PhD thesis Report

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Title: New Results in (Ob)structural Graph Theory
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1 Description of the field

The general contribution of the PhD thesis of Łukasz Bożyk resides in in the field of structural and algorithmic graph theory and, in particular, in the study of min-max dualities for graph parameters as well as their algorithmic applications. The study of graph parameters is a vibrant field of graph algorithms as such parameters that, when bounded, may lead to the design of efficient algorithm for several problems.

The main focus of the thesis, is the proof of universal obstructions characterizations for parameters in graphs and digraphs. This consist in detecting generic structures whose "containment" in a (di)graph certifies a lower bound for the parameter, while their absence implies that the value of the parameter is small. The results of the thesis are accompanied with interesting algorithmic applications, mostly related to the design of parameterized algorithms.

2 Parametric dualities

A (di)graph parameter is a function p mapping (digraphs) to numbers. Also a containment relation between graphs is defined with the help of a partial ordering relation $\leq$ on graphs. The parameter p is $\leq$-*monotone* if $G_1 \leq G_2 \Rightarrow \mathsf{p}(G_1) \leq \mathsf{p}(G_2)$. A parameterized graph $\mathcal{G} = \langle G_k \rangle_{k\in\mathbb{N}}$ serves as a *universal* $\leq$-*obstruction* for p if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph G where $\mathsf{p}(G) \geq f(k)$, $\leq$-contains G_k and, moreover, the value of the parameter on G_k is at least k. We refer $f : \mathbb{N} \to \mathbb{N}$ to the *gap function* of this universal obstruction. A considerable part of structural graph theory is related to the following questions:

> Given a $\leq$-monotone parameter p, is there a universal $\leq$-obstruction for p?
> In case of a positive answer to the above, what is the best possible gap function? (1)

The research program of providing an answer to the questions in (1) for concrete (interesting) instantiations of p has attracted a lot of research both in combinatorics and in algorithms. Such theorems detect regular patterns that determine the asymptotical behavior of graph parameters and reveal structural min-max dualities that have numerous applications in the design of algorithms, mostly parameterized and approximation.

3 Results

The two main combinatorial results of the first part of the thesis are the following.

(A) When $\leq_{\text{im}}$ is the immersion relation in tournaments and p_H is the minimum number of arcs in a tournament whose removal $\leq$-excludes H. According to the results of the thesis the $\leq_{\text{im}}$-universal obstruction G_k for p consists of k arc-disjoint copies of H. Moreover, the gap is $O_k(k^3)$.

(B) When $\leq_{\text{tp}}$ is the topological minor relation in tournaments and p_H is the minimum number of vertices in a tournament whose removal $\leq$-excludes H. Here the author proves that the $\leq_{\text{tp}}$-universal obstruction for p consists of k vertex disjoint copies of H. The gap is $O_k(k \log k)$.

Both above results belong to the more general topic of proving min-max dualities between coverings and packings of graphs patterns. Such properties are known as Erdős-Pósa dualities as they date back to the seminar result of Erdős-Pósa, asserting that every graph not containin k vertex-disjoint cycles (i.e., a cycle packing) contains $O(k \log k)$ vertices meeting all of its cycles (i.e., a cycle covering). This result has been later extended by Robertson and Seymour for the case where the packed and covered objects are graphs containing some planar graph H as a minor. Also they proved that such dualities fail when H is not planar. The study of analogous dualities for other containment relations, different than the minor ones is an interesting and non-trivial avenue of research and their study in directed graphs is so far more than incomplete. The above results (A) and (B) are a clear advance in this direction as they resolve the question for tournaments for the immersion and the topological minor relations, and also they do it with quite reasonable gap functions, improving previous results on special cases of these questions. This is done in Chapter 3 of the thesis.

In Chapter 4 the following algorithmic problem, related to to the duality of result (A), is examined (for every fixed digraph H):

> Given a tournament G and an integer k, are there $\leq k$ edges in G that when removed, the resulting directed graph $\leq_{\text{im}}$-excludes the graph H? (2)

The above problem belongs in the general family of *modification problems* where edge deletion is the modification operation that we are applying towards obtaining some desired property. That way, every Erdős-Pósa property generates a modification problem, when the cover parameter involves the removal of some set of vertices or edges.

The fourth chapter of the thesis is dedicated to the design of a polynomial kernel for problem (2) for every choice of H, i.e., a polynomial time algorithm that given an instance G, k, outputs an equivalent instance G', k' whose size is a polynomial function of k.

Whether a polynomial kernel exists for (2) (and for which instantiations of H) has been an interesting open problem. The thesis resolved it for every H. The proof is based on a novel and non-trivial adaptation of the so called *protrusion replacement technique* on tournaments. It appears that this is the first time this technique has been employed in order to deal with directed graphs.

The second part of the thesis studies the graph parameter of tree-cut width and it proves structural results related to possible counterparts of tree-cut width on directed graphs. Here

the containment relation is the immersion relation $\leq_{\text{im}}$.

One of the most important graph parameters in both structural and algorithmic graph theory is treewidth as it serves as a measure of the topological resemblance of a graph to the structure of a tree. In an attempt to define an edge-analogue of treewidth, Wollan defined a new parameter called *tree-cut width* that is $\leq_{\text{im}}$-monotone. This parameter has algorithmic applications for "edge-based" problems where general techniques for treewidth fail. According to Wollan, the universal obstruction for tree-cut width is the, so called *k-wall*, and this answered question (1) for this parameter. The definition of tree-cut is based on a tree-like decomposition of a graph where *two* quantities are minimized: the first is the *adhesion-width* expressing the max number of crossing edges corresponding to the edges of the underlying tree and the *bag-width* that expresses how edges are arranged around the vertices of the underlying tree. The thesis studies in depth this bi-parametric nature of tree-cut width.

Chapter 5 is dedicated to the bi-parametric study of tree-cut width and provides min-max duality results linking these two components of its definition to obstructing structures that can be seen as edge-analogues of the corresponding obstructing structures of treewidth. These are the appropriate "edge-based" notions of brambles and tangles. Also the min-max theorem is further supported by a bi-parametric graph searching game. The bi-parametric viewpoint adopted for the study of tree-cut is novel, interesting, and natural.

At this point it should be stressed that the only previous bi-parametric study for graph parameters was the one by Geelen and Joeris for the case of treewidth, where the answered question (1) by providing biparametric universal obstructions. The bi-parametric analogue of question (1) for tree-cut width remains a wide open problem. In my opinion, the results of the thesis provide the correct combinatorial ground for its resolution. Also, the proofs of the min-max dualities of Chapter 5 it may inspire bi-parametric analogues of other graphs parameters.

Chapter 6 can be seen as a brave attempt to define a natural analogue of tree-cut width for directed graphs and provide an answer to question (1), as the one given by Wollan for the undirected case. We stress that such an directed analogue has been defined for treewidth by Johnson, Robertson, Seymour, and Thomas and related research has revealed both min-max dualities and universal obstructions. How to develop the counterpart of this theory for tree-cut width is an important question. Chapter 6 proves a series of min-max results that reveal some structural prerequisites for such a theory and provides useful insight on the the structural challenges of this program.

The third part of the thesis is dedicated to the study of the following problem.

> Given a graph G and an integer k, are there $\leq k$ vertices in G that when removed, the resulting graph is a bipartite permutation graph? (3)

Again as in the case of the problem in (2) the above problem is a graph modification where the modification operation is vertex removal and the target property is the one of being a bipartite permutation graph. The main results of Chapter 7 are the following two.

- There is an $O(9^k \cdot |G|^9)$-time algorithm for instances (G,k) of the problem in (3).
- There exists a polynomial-time 9-approximation algorithm for the problem of computing, given a graph G, the minimum k for which (G, k) is a yes-instance of the problem in (3).

The above results are based on a "ordering based" characterization of bipartite permutation graphs given by Spinrad, Brandstädt, and Stewart. The proofs are based to a clever use of this equivalence.

4 Conclusion

The PhD thesis of Łukasz Bożyk provides a series of combinatorial results on dualities on (di)graph width parameters and a series of related algorithmic results on graph modification problems. The results of the thesis are strong, non-trivial, original, and innovative. They deal with important existing open questions and they constitute a solid advance in science. I believe that the techniques and/or ideas developed in this thesis should serve as a necessary guide to any further investigations on this topic.

The overall presentation of the thesis is very good. The thesis, besides its hight technical quality, is is very well written and structured and it is clear that the candidate had the reader in mind, providing important useful insight on the ideas and techniques that are used for the proofs.

Given the above, I consider that the thesis of Łukasz Bożyk is sufficient to grant a PhD.

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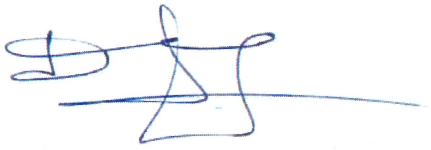

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