Second report for the PhD Thesis "Algebraic contact manifolds, their generalizations and applications"

(Author: Robert Śmiech)

Conclusions.

This is the second report of the doctoral thesis of Robert Śmiech. The thesis contains a number of results on holomorphic contact manifolds, their generalizations, and some related topics, on which the candidate has worked during his PhD studies.

In this second version the author has committed the changes indicated by the referees in the previous reviews. In my opinion this version is better written than the previous one. The author has added some details, has improved some proofs, and has made more evident what are his original contributions to the topic.

I specially appreciate Theorem 4.1.3, which presents a relation among the problem of characterizing adjoint varieties (the algebraic LeBrun–Salamon conjecture), and the Effective Non-Vanishing Conjecture of Kawamata.

In my previous report I already considered Śmiech's thesis adequate for a PhD degree in mathematics. I think that, in this improved version, the dissertation has gained depth, soundness and clarity, and I confirm that is is certain sufficient to grant a PhD in Mathematics.

Trento, March 12, 2024

Luis E. Solá Conde

Report for the PhD Thesis "Algebraic contact manifolds, their generalizations and applications" (Author: Robert Śmiech)

Description.

The thesis under review contains results obtained by the PhD candidate during his doctoral studies. The main topic of the thesis is the study of projective contact manifolds, which are algebro-geometric relatives of quaternion-Kähler manifolds. In fact, via the twistor space construction, the classification of the latter would reduce to the classification of the former. The main conjecture in this direction is the LeBrun–Salamon conjecture, which predicts that the only complex contact Fano manifolds are rational homogeneous. The conjecture is known to be true when the Picard number of the variety is larger than one, and one expects to find a unique contact Fano manifold of Picard number one for each simple complex Lie algebra different from $\mathfrak{sl}(n)$.

In the case of Picard number one, one would like to understand why the existence of a contact structure on a Fano manifold implies the existence of a large group of automorphisms of the manifold. If one further assumes that the contact Fano manifold X is not a projective space, the Lie algebra $\mathfrak{g}$ associated to such group is known to be isomorphic the space of global sections of the ample generator L of the Picard group of the variety. Then a possible strategy towards the LeBrun–Salamon conjecture is to prove first that L has a large number of sections, and use this to prove that the rank of the Lie algebra $\mathfrak{g}$ (which is known to be semisimple under the hypotheses of the original LeBrun–Salamon conjecture) is large enough. Then the theory of torus actions on algebraic varieties allows us to prove that the contact variety is rational homogeneous via equivariant K-theory. These ideas have been briefly described in the fourth chapter of the thesis.

A part of the thesis (Chapter 3) is devoted to the study of lower bounds for the dimension of the space of global sections of the smallest divisor L on a Fano manifold X satisfying that the antocanonical divisor $-K_X$ is linearly equivalent to a multiple of L (which is the thesis is called fundamental divisor of X). These results, which are interesting independently of their potential implications to the case of contact Fano manifolds, were obtained in collaboration with A. Höring.

Chapter 5 of the thesis deals with two generalizations of the concept of projective contact manifold. First of all, one may consider generically contact manifolds, that come with a twisted one form whose kernel is maximally non integrable on an open set. One may obtain one such manifold, for instance, by considering certain birational modifications of projective contact manifolds. The author surveys on the Master thesis of B. Mroczek, who has proved that a generically contact structure on a projective space is a contact structure. Then he consider singular contact varieties (requiring rational singularities, and a twisted 1-form which is contact on the smooth locus of the variety). Examples of singular contact varieties are, for instance, the projectivization of nilpotent orbits on simple Lie algebras. Contact varieties are closely related to the singular symplectic varieties, that have been studied by Beauville, Kaledin, et al. In fact, one can construct a symplectic variety out of a contact variety by a process called symplectization. In this way, for instance, one may infer the existence of a singular stratification compatible with the contact structure on every contact variety (Theorem 5,4,5). The author studies also how to construct contact varieties by quotienting a given contact variety by the action of a finite group of contactomorphisms, and proves some partial results on the induced 1-form on resolutions of singularities of a contact variety.

Finally, the thesis is concluded with a chapter devoted to (complex) symplectic Monge–Ampère differential equations. They are determined by hyperplane sections of Lagrangian Grassmannians, a fact that allows us to use algebro-geometric tools to study them. The author reports here on some results obtained in collaboration with Gutt, Manno and Moreno.

Remarks.

(1) General remarks: indentation is not used properly all along the thesis, and English grammar should be reviewed thoroughly.

(2) Page 9, paragraph 2: "allow in particular to reduce the classification problem ... to the question of nonemptiness..." I would rather say that the existence of a large enough number of independent sections of the fundamental linear system of a Fano contact manifold can be used to prove its homogeneity, by means of certain tools of the theory of torus actions. By the way, the concept of fundamental linear system/divisor/line bundle, that appears here for the first time, should be explained (or a cross-reference to its definition should be included).

(3) Page 11, first paragraph after the fourth statement: I do not think that "very rigid" means that "their class is not closed with respect to taking quotients or birational modifications.

(4) Page 12, Remark 1.2.2, line 5: in complex algebraic category $\rightarrow$ in the complex algebraic category.

(5) Page 14: "The letter L always means a line bundle......then this is a contact line bundle". L was used previously to denote the fundamental line bundle of a Fano manifold. Please explain.

(6) Page 15, Definition 2.1.2: the canonical sheaf should be defined.

(7) Page 31, Section 3.4. Structure of THE general element.

(8) Page 33. Consequently, the result for *subsequent* dimensions. What do you mean? What is the initial dimension you work with? I think this statement would deserved a more detailed proof in a PhD thesis (and not only a list of the ingredients you use).

(9) Page 44, statement of 4.5.5: "Moreover, if the actions...". The word "Moreover" makes the reader think that you still need the vanishing of the higher cohomology groups of L, L', which is not the case.

(10) Page 57, statement of Proposition 5.4.14. Delete "we can put $j = i$".

(11) Page 57, example 5.4.15: I do not think that noting that this is the "author's favourite example" is needed in the dissertation. Instead, the author could include a few words explaining why this example is particularly interesting (which is explain briefly at the end of the example).

(12) Page 64. The embedding of J^2 into a Lagrangian Grassmannian bundle, which is fundamental later, should be made more apparent in the text (perhaps by adding a definition or statement). By the way, this is later called "LGr-bundle" in some places. Please, introduce clearly the notation.

(13) Page 65, paragraph after Definition 6.3.3. This paragraph is poorly written and very difficult to understand: "of LGr-bundle" (you mean the one introduced in the previous page?), "is a hyperplane section" (of what?), etc.

Conclusions.

The study of contact manifolds and their generalizations is a wide subject that has attracted many differential and algebraic geometers in the last years, on which research is still very active. The thesis focuses mostly in the algebro-geometric point of view, and its relation with Mori theory, presenting a reasonably well-written introduction to the topic (Chapters 1 and 2), reviewing some recent results and tools (Chapter 4), and presenting the original contributions of the author (Chapters 3 and 5).

Chapter 6 is, in my honest opinion, quite disconnected from the main topic of the dissertation. The author's asserts in the introduction that "contact structures arise naturally in the context of the theory of partial differential equations"; I do not think that assertion has been sufficiently justified in the thesis, besides the fact that the space of 1-jets of a manifold supports a contact structure. On the other hand, the results contained in the chapter are interesting, and show that the author is capable of working in different areas of mathematics.

The material presented in the thesis is reasonably well-written. However, I would expect to find more details in some parts of it: in my opinion some proofs are too brief, and some ideas could have been explored in a deeper way.

Overall, the dissertation by Robert Śmiech constitutes a honest scientific contribution to the field, and I would certainly consider it as adequate for a PhD degree in mathematics.

Trento, July 18, 2023

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