |m|^{2p-2} m \big| \, dy \\ &\leqslant \frac{C}{\varepsilon} \fint_{B(x,\varepsilon)} |\nabla f - m| \big(|\nabla f|^{2p-2} + |m|^{2p-2} \big) \, dy \\ &\leqslant \frac{C}{\varepsilon} \left(\fint_{B(x,\varepsilon)} |\nabla f - m|^s \, dy \right)^{1/s} \left(\fint_{B(x,\varepsilon)} |\nabla f|^{2(p-1)s'} \, dy \right)^{1/s'} \\ &\leqslant C \left(\fint_{B(x,\varepsilon)} |\nabla^2 f|^{s_*} \, dy \right)^{1/s_*} \left(\fint_{B(x,\varepsilon)} |\nabla f|^{2(p-1)s'} \, dy \right)^{1/s'}. \end{aligned} \tag{2.4}$$

It is convenient here to choose $s > 1$ so that

$$2(p-1)s' < 2p \qquad \text{and} \qquad 1 \leqslant s_* = \frac{ns}{n+s} < p.$$

These requirements are satisfied, for example, for $s = pn/(n-1)$. Now, (2.4) implies that

$$\sup_{\varepsilon>0} |g * \varphi_\varepsilon(x)| \leqslant C\Psi_1(x)\Psi_2(x), \tag{2.5}$$

where

$$\Psi_1 \colon = \big[M\big(|\nabla^2 f|^{s_*}\big)\big]^{1/s_*}, \qquad \Psi_2 \colon = \big[M\big(|\nabla f|^{2(p-1)s'}\big)\big]^{1/s'},$$

and $M(\ldots)$ denotes the Hardy–Littlewood maximal function; that is,

$$Mv(x) \colon = \sup_{\varepsilon>0} \fint_{B(x,\varepsilon)} |v(y)|\, dy \qquad \text{for } v \in L^1_{\mathrm{loc}}.$$

Since $|\nabla^2 f|^{s_*} \in L^{p/s_*}$ and $p/s_* > 1$, the Hardy–Littlewood maximal theorem [**14**, p. 13] yields $\Psi_1 \in L^p(\mathbb{R}^n)$ and

$$\|\Psi_1\|_{L^p} \leqslant C(n,p)\|\nabla^2 f\|_{L^p}. \tag{2.6}$$

Similarly, $\Psi_2 \in L^{p'}(\mathbb{R}^n)$ and

$$\|\Psi_2\|_{L^{p'}} \leqslant C(n,p)\|\nabla f\|^{2p-2}_{L^{2p}}. \tag{2.7}$$

Combining these two bounds with (2.5), we conclude that $g_* = \sup_{\varepsilon>0} |g * \varphi_\varepsilon| \in L^1$; (2.2) follows from the Hölder inequality.

Step 2. The general case follows from an approximation argument. Since $C_0^\infty(\mathbb{R}^n)$ is *not* dense in $\mathrm{BMO}(\mathbb{R}^n)$, we provide some details. Let $Q(j) = [-j,j]^n \subset \mathbb{R}^n$. Fix two nonnegative cutoff functions, $\psi \in C_0^\infty(Q(1))$ with $\int \psi = 1$, and $\gamma \in C_0^\infty(Q(2))$ with $0 \leqslant \gamma \leqslant 1$ and $\gamma \equiv 1$ on $Q(1)$. Let $\psi_j(x) = j^n\psi(jx)$ and $\gamma_j(x) = \gamma(x/j)$. For $f \in W^{2,p}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ we set

$$f_j(x) = \psi_j * f(x) \quad \text{and} \quad h_j(x) = (f_j(x) - c_j)\gamma_j(x), \qquad j = 1, 2, \ldots,$$

where $c_j = (f_j)_{Q(j)}$ is the average of f_j on $Q(j)$. Then $h_j \in C_0^\infty(\mathbb{R}^n)$ and

$$\|\nabla^2 h_j - \nabla^2 f\|_{L^p} \to 0 \qquad \text{as } j \to \infty; \tag{2.8}$$

$$\|h_j\|_{\mathrm{BMO}} \leqslant C\|f\|_{\mathrm{BMO}}. \tag{2.9}$$

Estimate (2.8) is standard. To obtain (2.9), fix j and note that for every cube $Q \subset \mathbb{R}^n$ there is a cube $\widetilde{Q}$ such that

$$Q \cap Q(2j) = \widetilde{Q} \cap Q(2j) \quad \text{and} \quad |\widetilde{Q}| \leqslant (4j)^n.$$

Thus, since h_j is supported in $Q(2j)$, we can compute $\|h_j\|_{\mathrm{BMO}}$, taking into account only those cubes Q that are contained in $Q(6j)$. For such Q, we set

$$a_Q = (f_j - c_j)_Q(\gamma_j)_Q$$

and we estimate

$$\begin{aligned}
\fint_Q |h_j - a_Q|\, dx &= \fint_Q \big|(f_j - c_j)\gamma_j - (f_j - c_j)_Q(\gamma_j)_Q\big|\, dx \\
&\leqslant \fint_Q |f_j - (f_j)_Q|\,|\gamma_j|\, dx + |(f_j)_Q - c_j| \fint_Q |\gamma_j - (\gamma_j)_Q|\, dx \\
&\leqslant \|f_j\|_{\mathrm{BMO}} + |(f_j)_Q - (f_j)_{Q(j)}|\ \mathrm{diam}\, Q\ \sup|\nabla\gamma_j|.
\end{aligned}$$

Now, we write

$$|(f_j)_Q - (f_j)_{Q(j)}| \leqslant |(f_j)_Q - (f_j)_{Q(6j)}| + |(f_j)_{Q(6j)} - (f_j)_{Q(j)}|$$

and we apply (1.8) to conclude that

$$\begin{aligned} \fint_Q |h_j - a_Q|\, dx &\leqslant C(n)\|f_j\|_{\mathrm{BMO}}\left(1 + \log\frac{(6j)^n}{|Q|}\right)\frac{\operatorname{diam} Q}{j} \\ &\leqslant C(n)\|f_j\|_{\mathrm{BMO}} \\ &\leqslant C(n)\|f\|_{\mathrm{BMO}}. \end{aligned}$$

(The last inequality follows easily from the definition of convolution.) This yields inequality (2.9).

Combining (2.8) and (2.9) with the first step of the proof, we see that $(\nabla h_j)_{j=1,2,\dots}$ is a Cauchy sequence in L^{2p}. Since $\nabla h_j \to \nabla f$ almost everywhere at least for a subsequence, we obtain the desired conclusion upon passing to the limit $j \to \infty$. □

Proof of Theorem 1.2. We proceed by double induction with respect to m and k. As in the previous proof, it is enough to obtain the desired inequality for $f \in C_0^\infty$.

The case $m = 1$, $k = 2$, is contained in Theorem 1.1. Thus, we assume that (1.5) holds for some fixed $1 \leqslant m < k$ and all $p > 1$.

Let $p > 1$ and $r := ((k+1)/m)p$. Set $q = ((k+1)/k)p$. By (1.1) we have

$$\|\nabla^k f\|_{L^q} \leqslant C\|\nabla^m f\|_{L^r}^{1-\theta}\|\nabla^{k+1} f\|_{L^p}^{\theta}, \tag{2.10}$$

with $\theta = (k-m)/(k-m+1)$, whereas the inductive hypothesis yields

$$\|\nabla^m f\|_{L^r} \leqslant C\|f\|_{\mathrm{BMO}}^{1-m/k}\|\nabla^k f\|_{L^q}^{m/k}. \tag{2.11}$$

Using (2.10) to estimate the right-hand side of (2.11), and cancelling an appropriate power of $\|\nabla^m f\|_{L^r}$, we obtain (1.5) with k replaced by $k+1$. This is the first induction step.

To obtain the second induction step, fix $p > 1$, assume that $m + 1 < k$, and let $r := (k/m)p$ and $q := (k/(m+1))p$. We estimate the right-hand side of

$$\|\nabla^{m+1} f\|_{L^q} \leqslant C\|\nabla^m f\|_{L^r}^{1-\theta}\|\nabla^k f\|_{L^p}^{\theta},$$

where $\theta = 1/(k-m)$, invoking the inductive hypothesis

$$\|\nabla^m f\|_{L^r} \leqslant C\|f\|_{\mathrm{BMO}}^{1-m/k}\|\nabla^k f\|_{L^p}^{m/k}.$$

As before, a routine computation leads to (1.5), this time with m replaced by $m+1$. The whole proof is now complete. □

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