Paraconsistent and Approximate Semantics for the OWL 2 Web Ontology Language

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Motivations

- Semantic Web is promising.
- The Web Ontology Language (OWL)
  - a family of knowledge representation languages
  - a standard recommended by W3C for Semantic Web
  - OWL 2: the new version announced in October 2009

Motivations

- A problem of knowledge representation:
  - vagueness & inconsistency
- Rough set theory:
  - a mathematical approach to vagueness
  - Rough concepts deal with concept approximation.
- Paraconsistent reasoning:
  - an approach to dealing with inconsistency
  - a kind of approximate reasoning
- Use rough concepts and paraconsistent reasoning for OWL 2.

Outline

1. The Description Logic $\mathcal{SROIQ}$
   - Syntax and Semantics
   - Knowledge Bases
   - Conjunctive Queries
2. Rough Concepts in Description Logic
   - Approximating Concepts
   - Example
3. Paraconsistent Semantics for $\mathcal{SROIQ}$
   - Dealing with Inconsistencies
   - Our Paraconsistent Semantics for $\mathcal{SROIQ}$
   - The Relationship between the Semantics

The Description Logic (DL) $\mathcal{SROIQ}$: Introduction

About $\mathcal{SROIQ}$

- a logical base of OWL 2
- a decidable fragment of first-order logic
- with automated reasoning techniques

Elements of DL

- **individuals** : objects
- **concepts** : classes of objects
- **roles** : binary relations between objects
  - e.g., similarity relations are special roles

$\mathcal{SROIQ}$: Interpretations

An **interpretation** $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ consists of:

- a non-empty set $\Delta^\mathcal{I}$ (the **domain**)
- a function $\cdot^\mathcal{I}$ (the **interpretation function**) that maps
  - every **individual name** a to $a^\mathcal{I} \in \Delta^\mathcal{I}$
  - every **concept name** A to $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$
  - every **role name** r to $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$
  - $\top$ to $\top^\mathcal{I} = \Delta^\mathcal{I}$, and $\bot$ to $\bot^\mathcal{I} = \emptyset$.

If A is a **nominal** then $A^\mathcal{I}$ is a singleton set.
For the **universal role** U, it is required that $U^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

$\mathcal{SROIQ}$: Inverse Roles and Complex Concepts

Syntax	Example	Semantics w.r.t. $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$
r^-	$hasChild^-$	$(r^-)^{\mathcal{I}} = \{\langle x, y \rangle \mid \langle y, x \rangle \in r^{\mathcal{I}}\}$
$\neg C$	$\neg Male$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
$C \sqcap D$	$Human \sqcap Male$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqcup D$	$Mother \sqcup Father$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\forall R.C$	$\forall hasChild.Doctor$	$\{x \mid \forall y.\langle x, y \rangle \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\}$
$\exists R.C$	$\exists hasChild.Human$	$\{x \mid \exists y.\langle x, y \rangle \in R^{\mathcal{I}} \land y \in C^{\mathcal{I}}\}$
$\exists S.\mathtt{Self}$	...	$\{x \mid \langle x, x \rangle \in S^{\mathcal{I}}\}$
$\geq n\, S.C$	$\geq 2\, hasChild.Male$	...
$\leq n\, S.C$	$\leq 1\, hasChild.Female$	...

where S is a "simple role".

$\mathcal{SROIQ}$: Knowledge Bases

A knowledge base consists of:

- **RBox** (axioms about roles)

$$hasChild \sqsubseteq hasDescendant$$
$$hasDescendant \circ hasDescendant \sqsubseteq hasDescendant$$
$$hasParent \doteq hasChild^{-}$$

- **TBox** (definitions of concepts and terminological axioms)

$$Parent \doteq Human \sqcap \exists hasChild.Human$$
$$Father \doteq Parent \sqcap Male$$
$$Mother \doteq Parent \sqcap Female$$

- **ABox** (data about instances)

$$John : Father$$
$$Mary : Mother$$
$$hasChild(John, Jack)$$

$\mathcal{SROIQ}$: RBoxes, TBoxes and ABoxes

- An **RBox** is a finite set of axioms of the form:
  - $R_1 \circ \ldots \circ R_k \sqsubseteq S$, or
  - $\mathtt{Ref}(R)$, $\mathtt{Irr}(R)$, $\mathtt{Sym}(R)$, $\mathtt{Tra}(R)$, or $\mathtt{Dis}(R, S)$.
- A **TBox** is a finite set of axioms of the form $C \sqsubseteq D$.
  - An axiom $C = D$ can be expressed as: $C \sqsubseteq D$ and $D \sqsubseteq C$.
- An **ABox** is a finite set of **individual assertions** of the form:
  - $a \dot{\neq} b$, $C(a)$, $R(a, b)$, or $\neg S(a, b)$.
- Some restrictions are required to guarantee decidability.
- The semantics of boxes (in particular, the definition of $\mathcal{I} \models \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$) is as usual.

$\mathcal{SROIQ}$: Conjunctive Queries

- A **conjunctive query** is an expression of the form $\varphi_1 \wedge \ldots \wedge \varphi_k$ where each φ_i is an individual assertion.
- A query φ is a **logical consequence** of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, denoted by $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models \varphi$, if every model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ satisfies φ.

Rough Set Theory and Description Logic

- Rough set theory: Pawlak, 1982
- Characterizing approximations by modal operators:
  - e.g., Y.Y. Yao, 1996
- Extending DLs with rough concepts: Schlobach et al., 2007

Rough Concepts

- R : a role standing for a **similarity predicate**,
  $\mathcal{I}$: an interpretation, $x \in \Delta^{\mathcal{I}}$

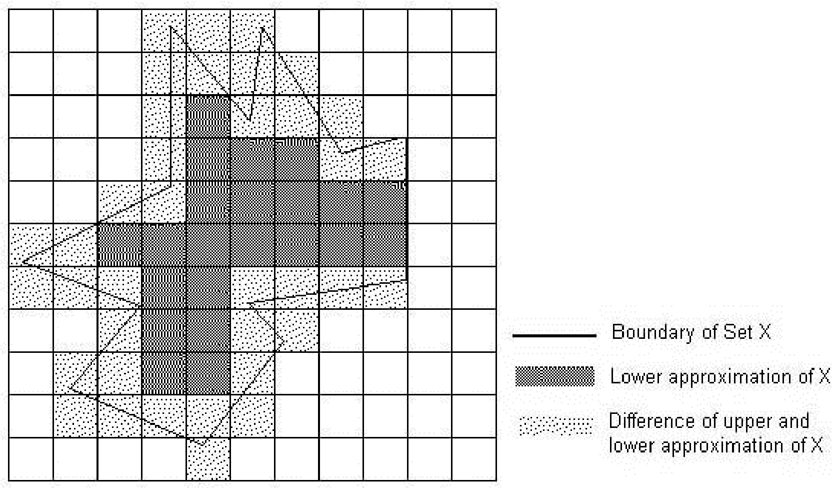
- the **neighborhood** of x w.r.t. R :
  $$n_R(x) \stackrel{\text{def}}{=} \{y \in \Delta^{\mathcal{I}} \mid \langle x, y \rangle \in R^{\mathcal{I}}\}$$

- the **lower approximation** of a concept C w.r.t. R :
  $$(\underline{C}_R)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid n_R(x) \subseteq C^{\mathcal{I}}\}$$

- the **upper approximation** of a concept C w.r.t. R :
  $$(\overline{C}_R)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid n_R(x) \cap C^{\mathcal{I}} \neq \emptyset\}$$

- $\langle \underline{C}_R, \overline{C}_R \rangle$ is called the **rough concept** of C w.r.t. R

illustration



Characterizations

Proposition [Schlobach et al., 2007; Y.Y. Yao, 1996; . . .]

$$(\underline{C}_R)^{\mathcal{I}} = (\forall R.C)^{\mathcal{I}} \text{ and } (\overline{C}_R)^{\mathcal{I}} = (\exists R.C)^{\mathcal{I}}$$

Correspondence (for a similarity predicate R)

- $(\underline{C}_R)^{\mathcal{I}} \subseteq (\overline{C}_R)^{\mathcal{I}}$: $\top \sqsubseteq \exists R.\top$
- **reflexivity** : $\mathtt{Ref}(R)$
- **symmetry** : $\mathtt{Sym}(R)$
- transitivity : $\mathtt{Tra}(R)$

Example

$\mathcal{A} = \{$*University*(*UW*), *has-name*(*UW*, *"University of Warsaw"*),
Institute(*IIUW*), *is-part-of*(*IIUW*, *UW*),
has-name(*IIUW*, *"Institute of Informatics, University of Warsaw"*),
Institute(*IMUW*), *is-part-of*(*IMUW*, *UW*),
has-name(*IMUW*, *"Institute of Mathematics, University of Warsaw"*),
works-at(*LANguyen*, *IIUW*), *teaches*(*LANguyen*, *SemanticWeb*),
has-name(*LANguyen*, *"Anh Linh Nguyen"*),
works-at(*HSNguyen*, *IMUW*), *teaches*(*HSNguyen*, *DataMining*),
has-name(*HSNguyen*, *"Hung Son Nguyen"*),
similar-name(*"Nguyen"*, *"Hung Son Nguyen"*),
similar-name(*"Nguyen"*, *"Anh Linh Nguyen"*),
similar-name(*"Nguyen"*, *"Linh Anh Nguyen"*),
similar-name(*"Anh Linh Nguyen"*, *"Linh Anh Nguyen"*),
University-of-Warsaw(*UW*),
Name-Linh-Anh-Nguyen(*"Linh Anh Nguyen"*)$\}$

Example

Knowledge Base

- $\mathcal{A} = \ldots$
- $\mathcal{R} = \{\textit{works-at} \circ \textit{is-part-of} \sqsubseteq \textit{works-at},$
  $\mathtt{Tra}(\textit{is-part-of}),\ \mathtt{Ref}(\textit{similar-name}),\ \mathtt{Sym}(\textit{similar-name})\}$
- $\mathcal{T} = \{\exists \textit{works-at}.\textit{University} \sqcap \exists \textit{teaches}.\top \sqsubseteq \textit{Academic-Teacher},$
  $\textit{Academic-Teacher} \sqsubseteq \textit{Teacher}\}$

Query

$$?x \;:\; \textit{Teacher} \sqcap \exists \textit{works-at}.\textit{University-of-Warsaw} \sqcap \exists \textit{has-name}.\textit{Name-Linh-Anh-Nguyen}$$

no results $\longrightarrow$ replace the above highlighted concept by

$$\exists \textit{similar-name}.\textit{Name-Linh-Anh-Nguyen}$$

Example

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The Problem with Inconsistencies

- Ontologies: distributed, dynamically growing, and hence easily affected by inconsistencies.
- When a knowledge base KB is inconsistent, the set $Cons(KB)$ of **logical consequences** of KB (w.r.t. the traditional semantics) contains all sentences.

Example

$$
\begin{aligned}
KB_1 &= \{Bird \sqsubseteq Fly\} \\
KB_2 &= KB_1 \cup \{Penguin \sqsubseteq Bird, \quad Penguin \sqsubseteq \neg Fly\} \\
KB_3 &= KB_2 \cup \{Bird(a), \quad Penguin(tweety)\}
\end{aligned}
$$

KB_3 is inconsistent. Using the traditional semantics, every query is a logical consequence of KB_3.

Which queries should be logical consequences of KB_3?

$Bird(tweety)$?

$Fly(a)$? $\qquad \neg Fly(a)$?

$Fly(tweety)$? $\qquad \neg Fly(tweety)$?

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Dealing with Inconsistencies

- Tolerate inconsistencies by paraconsistent reasoning.
- Define a paraconsistent semantics $\mathfrak{s}$ such that the set $Cons_{\mathfrak{s}}(KB)$ of logical consequences of KB w.r.t. semantics $\mathfrak{s}$ satisfies:
  - $Cons_{\mathfrak{s}}(KB) \subseteq Cons(KB)$
  - $Cons_{\mathfrak{s}}(KB)$ contains mainly only "meaningful" logical consequences of KB
  - $Cons_{\mathfrak{s}}(KB)$ approximates $Cons(KB)$ as much as possible.

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Dealing with Inconsistencies for DLs

- Many-valued semantics:
  - Four-valued semantics:
    - Meghini and Straccia 1996; Ma et al. 2008 & 2009;
    - based on Belnap's four-valued logic
  - Three-valued semantics:
    - Nguyen and Szałas, 2010: for the DL $\mathcal{SHIQ}$ (of OWL 1)
- Constructive (intuitionistic) semantics:
  - Odintsov and Wansing, 2008: for the DL $\mathcal{ALC}$

Our Paraconsistent Semantics for $\mathcal{SROIQ}$

We define **paraconsistent semantics** $\mathfrak{s}$ for $\mathcal{SROIQ}$, which are characterized by four parameters $\langle \mathfrak{s}_{\mathtt{C}}, \mathfrak{s}_{\mathtt{R}}, \mathfrak{s}_{\forall\exists}, \mathfrak{s}_{\mathtt{GCI}} \rangle$ standing for:

- $\mathfrak{s}_{\mathtt{C}}$: using 2-, 3-, or 4-valued semantics for concept names
- $\mathfrak{s}_{\mathtt{R}}$: using 2-, 3-, or 4-valued semantics for role names
- $\mathfrak{s}_{\forall\exists}$: interpreting concepts $\forall R.C$ and $\exists R.C$ in two ways
- $\mathfrak{s}_{\mathtt{GCI}}$: using weak, moderate, or strong semantics for terminological axioms (i.e. General Concept Inclusions).

Our Paraconsistent Semantics for $\mathcal{SROIQ}$ (2)

- $\mathfrak{s} = \langle \mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_{\mathrm{R}}, \mathfrak{s}_{\forall\exists}, \mathfrak{s}_{\mathrm{GCI}} \rangle \in \mathfrak{S}$, where
  $\mathfrak{S} = \{2,3,4\} \times \{2,3,4\} \times \{+, +-\} \times \{w, m, s\}$
- An $\mathfrak{s}$-**interpretation** $\mathcal{I}$ has the interp. function mapping
  - every concept name A to a pair $A^{\mathcal{I}} = \langle A_+^{\mathcal{I}}, A_-^{\mathcal{I}} \rangle$ of subsets of $\Delta^{\mathcal{I}}$ such that
    - if $\mathfrak{s}_{\mathrm{C}} = 2$ then $A_+^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus A_-^{\mathcal{I}}$
    - if $\mathfrak{s}_{\mathrm{C}} = 3$ then $A_+^{\mathcal{I}} \cup A_-^{\mathcal{I}} = \Delta^{\mathcal{I}}$
  - every role name r to a pair $r^{\mathcal{I}} = \langle r_+^{\mathcal{I}}, r_-^{\mathcal{I}} \rangle$ of binary relations on $\Delta^{\mathcal{I}}$ such that
    - if $\mathfrak{s}_{\mathrm{R}} = 2$ then $r_+^{\mathcal{I}} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus r_-^{\mathcal{I}}$
    - if $\mathfrak{s}_{\mathrm{R}} = 3$ then $r_+^{\mathcal{I}} \cup r_-^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

Our Paraconsistent Semantics for $\mathcal{SROIQ}$ (3)

The intuition behind $A^{\mathcal{I}} = \langle A_+^{\mathcal{I}}, A_-^{\mathcal{I}} \rangle$ is that:

- $A_+^{\mathcal{I}}$ gathers positive evidence about A
- $A_-^{\mathcal{I}}$ gathers negative evidence about A.

Thus, $A^{\mathcal{I}}$ can be treated as the function from $\Delta^{\mathcal{I}}$ to $\{\mathfrak{t}, \mathfrak{f}, \mathfrak{i}, \mathfrak{u}\}$:

$$A^{\mathcal{I}}(x) \stackrel{\text{def}}{=} \begin{cases} \mathfrak{t} & \text{for } x \in A_+^{\mathcal{I}} \text{ and } x \notin A_-^{\mathcal{I}} \\ \mathfrak{f} & \text{for } x \in A_-^{\mathcal{I}} \text{ and } x \notin A_+^{\mathcal{I}} \\ \mathfrak{i} & \text{for } x \in A_+^{\mathcal{I}} \text{ and } x \in A_-^{\mathcal{I}} \\ \mathfrak{u} & \text{for } x \notin A_+^{\mathcal{I}} \text{ and } x \notin A_-^{\mathcal{I}} \end{cases}$$

Similarly for $r^{\mathcal{I}} = \langle r_+^{\mathcal{I}}, r_-^{\mathcal{I}} \rangle$.

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The intuition behind $A^\mathcal{I} = \langle A^\mathcal{I}_+, A^\mathcal{I}_- \rangle$ is that:

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Similarly for $r^\mathcal{I} = \langle r^\mathcal{I}_+, r^\mathcal{I}_- \rangle$.

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Similarly for $r^{\mathcal{I}} = \langle r^{\mathcal{I}}_{+}, r^{\mathcal{I}}_{-} \rangle$.

Our Paraconsistent Semantics for $\mathcal{SROIQ}$ (4)

The interpretation function $\cdot^{\mathcal{I}}$ maps

- an inverse role R to a pair $R^{\mathcal{I}} = \langle R_+^{\mathcal{I}}, R_-^{\mathcal{I}} \rangle$
  defined by $(r^-)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle (r_+^{\mathcal{I}})^{-1}, (r_-^{\mathcal{I}})^{-1} \rangle$
- a complex concept C to a pair $C^{\mathcal{I}} = \langle C_+^{\mathcal{I}}, C_-^{\mathcal{I}} \rangle$ defined as follows:
  - $\top^{\mathcal{I}} \stackrel{\text{def}}{=} \langle \Delta^{\mathcal{I}}, \emptyset \rangle$, $\bot^{\mathcal{I}} \stackrel{\text{def}}{=} \langle \emptyset, \Delta^{\mathcal{I}} \rangle$
  - $(\neg C)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_-^{\mathcal{I}}, C_+^{\mathcal{I}} \rangle$
  - $(C \sqcap D)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_+^{\mathcal{I}} \cap D_+^{\mathcal{I}}, C_-^{\mathcal{I}} \cup D_-^{\mathcal{I}} \rangle$
  - $(C \sqcup D)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_+^{\mathcal{I}} \cup D_+^{\mathcal{I}}, C_-^{\mathcal{I}} \cap D_-^{\mathcal{I}} \rangle$
  - ...

  where $(\forall R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}}$ are dependent on $\mathfrak{s}_{\forall\exists}$.

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  - $(\neg C)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_{-}^{\mathcal{I}}, C_{+}^{\mathcal{I}} \rangle$
  - $(C \sqcap D)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_{+}^{\mathcal{I}} \cap D_{+}^{\mathcal{I}}, C_{-}^{\mathcal{I}} \cup D_{-}^{\mathcal{I}} \rangle$
  - $(C \sqcup D)^{\mathcal{I}} \stackrel{\text{def}}{=} \langle C_{+}^{\mathcal{I}} \cup D_{+}^{\mathcal{I}}, C_{-}^{\mathcal{I}} \cap D_{-}^{\mathcal{I}} \rangle$
  - ...

  where $(\forall R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}}$ are dependent on $\mathfrak{s}_{\forall\exists}$.

Example

Consider a Semantic Web service supplying information about stocks. Assume that a web agent looks for low risk stocks, promising big gain. The agent's query can be expressed by

$$(LR \sqcap BG)(x)$$

where LR and BG stand for "low risk" and "big gain", respectively.

Example (2)

For simplicity, assume that the service has a knowledge base consisting only of the following concept assertions (perhaps provided by different experts/agents):

$$LR(s_1), \neg LR(s_1), \neg LR(s_2), \neg BG(s_2), LR(s_3), BG(s_3).$$

We then consider the interpretation $\mathcal{I}$ with:

$$\begin{aligned} LR^{\mathcal{I}} &= \langle \{s_1, s_3\}, \{s_1, s_2\} \rangle \\ BG^{\mathcal{I}} &= \langle \{s_1, s_3\}, \{s_2\} \rangle. \end{aligned}$$

Example (3)

$LR^{\mathcal{I}} = \langle\{s_1, s_3\}, \{s_1, s_2\}\rangle$ and $BG^{\mathcal{I}} = \langle\{s_1, s_3\}, \{s_2\}\rangle$.

Using any semantics $\mathfrak{s} \in \mathfrak{S}$ with $\mathfrak{s}_C = 3$, we have that

$$(LR \sqcap BG)^{\mathcal{I}} = \langle LR_+^{\mathcal{I}} \cap BG_+^{\mathcal{I}}, LR_-^{\mathcal{I}} \cup BG_-^{\mathcal{I}}\rangle = \langle\{s_1, s_3\}, \{s_1, s_2\}\rangle,$$

meaning that:

$$(LR \sqcap BG)^{\mathcal{I}}(s_1) = \mathfrak{i}, \;\; (LR \sqcap BG)^{\mathcal{I}}(s_2) = \mathfrak{f}, \;\; (LR \sqcap BG)^{\mathcal{I}}(s_3) = \mathfrak{t}.$$

Our Paraconsistent Semantics for $\mathcal{SROIQ}$ (5)

$\mathcal{I} \models_{\mathfrak{s}} (C \sqsubseteq D)$ if

- case $\mathfrak{s}_{\mathtt{GCI}} = w$: $C_{-}^{\mathcal{I}} \cup D_{+}^{\mathcal{I}} = \Delta^{\mathcal{I}}$
- case $\mathfrak{s}_{\mathtt{GCI}} = m$: $C_{+}^{\mathcal{I}} \subseteq D_{+}^{\mathcal{I}}$
- case $\mathfrak{s}_{\mathtt{GCI}} = s$: $C_{+}^{\mathcal{I}} \subseteq D_{+}^{\mathcal{I}}$ and $D_{-}^{\mathcal{I}} \subseteq C_{-}^{\mathcal{I}}$.

$\mathcal{I} \models_{\mathfrak{s}} (R_1 \circ \ldots \circ R_k \sqsubseteq S)$ if $R_{1+}^{\mathcal{I}} \circ \ldots \circ R_{k+}^{\mathcal{I}} \subseteq S_{+}^{\mathcal{I}}$

...

$\mathcal{I} \models_{\mathfrak{s}} C(a)$ if $a^{\mathcal{I}} \in C_{+}^{\mathcal{I}}$

$\mathcal{I} \models_{\mathfrak{s}} R(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R_{+}^{\mathcal{I}}$

$\mathcal{I} \models_{\mathfrak{s}} \neg S(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in S_{-}^{\mathcal{I}}$

Some Properties

- De Morgans laws hold for the constructors.
- If $\mathfrak{s}_{\mathrm{C}} \in \{2, 3\}$ and $\mathfrak{s}_{\mathrm{R}} \in \{2, 3\}$ then $\mathfrak{s}$ is a 3-valued semantics,
  - i.e. $C_+^{\mathcal{I}} \cup C_-^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $R_+^{\mathcal{I}} \cup R_-^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ always hold.
- If $\mathfrak{s}_{\mathrm{C}} = 2$ and $\mathfrak{s}_{\mathrm{R}} = 2$ then $\mathfrak{s}$ coincides with the traditional (2-valued) semantics.

The Relationship between the Semantics

- Let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S} = \{2,3,4\} \times \{2,3,4\} \times \{+, +-\} \times \{w, m, s\}$, $\mathfrak{s} = \langle \mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_{\mathrm{R}}, \mathfrak{s}_{\forall\exists}, \mathfrak{s}_{\mathtt{GCI}} \rangle$, $\mathfrak{s}' = \langle \mathfrak{s}'_{\mathrm{C}}, \mathfrak{s}'_{\mathrm{R}}, \mathfrak{s}'_{\forall\exists}, \mathfrak{s}'_{\mathtt{GCI}} \rangle$.
- Define $\mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$ according to $w \sqsubseteq m \sqsubseteq s$ and $w \sqsubseteq s$.

Ordering Semantics

Define that $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ if:

- $\mathfrak{s}'_{\mathrm{C}} \leq \mathfrak{s}_{\mathrm{C}}, \mathfrak{s}'_{\mathrm{R}} \leq \mathfrak{s}_{\mathrm{R}}, \mathfrak{s}_{\forall\exists} = \mathfrak{s}'_{\forall\exists}$, and $m \sqsubseteq \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$; or
- $\mathfrak{s}'_{\mathrm{C}} \leq \mathfrak{s}_{\mathrm{C}} \leq 3, \mathfrak{s}'_{\mathrm{R}} \leq \mathfrak{s}_{\mathrm{R}} \leq 3, \mathfrak{s}_{\forall\exists} = \mathfrak{s}'_{\forall\exists}$, and $\mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$; or
- $\mathfrak{s}'_{\mathrm{C}} \leq \mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_{\mathrm{R}} = \mathfrak{s}'_{\mathrm{R}} = 2$, and $m \sqsubseteq \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$; or
- $\mathfrak{s}'_{\mathrm{C}} \leq \mathfrak{s}_{\mathrm{C}} \leq 3, \mathfrak{s}_{\mathrm{R}} = \mathfrak{s}'_{\mathrm{R}} = 2$, and $\mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$; or
- $\mathfrak{s}_{\mathrm{C}} = \mathfrak{s}'_{\mathrm{C}} = 2$ and $\mathfrak{s}_{\mathrm{R}} = \mathfrak{s}'_{\mathrm{R}} = 2$.

The Relationship between the Semantics

Theorem

Let $\mathfrak{s} \sqsubseteq \mathfrak{s}'$. Then $\mathfrak{s}$ is weaker than or equal to $\mathfrak{s}'$.
That is, for any knowledge base KB, $Cons_{\mathfrak{s}}(KB) \subseteq Cons_{\mathfrak{s}'}(KB)$.

Postulate

If $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ and KB is $\mathfrak{s}'$-satisfiable, then it is better to use $\mathfrak{s}'$ than $\mathfrak{s}$.

A Translation into the Traditional Semantics

A translation $\pi_{\mathfrak{s}}$, for the case $\mathfrak{s}_{\mathrm{C}} \in \{3, 4\}$, $\mathfrak{s}_{\mathrm{R}} \in \{2, 4\}$, $\mathfrak{s}_{\forall\exists} = +$ such that $KB \models_{\mathfrak{s}} \varphi$ iff $\pi_{\mathfrak{s}}(KB) \models \pi_{\mathfrak{s}}(\varphi)$ (see the paper for details).

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A Translation into the Traditional Semantics: Example

Let $\mathfrak{s}$ be any semantics with $\mathfrak{s}_C = 3$ and $\mathfrak{s}_{GCI} = m$.
We have $\pi_{\mathfrak{s}}(\langle \mathcal{T}, \mathcal{A} \rangle) = \langle \mathcal{T}', \mathcal{A}' \rangle$, where:

$$\mathcal{T} : \{Bird \sqsubseteq Fly,\; Penguin \sqsubseteq Bird,\; Penguin \sqsubseteq \neg Fly\}$$

$$\mathcal{A} : \{Bird(a),\; Penguin(tweety)\}$$

$$\mathcal{T}' : \{Bird_+ \sqsubseteq Fly_+,\; Penguin_+ \sqsubseteq Bird_+,\; Penguin_+ \sqsubseteq Fly_-\}$$

$$\mathcal{A}' : \{Bird_+(a),\; Penguin_+(tweety)\}.$$

We also have that

$$\begin{aligned}
\pi_{\mathfrak{s}}(Bird(tweety)) &= Bird_+(tweety)\\
\pi_{\mathfrak{s}}(Fly(tweety)) &= Fly_+(tweety)\\
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Conclusions

- We introduced and studied a number of different paraconsistent semantics for $\mathcal{SROIQ}$ in a uniform way, which approximate the traditional semantics better than the 4-valued semantics studied by other authors for DLs.
- We also study the relationship between the semantics and paraconsistent reasoning in $\mathcal{SROIQ}$ through a translation into the traditional two-valued semantics. Such a translation allows one to use existing tools and reasoners to deal with inconsistent knowledge.

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Thanks for your attention!